WALRASIAN ALLOCATIONS WITHOUT PRICE-TAKING BEHAVIOR[1]

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Abstract

Consider an exchange economy with a *finite* number of agents, who are arbitragers, in that they try to upset allocations imagining plausible beneficial trades. Their thought process is interactive, in that agents are conscious that the others are also going through the same steps. With this introspective process, each agent constructs a *supermarket*, i.e., a set of bundles that he considers achievable, in the sense that a sequence of plausible trades with other agents yields those bundles. We shed additional light on a result of Dagan (1996), by showing that Walrasian allocations can be characterized also as those where each agent chooses optimally from his supermarket. In addition, we extend the analysis to economies without short sales, where the characterization of Walrasian allocations is also obtained. Our analysis provides a different behavioral assumption for Walrasian allocations and connects with the core convergence theorem.

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1 Introduction

When we teach the Walrasian model of competition to undergraduates in economics, we often stumble with the same old difficulty: essentially, the model is not "closed," since it does not explain the source of a fundamental endogenous variable, the equilibrium price. For the one-market model of supply and demand, we often give an explanation that relies on agents that are not price-takers. If the market price were not the competitive one, but lower, some of the unsatisfied consumers would realize they could attract a seller by offering an ϵ over and above the market price. Similarly for sellers if the market price were higher than the equilibrium one.

Three features characterize this interpretation. (1) The agents in the economy are not price-takers, but arbitragers. (2) They are optimistic about their arbitrage possibilities. In particular, they ignore potential infeasibilities in their thought process: the unsatisfied consumers of the previous paragraph do not take into account that only a few more units will be sold at the market price plus ϵ. Thus, if they all hold the same belief, many of them are bound to be disappointed. And (3) the only allocation that survives the presence of these arbitragers in the market is the competitive one.

In the same spirit, our objective in this paper is to describe Walrasian allocations without price-taking behavior in *finite* economies where many goods are exchanged. This will constitute an attempt to "close" the Walrasian model through providing a different behavioral assumption behind competitive allocations.

In the last three decades, the game theoretic literature has succeeded in providing some insightful answers to the same question. These answers come both from the cooperative theory, with the well-known core and value equivalence principles (e.g., Debreu and Scarf (1963), Aumann (1964) and Aumann and Shapley (1974)); and from the non-cooperative theory, with either auction-like centralized procedures (as in Dubey (1982) and Dubey, Mas-Colell, and Shubik (1980)) or with the more recent study of sequential decentralized trading models (such as Gale (1986) or Dagan, Serrano, and Volij (1996)). In all of this work, prices do not appear explicitly and Walrasian allocations arise as the outcomes of either coalitional interaction or matching and bargaining processes in *large* economies.

The approach in this paper can be viewed as axiomatic, where the axiom is the arbitrager behavior imposed on agents. We follow Schmeidler and Vind (1972), Vind (1977), McLennan and Sonnenschein (1991) Theorem A, Makowski and Ostroy (1995) and Dagan (1996) in their specifications of equilibria based on abstract sets of net trades. Following Dagan (1996), we shall say that an arbitrage-free equilibrium is a list of bundles and sets of choice, one for each agent, such that: (1) the list of bundles is feasible for the economy as a whole, (2) the bundle assigned to each agent is top ranked in his preferences among the bundles in the agent's choice set, and (3) an agent's choice set must contain bundles that the agent can achieve after a certain process of recontracting. However, unlike in all these papers, our choice sets are not given exogenously to the agents.

We investigate the implications of an introspective process followed by the agents, which is based on sequences of hypothetical trades. For a two-agent economy the process runs as follows. Imagine an Edgeworth box and suppose an allocation x has been proposed and is under discussion. Before agents give consent to it, x is put to an "arbitrage-free" test. Originally, each agent i counts on his own endowment $Z_i^x(0)$ as being feasible for him. However, once the allocation x is being discussed, agent i may make the following proposal to agent j: "why don't you lend me your resources $Z_j^x(0)$? I will pay you back for them with a bundle that you prefer weakly to x." All trade under these contracts give rise to the set of bundles $Z_i^x(1)$ that agent i considers achievable, in the sense that they can be obtained through acceptable trade in the eyes of agent j. With the same arguments, agent j will think that the bundles in the set $Z_j^x(1)$ are achievable for him. Of course, agents can look further and their thought process could continue. Specifically, given that agent i believes that the bundles in the set $Z_i^x(t)$ are achievable for him and that so are those in the set $Z_j^x(t)$ for j, agent i may think he can repeat the speech above: "why don't you lend me your "resources" $Z_j^x(t)$? You think you can achieve all those bundles, right? I will pay you back for them with a bundle that you prefer weakly to x." This constructs the set $Z_i^x(t+1)$, and so on *ad infinitum.* This process of interactive introspection yields what we call *supermarkets* –the limit as $t \to \infty$ of $Z_i^x(t)$–, over which agents will make their final consumption decisions.[1]

After they construct these achievable sets based on exploiting all arbitrage opportunities, will

[1]Vind (1977) calls "markets" to his sets, but we are in America.

they agree to the allocation x? It turns out that agents' optimal choices over the supermarkets will be the bundles in the allocation x if and only if x is Walrasian. One message of our paper is that even if the world were populated by these arbitragers, the only allocations immune to their potentially upsetting thought process are the ones that Walras identified, but with no appeal to his auctioneer. Thus, if x is not Walrasian, at least one agent will believe that he can do better by exploiting the opportunities offered by x; an outside observer would not find x as the final state of the economy.

Before making their consumption decisions, our agents explore the bundles that one could achieve through reasonable contracts with the rest of the economy, and reasonable contracts over those contracts, and so on. Their strategizing is boundedly rational in one respect, though. Just like price-takers, who believe that any bundle below the price hyperplane is feasible, our agents do not realize the frequent infeasibilities that their thought process involves, as contracts in the second and later iterations are built upon the fulfillment of previous contracts in the thought process.

Based on the supermarkets, we provide a restatement and new proof of Dagan (1996) characterization of Walrasian allocations. The new proof is simple and, in addition, provides an alternative view of the result. In Dagan (1996), the auctioneer, instead of calling out prices, gives each agent a set of bundles from which to choose, which makes agents "choice set-takers." In contrast, our iterative construction is a way to replace the workings of the auctioneer.[2]

All previous results in this literature, including Dagan's and our Theorem 2, allow for unlimited short sales. The assumption of unlimited short sales is very strong, particularly outside of finance. In general, we are used to think of budget sets as subsets of the consumption set. We incorporate this restriction on our supermarkets and extend the analysis to economies without short sales. It turns out that the characterization result cannot be found under the assumptions made for Theorem 2, which included continuity and monotonicity of preferences, as well as in-

[2] Our constructive definition of the choice sets and that of Dagan (1996) resemble respectively the definitions of common knowledge due to Monderer and Samet (1989) and Aumann (1976). While the former looks at the iterative application of the operator "everybody knows," the latter gives a more compact definition, already in the limit, so to speak. Another comparison can be drawn with the two definitions of rationalizability (see Osborne and Rubinstein (1994), pp. 54-55).

teriority of endowments. We prove Theorems 3 and 4, which in addition require convexity of preferences.

Supermarkets are the smallest choice sets that can be coupled with a Walrasian allocation in an arbitrage-free equilibrium. They are in general strictly contained in a Walrasian budget set, although both sets essentially coincide in differentiable economies. When short sales are allowed, supermarkets inherit some other properties of budget sets: they can be written as the sum of each agent's initial endowments and a common (across agents) set of net trades, which is additive. In this sense, Walrasian allocations are anonymous, in that they offer the same set of net trades to each agent, with independence of his initial condition. We show that versions of these properties are retained in economies without short sales.

To understand the characterizations of Walrasian allocations by means of supermarkets, it is helpful to keep in mind their connections with the Debreu and Scarf (1963) core convergence theorem. Our agents' thought process resembles the trades that an agent could engage in had the economy been replicated. The two processes are distinct when one takes a finite number of iterations, but they surprisingly converge to the same set of bundles in the limit (Theorem 1). While in the core convergence theorem an agent seeks cooperation with an increasingly high number of agents, our agents envision increasingly complicated contracts among the same finite set of agents. The use of convexity in our characterization when short sales are precluded is another connecting point with core convergence.

Notice that a great deal of information is required by the agents in order to carry out the interactive thought process. They must know the endowments and preferences of the others. This contrasts with the informational parsimony of Walrasian allocations, as usually understood. The same large informational requirements are needed, though, for coalitionally beneficial trades that yield core allocations in large economies. We put aside this important issue in this paper, and leave it as an open question whether a characterization with less informational requirements is possible.

The paper is organized as follows: Section 2 is devoted to notation and preliminaries. The supermarkets and their relationship to replica economies are the subject of Section 3. When short sales are allowed, Section 4 reviews Dagan's characterization of Walrasian allocations by means of arbitrage-free equilibria and reinterprets it in terms of supermarkets. The relationship between supermarkets and budget sets is investigated in section 5. Section 6 studies economies without short sales and contains another characterization of Walrasian allocations that makes use of our construction. Section 7 concludes by discussing related literature.

2 Notation and Preliminaries

We denote by $\mathbb{R}^l$ the l-dimensional Euclidean space, by $\mathbb{R}^l_+$ its non-negative orthant and by $\mathbb{R}^l_{++}$ the interior of $\mathbb{R}^l_+$. Given two non-empty subsets A and B of $\mathbb{R}^l$, we denote by $A + B = \{x\ in \mathbb{R}^l \mid \exists a \in A, b \in B \text{ such that } x = a + b\}$. We denote by $A - B$ the set $A + (-B)$. We also postulate that $A + \emptyset = A$. Also, for $a \in \mathbb{R}^l$ and $B \subseteq \mathbb{R}^l$, we denote by $a + B$ the set $\{a\} + B$. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{Z}$ the set of integers.

Given x and $y \in \mathbb{R}^l$, we write $x \gg y$ whenever $x_i > y_i$, $i = 1, \ldots, n$ and $x \geq y$ whenever $x_i \geq y_i$, $i = 1, \ldots, n$.

An exchange economy is a system $\mathcal{E} = \langle N, (X_i, \succeq_i, \omega_i)_{i \in N} \rangle$, where N is a finite set that contains at least two agents; for each agent $i \in N$, $X_i \subseteq \mathbb{R}^l$ is i's consumption set, $\succeq_i$ is his reflexive preference relation over bundles in X_i, and $\omega_i \in \mathbb{R}^l$ is his initial endowment.

We refer to non-empty subsets S of N as coalitions. Let S be a coalition. An S-allocation in $\mathcal{E}$ is a list of bundles $(x_i)_{i \in S}$ such that $x_i \in X_i \ \ \forall i \in S$ and $\sum_{i \in S} x_i = \sum_{i \in S} \omega_i$. We refer to N-allocations simply as allocations and we denote the set of allocations in $\mathcal{E}$ by $\mathcal{A}(\mathcal{E})$.

For every $i \in N$ and $x_i \in X_i$, define the preferred and the weakly preferred sets as follows:

$$P_i(x_i) = \{z_i \in X_i \mid z_i \succ_i x_i\}$$

$$W_i(x_i) = \{z_i \in X_i \mid z_i \succeq_i x_i\}.$$

where $\succ_i$ is i's strict preference relation, defined as usual.

An allocation $(x_i)_{i \in N}$ in $\mathcal{E}$ is said to be *improved upon* by a coalition S if there exists $i \in S$ with

$$P_i(x_i) \cap \left[\sum_{k \in S} \omega_k - \sum_{k \in S \setminus \{i\}} W_k(x_k) \right] \neq \emptyset.$$

An allocation in $\mathcal{E}$ is said to be a *core allocation* if it cannot be improved upon by any coalition.

Given an economy $\mathcal{E}$, we can define its replica economies as follows. For every $m \in I\!N$, let $\overline{m} = \{1, 2, ..., m\}$. Let $\mathcal{E} = \langle N, (X_i, \succeq_i, \omega_i)_{i \in N} \rangle$ be an economy and let $x = (x_i)_{i \in N}$ be an allocation in $\mathcal{E}$. The m-replica of $\mathcal{E}$ is the economy $\mathcal{E}^m = \langle N \times \overline{m}, (X_{(i,j)}, \succeq_{(i,j)}, \omega_{(i,j)})_{(i,j) \in N \times \overline{m}} \rangle$, where for all $(i,j) \in N \times \overline{m}$, $X_{(i,j)} = X_i$, $\succeq_{(i,j)} = \succeq_i$, and $\omega_{(i,j)} = \omega_i$. The m-replica of x is $x^m = (x_{(i,j)})_{(i,j) \in N \times \overline{m}}$, where $x_{(i,j)} = x_i \ \forall (i,j) \in N \times \overline{m}$. An allocation x in $\mathcal{E}$ is a *shrunk core allocation* if x^m is a core allocation of $\mathcal{E}^m$ $\forall m \in I\!N$.

An allocation x in $\mathcal{E}$ is *Walrasian* if there exists $p \in \mathbb{R}^l \backslash \{0\}$, $p \sum_{k \in N} (x_k - \omega_k) = 0$, such that if $x'_k \in P_k(x_k)$, then $px'_k > p\omega_k$.

3 Supermarkets and Replica Economies

Consider for example a two-agent two-good exchange economy and suppose that the allocation x is under discussion. Denote agent 1's initial endowment $\{\omega_1\}$ by $Z_1^x(0)$ and agent 2's endowment $\{\omega_2\}$ by $Z_2^x(0)$. Suppose agent 1 proposes a trade in which agent 2 would give agent 1 his endowments in exchange for a bundle that leaves agent 2 no worse than at x. Denote by $Z_1^x(1)$ the set of bundles that agent 1 can achieve by means of these trades. Similarly, construct $Z_2^x(1)$ from proposals that agent 2 could make to agent 1. These sets of choice generated by the first round of arbitrage are the ones underlying core allocations, i.e., a core allocation x can be understood as one where each agent i maximizes over $Z_i^x(1)$ (see Dagan (1996) for more details).

The bundles in the set $Z_1^x(1)$ are considered achievable by agent 1 in the sense that they are delivered by contracts that agent 2 can reasonably accept. Similarly for agent 2 and the bundles in $Z_2^x(1)$. This is known by both agents. In taking arbitrage to its ultimate logical consequences, agents could envision more complicated contracts that would be still acceptable by the trading partner and further expand each agent's set of achievable bundles. Namely, agent 1 thinks that the bundles in $Z_2^x(1)$ can be obtained by agent 2 somehow and that he can acquire them provided that he leaves agent 2 reasonably happy. This new contracts give rise to the bundles in the set $Z_1^x(2)$, which agent 1 considers achievable. Similarly, agent 2 calculates the set $Z_2^x(2)$, and so on. Thus, in order to see what is the set of bundles from which he can consume, an agent imagines an infinite process of mutually beneficial trades (new rounds of contracts built on previous ones). The agents are too optimistic in assessing their arbitrage opportunities because the thought process involves infeasibilities: those contracts envisioned in iterations beyond the first one are built upon the fulfillment of previous contracts. The final sets of choice so generated are what we call "supermarkets" They are formally defined in the next paragraph.

Definition 1 Fix an allocation x in $\mathcal{E}$ and an agent $i \in N$. Let Z_i^x be defined recursively as follows:

$$Z_i^x(0) = \{\omega_i\} \qquad \forall i \in N,$$

for $t > 0$ define

$$Z_i^x(t) = \bigcup_{\substack{S \subseteq N \\ i \in S}} \Big[\sum_{k \in S} Z_k^x(t-1) - \sum_{k \in S \setminus \{i\}} W_k(x_k) \Big], \qquad i \in N$$

and finally,

$$Z_i^x = \bigcup_{t \in I\!N} Z_i^x(t).$$

Next we turn to define the set of feasible bundles for an agent when an allocation x is fixed, the economy is replicated any number m of times and cooperation is sought with any subset of agents.

Definition 2 Fix an allocation x in $\mathcal{E}$ and an agent $i \in N$. Define the following sets:

$$A_i^x(0) = \{\omega_i\} \qquad i \in N$$

$$A_i^x(m) = \bigcup_{\substack{S \subseteq N \times \overline{m} \\ (i,1) \notin S}} \Big[\omega_i + \sum_{(k,\ell) \in S} (\omega_k - W_k(x_k)) \Big] \qquad i \in N$$

$$A_i^x = \bigcup_{m \in I\!N} A_i^x(m).$$

The sets $A_i^x(m)$ are useful to study the core of replica economies. That is, by the definition of the core of an economy, the m-replica, x^m, of an allocation $x \in \mathcal{A}(\mathcal{E})$ is in the core of the m-replica $\mathcal{E}^m$, of $\mathcal{E}$, if for all $i \in N$,

$$P_i(x_i) \cap A_i^x(m) = \emptyset.$$

Note also that

$$a \in A_i^x(m) \Longleftrightarrow a \in \omega_i + \sum_{k \in S} n_k(\omega_k - W_k(x_k))$$

for some integers $0 < n_k \leq m$ for $k \in S \backslash \{i\}$, $0 \leq n_i < m$, for some $S \subseteq N$ with $i \in S$.

The interactive sets of choice $Z_i^x(t)$ differ from the sets of bundles $A_i^x(m)$ that an agent i could consider feasible if the economy $\mathcal{E}$ is replicated a number of times. We next present an example to illustrate this point.

Example 1 Consider a two-person, two-good exchange economy, where agents have identical preferences represented by the utility function $u(x_i) = \min\{x_i^1, x_i^2\}$, $i = 1, 2$ and endowments given by $\omega_1 = (0, 2)$ and $\omega_2 = (2, 0)$. See Figure 1.

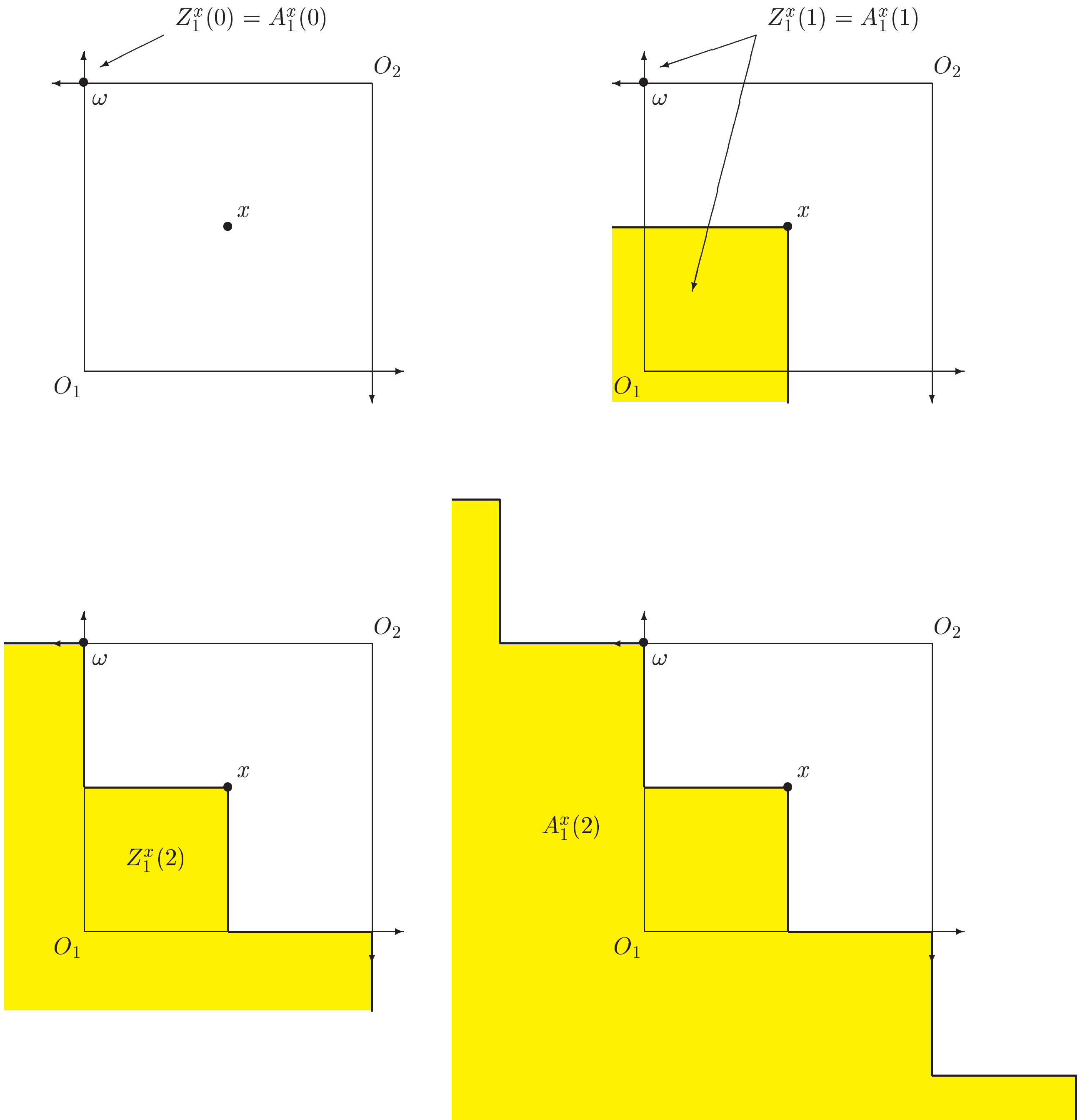
$Z_1^x(0) = A_1^x(0)$
O_2
ω
x
O_1
$Z_1^x(1) = A_1^x(1)$
O_2
ω
x
O_1
O_2
ω
x
$Z_1^x(2)$
O_1
O_2
ω
x
$A_1^x(2)$
O_1

Figure 1.

Consider the allocation x that assigns the bundle (1,1) to each agent. It is easy to see that $Z_1^x(0) = A_1^x(0) = \{(0,2)\}$ and $Z_2^x(0) = A_2^x(0) = \{(2,0)\}$. Next, it can be easily checked that

$$Z_1^x(1) = A_1^x(1) = \{(0,2)\} \cup ((1,1) - \mathbb{R}_+^2)$$

and

$$Z_2^x(1) = A_2^x(1) = \{(2,0)\} \cup ((1,1) - \mathbb{R}_+^2).$$

However, it can be checked that whereas

$$Z_1^x(2) = Z_2^x(2) = \{(0,2) - \mathbb{R}_+^2\} \cup ((1,1) - \mathbb{R}_+^2) \cup ((2,0) - \mathbb{R}_+^2),$$

$$A_1^x(2) = ((-1,3) - \mathbb{R}_+^2) \cup ((0,2) - \mathbb{R}_+^2) \cup ((1,1) - \mathbb{R}_+^2) \cup ((2,0) - \mathbb{R}_+^2)$$

and

$$A_2^x(2) = ((3,-1) - \mathbb{R}_+^2) \cup ((2,0) - \mathbb{R}_+^2) \cup ((1,1) - \mathbb{R}_+^2) \cup ((0,2) - \mathbb{R}_+^2).$$

The reader can check in Example 1 that, as long as the iterations are finite, the sets $Z_i^x(t)$ and $A_i^x(m)$ are not the same when $t = m > 2$. In the replica exercise, when agent i cooperates with agent k, he "buys" ω_k, while in the introspective process, he "buys" $Z_k^x(t)$, which in general contains more bundles. On the other hand, the replicas include coalitions that consist of only copies of agent i. These are ruled out in our thought process. These two reasons explain why in general there is no inclusion in either direction between $Z_i^x(t)$ and $A_i^x(m)$ when $t = m$. The difference between the two processes disappears in the limit, as the next result shows.

Theorem 1 Let $\mathcal{E}$ be an economy and let $x = (x_1, \ldots, x_n)$ be an allocation in $\mathcal{E}$. For all $i \in N$, $Z_i^x = A_i^x$.

Proof : The proof comprises the following two lemmas:

Lemma 1 Let $\mathcal{E}$ be an economy and let $x = (x_1, \ldots, x_n)$ be an allocation in $\mathcal{E}$. For all $i \in N$, $Z_i^x \subseteq A_i^x$.

Proof : Since the allocation x is fixed, we shall drop the superscript everywhere in what follows. The proof is by induction. Since $Z_i(1) = A_i(1)$, it is clear that $Z_i(1) \subseteq A_i$ for all $i \in N$. Assume now that $Z_i(t) \subseteq A_i$ for all $i \in N$. We want to show that $Z_i(t+1) \subseteq A_i$. So let $z \in Z_i(t+1)$. By definition of $Z_i(t+1)$, we have that $z \in \sum_{k \in F} Z_k(t) - \sum_{k \in F \setminus \{i\}} W_k(x_k)$ for some $F \subseteq N$ with $i \in F$. By the induction hypothesis, $Z_k(t) \subseteq A_k$ for all $k \in F$. By definition of A_k , $Z_k(t) \subseteq A_k(m_k)$ for some m_k, for all $k \in F$. Denoting $m = \max\{m_k : k \in F\}$ and recalling that $\{A_k(m)\}_{m \in I\!N}$ is an increasing sequence, we have $z \in \sum_{k \in F} A_k(m) - \sum_{k \in F \setminus \{i\}} W_k(x_k)$ for some $F \subseteq N$ with $i \in F$. By definition of $A_k(m)$, there must be sets $S_k \subseteq N \times \overline{m}$ with $(k,1) \in S_k$ such that $z \in \sum_{k \in F}(\sum_{(j,\ell) \in S_k} \omega_j - \sum_{(j,\ell) \in S_k \setminus (k,1)} W_j(x_j)) - \sum_{k \in F \setminus \{i\}} W_k(x_k)$ for some $F \subseteq N$ with $i \in F$. But this can be written as $\omega_i + \sum_{k \in F}(\omega_k - W_k(x_k))n_k$ for some integers $0 \leq n_k$, which means that $z \in A_i$. □

Lemma 2 Let $\mathcal{E}$ be an economy and let $x = (x_1, \ldots, x_n)$ be an allocation in $\mathcal{E}$. For all $i \in N$, $A_i^x \subseteq Z_i^x$.

Proof : Again the superscript x is dropped everywhere in what follows. Let $a \in A_i$. By definition of A_i, this means that

$$a \in \omega_i + \sum_{k \in F} (\omega_k - W_k(x_k))n_k$$

for some $F \subseteq N$ and some positive integers n_k. We shall show that Z_i contains the latter set.

By definition of Z_i, we have that

$$Z_k + Z_{k+1} - W_{k+1}(x_{k+1}) \subseteq Z_k \qquad \forall k \in N$$

where $k+1$ means $k+1 \bmod |N|$. Applying the same inclusion iteratively n_k times we have

$$Z_k + (n_k + 1)(Z_{k+1} - W_{k+1}(x_{k+1})) \subseteq Z_k \quad \forall k \in N. \tag{1}$$

By the definition of Z_i we have

$$Z_i + \sum_{k \in N \setminus \{i\}} (Z_k - W_k(x_k)) \subseteq Z_i. \tag{2}$$

Pick an individual $j \in N\backslash\{i\}$ (since there are at least two agents in the economy, this can be done). By the definition of Z_j we have

$$Z_j + Z_i - W_i(x_i) \subseteq Z_j \tag{3}$$

Substituting (3) into (2) we have

$$Z_i + \sum_{k \in N} (Z_k - W_k(x_k)) \subseteq Z_i. \tag{4}$$

Substituting (1) into (4) we get

$$Z_i + \sum_{k \in N} [Z_k + (Z_{k+1} - W_{k+1}(x_{k+1}))(n_k + 1) - W_k(x_k)] \subseteq Z_i$$

which can be rewritten as

$$Z_i + \sum_{k \in N} (Z_k - W_k(x_k))(n_k + 1) \subseteq Z_i.$$

Rearranging and taking into account that $n_k = 0$ for $k \in N\backslash F$,

$$Z_i + \sum_{k \in F} (Z_k - W_k(x_k))n_k + \sum_{k \in N} (Z_k - W_k(x_k)) \subseteq Z_i.$$

Since $0 \in \sum_{k \in N}(Z_k - W_k(x_k))$ and $\omega_k \in Z_k$ for all $k \in N$, we get

$$\omega_i + \sum_{k \in F} (\omega_k - W_k(x_k))n_k \subseteq Z_i$$

which is what we wanted to prove. □

This concludes the proof of Theorem 1. □

Note that the proof relies exclusively on set theoretic arguments. Virtually no assumptions on the economy are necessary to establish the result. Also, the theorem shows the equality between the two processes for *any* allocation.

4 Walrasian Allocations and Arbitrage-Free Equilibria

This section reviews results in Dagan (1996). It is included for the sake of completeness and to facilitate the understanding of the results in the next sections. Dagan's result is reinterpreted, by making use of the supermarkets Z_i^x. We consider economies satisfying the following assumptions:

A1. For all $i \in N$, the consumption sets $X_i = \mathbb{R}^l_+$

A2. For all $i \in N$, for all $x_i \in X_i$, the preferred set $P_i(x_i)$ is open relative to X_i (upper semicontinuity of preferences)

A3. For all $i \in N$ and for all $x_i, y_i \in X_i$, $y_i \gg x_i$ implies $y_i \in P_i(x_i)$ (monotonicity of preferences), and

A4. For all $i \in N, \omega_i \in \mathbb{R}^l_{++}$ (interiority of the endowments).

Next we present the definition of an arbitrage-free equilibrium, first found in Dagan (1996):

Definition 3 Let $\mathcal{E} = \langle N, (X_i, \succeq_i, \omega_i)_{i\in N}\rangle$ be an economy. An arbitrage-free equilibrium of $\mathcal{E}$ is a collection $\langle (x_i)_{i\in N},\ (C_i)_{i\in N}\rangle$ such that:

(i) $(x_i)_{i\in N} \in \mathcal{A}(\mathcal{E})$;

(ii) $\omega_i, x_i \in C_i \quad \forall i \in N$;

(iii) $\sum_{k\in S} C_k - \sum_{k\in S\setminus\{i\}} W_k(x_k) \subseteq C_i \quad \forall i \in S,\ \forall S \subseteq N$;

(iv) $P_i(x_i) \cap C_i = \emptyset \quad \forall i \in N$.

That is, an assignment of bundles and sets of choice to agents is an arbitrage-free equilibrium if the list of bundles is feasible for the economy, and the bundle assigned to each agent is top ranked according to his preference relation among the bundles in the given choice set. On the other hand, the choice set must contain the endowments and satisfy a recontracting condition, which says that each agent's set of choice must contain all bundles that can be achieved through acceptable trade with the other agents.

Remark: Note that a Walrasian equilibrium is a particular case of an arbitrage-free equilibrium, where for each agent i, the set C_i is the budget set

$$H_i^p = \{x_i \in \mathbb{R}^l : px_i \leq p\omega_i\},$$

where p is the Walrasian equilibrium price vector.

For our purposes, we will work with arbitrage-free equilibria where the choice sets C_i are the supermarkets. The following result demonstrates the relevance of the introspective construction.

Proposition 1 Let $\langle (x_i)_{i\in N}, (C_i)_{i\in N}\rangle$ be an arbitrage-free equilibrium. Then,

(i) $Z_i^x \subseteq C_i$

(ii) $\langle (x_i)_{i\in N}, (Z_i^x)_{i\in N}\rangle$ is an arbitrage-free equilibrium.

Proof : (i) We show that $Z_i^x(t) \subseteq C_i \quad \forall i \in N \quad \forall t \in I\!N$. The p roof is by induction. Since $\langle (x_i)_{i\in N}, (C_i)_{i\in N}\rangle$ is an arbitrage-free equilibrium, we have: $Z_i^x(0) = \{\omega_i\} \subseteq C_i \quad \forall i \in N$. Assume now that $Z_i^x(t-1) \subseteq C_i \quad \forall i \in N$ for some $t \in I\!N$. Since $\langle (x_i)_{i\in N}, (C_i)_{i\in N}\rangle$ is an arbitrage-free equilibrium, we have:

$$\sum_{k\in S} C_k - \sum_{k\in S\setminus\{i\}} W_k(x_k) \subseteq C_i \qquad \forall i \in S \quad \forall S \subseteq N.$$

By the induction hypothesis,

$$\sum_{k\in S} Z_k^x(t-1) - \sum_{k\in S\setminus\{i\}} W_k(x_k) \subseteq C_i \qquad \forall i \in S \quad \forall S \subseteq N.$$

Therefore,

$$\bigcup_{\substack{S\subseteq N\\ i\in S}} \Big[\sum_{k\in S} Z_k^x(t-1) - \sum_{k\in S\setminus\{i\}} W_k(x_k)\Big] \subseteq C_i \qquad \forall i \in N.$$

By definition of $Z_i^x(t)$,

$$Z_i^x(t) \subseteq C_i \qquad \forall i \in N.$$

(ii) We have to show that $\langle (x_i)_{i\in N}, (Z_i^x)_{i\in N}\rangle$ satisfies conditions (i)-(iv) of definition 3. Condition (i) is obvious: $(x_i)_{i\in N}$ is an allocation. To see that condition (ii) also holds, note that $\omega_i \in Z_i^x(0) \subseteq$

Z_i^x and $x_i \in Z_i^x(1) \subseteq Z_i^x$ for all $i \in N$. Condition (iv) follows from part (i) of this proposition, because $\langle (x_i)_{i\in N}, (C_i)_{i\in N}\rangle$ is an arbitrage-free equilibrium.

Finally, to show (iii) note that if

$$z \in \sum_{k\in S} Z_k^x - \sum_{k\in S\setminus\{i\}} W_k(x_k),$$

then by definition of Z_k^x,

$$z \in \sum_{k\in S} Z_k^x(t) - \sum_{k\in S\setminus\{i\}} W_k(x_k) \qquad \text{for some } t \in \mathbb{N}.$$

Note how this last step used the fact that the sequence $Z_k^x(t)$ is increasing in t for all $k \in N$. Therefore, by definition of $Z_i^x(t+1)$, we have $z \in Z_i^x(t+1) \subseteq Z_i^x$.

□

Proposition 1 means that there is no loss of generality in restricting attention to arbitrage-free equilibria of the form $\langle (x_i)_{i\in N}, (Z_i^x)_{i\in N}\rangle$, based on the supermarkets. With the help of the previous results, we present next an alternative proof of Theorem 1 in Dagan (1996)), which characterizes the set of allocations that can be supported by an arbitrage-free equilibrium.

Theorem 2 Let $\mathcal{E}$ be an economy satisfying assumptions A1-A4. An allocation $(x_i)_{i\in N}$ is Walrasian if and only if $\langle (x_i)_{i\in N}, (Z_i^x)_{i\in N}\rangle$ is an arbitrage-free equilibrium.

Proof : Let $x := (x_i)_{i\in N}$ be Walrasian. By the remark that follows definition 3 and by Proposition 1, $\langle (x_i)_{i\in N}, (Z_i^x)_{i\in N}\rangle$ is an arbitrage-free equilibrium.

Assume now that $\langle (x_i)_{i\in N}, (Z_i^x)_{i\in N}\rangle$ is an arbitrage-free equilibrium. By condition (i) of Definition 3, x is an allocation. By condition (iv) of Definition 3,

$$P_i(x_i) \cap Z_i^x = \emptyset \qquad \forall i \in N.$$

Since by Theorem 1, $A_i^x = Z_i^x$, we have

$$P_i(x_i) \cap A_i^x = \emptyset \qquad \forall i \in N,$$

which implies that x is a shrunk core allocation. Finally, by assumptions A1-A4 (which are used only here) and by Theorem 3 in Debreu and Scarf (1963), x is Walrasian. □

The following result highlights an important property of arbitrage-free equilibria.

Proposition 2 Let $\langle (x_i)_{i\in N}, (C_i)_{i\in N}\rangle$ be an arbitrage-free equilibrium. Then, for all $i \in N$, $C_i = \omega_i + Z$ for some common $Z \subseteq \mathbb{R}^l$. Furthermore, $Z = Z + Z$.

Proof : We first show a preliminary step: for all $i \in N$, $C_i + \sum_{k\in N}(C_k - W_k(x_k)) \subseteq C_i$. Pick $i \in N$. By condition (iii) in Definition 3, $\sum_{k\in N} C_k - \sum_{k\in N\setminus\{i\}} W_k(x_k) \subseteq C_i$. Choose now an agent $j \neq i$. Since there are at least two agents in the economy, this can be done. Again, by condition (iii) we have $C_j + C_i - W_i(x_i) \subseteq C_j$. Substituting this set inclusion into the previous one, we get what we wanted to show.

We shall show that $C_i = \omega_i + Z$ for $Z = \sum_{k\in N}(C_k - W_k(x_k))$. By (ii) in Definition 3 and the preliminary argument, $\omega_i + Z \subseteq C_i$.

Now let $a \in C_i$. Since $(x_k)_{k\in N}$ is an allocation, $0 = \sum_{k\in N}(\omega_k - x_k)$. Consequently, $a = a + \sum_{k\in N}(\omega_k - x_k)$. Rearranging, $a = \omega_i + (a - x_i) + \sum_{k\in N\setminus\{i\}}(\omega_k - x_k)$. Since $a \in C_i$, we have that $a \in \omega_i + \sum_{k\in N}(C_k - W_k(x_k))$.

It is straightforward to see that $Z = \sum_{k\in N}[C_k - W_k(x_k)]$ is additive. □

Proposition 2 says that, if $\langle (x_i)_{i\in N}, (C_i)_{i\in N}\rangle$ is an arbitrage-free equilibrium, agent i's set of choice C_i can be written as the sum of two sets: the initial endowments and a common (across agents) set of net trades. One example of such a set would be the half-space below the hyperplane through the initial endowments with a common price vector, but note that supermarkets will also satisfy this property. Arbitrage-free equilibria are therefore anonymous across agents in these environments, in the sense that, regardless of one's initial situation, the set of net trades that each agent considers available is independent of names. This is discussed in McLennan and Sonnenschein (1991) and it is stronger than the "fairness" of net trades of Schmeidler and Vind (1972).

5 Supermarkets and Budget Sets

In Proposition 1, it has been established that supermarkets Z_i^x are the smallest choice sets that can accompany an allocation in an arbitrage-free equilibrium. It follows from the additivity of the set Z of net trades in Proposition 2 and from condition (iv) in definition 3 that budget sets H_i^p are the largest choice sets that can appear in an arbitrage-free equilibrium.

As an illustration of the difference between Walrasian budget sets and the supermarkets, consider Example 1 once again. It can be checked that

$$Z_1^x = Z_2^x = \bigcup_{a \in \mathbb{Z}} [\{(a, 2-a)\} - \mathbb{R}_+^2],$$

which is considerably smaller than H_i^p, the budget half-space determined by the Walrasian prices $p^1 = p^2$. Note also that this set is neither a cone nor a convex set, unlike the sets constructed in Makowski and Ostroy (1995). Arbitrage need not yield a convex cone, but its "integer version," in finite economies.

Next we show that the difference between supermarkets and budget sets becomes insubstantial when preferences are differentiable.

Proposition 3 Suppose that $\mathcal{E}$ satisfies A1-A4 and that preferences are differentiable. Let x be a Walrasian allocation. Then, for all $i \in N$, the closure of the supermarket Z_i^x coincides with the budget set H_i^p.

Proof : The inclusion $Z_i^x \subseteq H_i^p$ follows from Proposition 1, part (i).

We now show that the interior of the budget set H_i^p is contained in Z_i^x. Let $y_i = x_i - z$ be in the interior of H_i^p, i.e., $pz > 0$. Partition the straight line segment z into a large number of small interv als. By differentiability, there exists $\lambda > 0$ small enough (without loss of generality, taken to be the inverse of an integer) such that for every $k \neq i$, $x_k + \lambda \frac{z}{|N|-1} \in W_k(x_k)$. This follows because the price hyperplane p supports the set $W_k(x_k)$ for all k. Thus, $x_k + \lambda \frac{z}{|N|-1} - x_k \in W_k(x_k) - Z_k^x(1)$. This implies that $\lambda z \in \sum_{k \neq i} [W_k(x_k) - Z_k^x(1)]$.

Therefore, $x_i - \lambda z \in Z_i^x(1) + \sum_{k \neq i} [Z_k^x(1) - W_k(x_k)] \subseteq Z_i^x(2)$.

By induction, assume now that $x_i - t\lambda z \in Z_i^x(t+1)$ (the previous steps show this for $t = 1$). Using that $-\lambda z \in \sum_{k \neq i}[Z_k^x(t+1) - W_k(x_k)]$, we obtain that $x_i - (t+1)\lambda z \in Z_i^x(t+2)$, and so on. This induction argument is completed in $1/\lambda$ steps to show that $y_i = x_i - z \in Z_i^x(\frac{1}{\lambda} + 1) \subseteq Z_i^x$.

□

6 Arbitrage-Free Equilibrium without Short Sales

The supermarkets Z_i^x use unlimited short sales since payments to agents for their resources are made by using the upper contour sets of the agents' preferences. Thus, it is clear that Dagan's result (which can be reinterpreted as Theorem 2) is based on a property of arbitrage-freeness that requires unlimited short sales. The same criticism applies to previous results in this literature. Let us illustrate this point with an example.

Example 2 Consider a two agent, two good economy where agent 1's initial endowment is given by $\omega_1 = (30, 30)$ and agent 2's initial endowment is $\omega_2 = (9, 9)$ (see figure 2). Assume that

$$\begin{aligned} P_1(30,30) &= \{(x_1,x_2) : (x_1,x_2) \gg (30,30)\} \cup \{(x_1,x_2) : (x_1,x_2) \gg (40,10)\} \cup \\ &\quad \{(x_1,x_2) : (x_1,x_2) \gg (10,40)\} \\ P_2(9,9) &= \{(x_1,x_2) : (x_1,x_2) \gg (9,9)\}. \end{aligned}$$

It is easy to see that the allocation $x = \langle(30,30),(9,9)\rangle$ is not Walrasian. According to Theorem 2, there must be one agent, i, with $P_i(x_i) \cap Z_i^x \neq \emptyset$. It is instructive to verify this. The reader can check that

$$Z_1^x(1) = \{(x_1,x_2) \in \mathbb{R}^2 : (x_1,x_2) \leq (30,30)\}$$

and that

$$Z_2^x(1) = \{(x_1,x_2) \in \mathbb{R}^2 : (x_1,x_2) \leq (9,9) \text{ or } (x_1,x_2) \leq (29,-1), \text{ or } (x_1,x_2) \leq (-1,29)\}.$$

Then, $(30,30) \in Z_1^x(1)$, $(29,-1) \in Z_2^x(1)$,and $(9,9) \in W_2(9,9)$. But then $(50,20) = (30,30) + (29,-1) - (9,9) \in Z_1^x(2)$. Since $(50,20) \in P_1(30,30)$, we have shown that $P_1(30,30) \cap Z_1^x \neq \emptyset$.

Note that in order to upset the allocation x it was necesary to make use of the bundle $(29, -1)$ which has a negative component. In other words, in the interactive process it was necessary to resort to short sales. This suggests the question of whether a characterization of Walrasian allocations can be found in which the choice sets are subsets of $\mathbb{R}^l_+$ and where the arbitrage involves no short sales. In general, the answer to this question will be negative. To show this, consider again the economy in the previous example.

Example 3 Consider the two-agent two-good pure exchange economy described in figure 2.

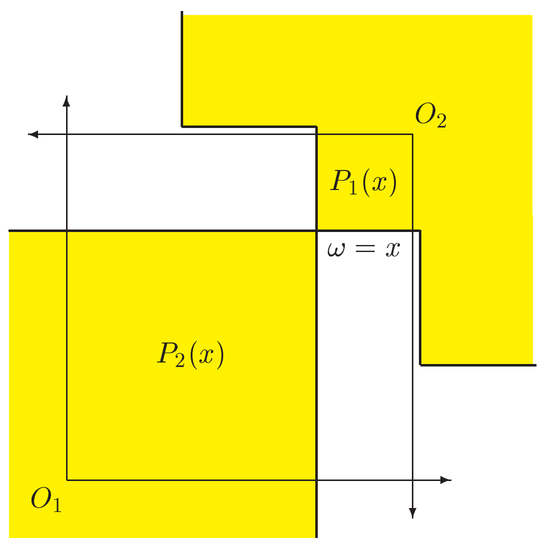

Figure 2.

Consider the following arbitrage-free condition where short sales are precluded:

$$[C_i + C_j - W_j(x_j)] \cap \mathbb{R}^2_+ \subseteq C_i \quad i \neq j, i, j = 1, 2.$$

Let $C_i = \{(x_1, x_2) \in \mathbb{R}^2_+ : (x_1, x_2) \leq \omega_i\}$ for $i = 1, 2$. Clearly, the endowment point satisfies this recontracting condition. In addition, it is a feasible allocation of the economy and both agents satisfy that $P_i(\omega_i) \cap C_i = \emptyset$. However, the endowments are not a Walrasian allocation, as shown in Example 2.

This example shows that the characterization of Walrasian allocations by means of arbitrage-free equilibria relies on the existence of short sales. We next show that, adding an extra assumption, one can do away with short sales.

A5. For all $i \in N$, for all $x_i \in X_i$, the preferred set $P_i(x_i)$ is convex (convexity of preferences).

Consider the following definition:

Definition 4 Let $\mathcal{E}$ be an economy. An arbitrage-free equilibrium without short sales is a pair $\langle (x_i)_{i\in N}, (C_i)_{i\in N} \rangle$ such that:

(i) $(x_i)_{i\in N} \in \mathcal{A}(\mathcal{E})$;

(ii) $\omega_i, x_i \in C_i \subseteq \mathbb{R}^l_+ \quad \forall i \in N$;

(iii) $(\sum_{k\in S} C_k - \sum_{k\in S\setminus\{i\}} W_k(x_k)) \cap \mathbb{R}^l_+ \subseteq C_i \quad \forall i \in S,\ \forall S \subseteq N$;

(iv) $P_i(x_i) \cap C_i = \emptyset \qquad \forall i \in N$.

That is, an arbitrage-free equilibrium without short sales constrains the choice sets to the non-negative orthant of $\mathbb{R}^l$ and imposes the same type of restrictions on the recontracting condition. Next we show that under the assumptions A1-A5, there exists a characterization of Walrasian allocations in terms of arbitrage-free equilibria without short sales.

Theorem 3 Let $\mathcal{E}$ be an economy satisfying assumptions A1-A5. Then, $(x_i)_{i\in N}$ is Walrasian if and only if it can be supported by an arbitrage-free equilibrium without short sales.

Proof : Let $x = (x_i)_{i\in N}$ be Walrasian. By Theorem 2 it can be supported by the arbitrage-free equilibrium $\langle (x_i)_{i\in N}, (Z_i^x)_{i\in N} \rangle$. It can be easily checked that $\langle (x_i)_{i\in N}, (Z_i^{x+})_{i\in N} \rangle$ is an arbitrage-free equilibrium without short sales where $Z_i^{x+} = Z_i^x \cap \mathbb{R}^l_+$.

Assume now that $\langle (x_i)_{i\in N}, (C_i)_{i\in N} \rangle$ is an arbitrage-free equilibrium without short sales. We want to show that $x = (x_i)_{i\in N}$ is Walrasian. By Theorem 2 it is enough to show that x can be supported by the arbitrage-free equilibrium $\langle (x_i)_{i\in N}, (Z_i^x)_{i\in N} \rangle$. Recall Theorem 1; it will be convenient to work with the sets A_i^x associated with replica economies. We therefore show that $\langle (x_i)_{i\in N}, (A_i^x)_{i\in N}\rangle$ is an arbitrage-free equilibrium.

Conditions (i), (ii) and (iii) in definition 3 are easily seen to be met. Therefore, we need to check condition (iv), i.e., $P_i(x_i) \cap A_i^x = \emptyset \quad \forall i \in N$.

Suppose not: there exists $i \in N$ and $y_i \in P_i(x_i) \cap A_i^x$. Notice that $y_i \in \mathbb{R}_+^l$. Because $y_i \in A_i^x$, there exists a replica of size m of the original economy $\mathcal{E}$ such that $y_i \in A_i(m)$. This implies that there exists a coalition S in this economy where for each $k \in N$ exactly $n_k \leq m$ (and $n_i \leq m-1$) copies of agent k are used by agent i, and such that:

$$y_i \in \omega_i + \sum_{k\in N} n_k[\omega_k - W_k(x_k)].$$

By assumptions A3 and A4, we can find $y_i' \leq \omega_i, y_i' \in C_i \subseteq \mathbb{R}_+^l$, and $p_{k,n} \in P_k(x_k)$ such that:

$$y_i = y_i' + \sum_{k\in N}\sum_{n=1}^{n_k}[\omega_k - p_{k,n}],$$

which can be rewritten as:

$$y_i = y_i' + \sum_{k\in N}\sum_{n=1}^{n_k}[\omega_k - p_{k,n}] + \sum_{k\in N}\lambda_k n_k(x_k - x_k),$$

for any positive constants λ_k, that is,

$$y_i = y_i' + \sum_{k\in N}[(\lambda_k+1)[(\sum_{n=1}^{n_k}\frac{1}{\lambda_k+1}\omega_k + \frac{\lambda_k}{\lambda_k+1}x_k) - \sum_{n=1}^{n_k}(\frac{1}{\lambda_k+1}p_{k,n} + \frac{\lambda_k}{\lambda_k+1}x_k)]].$$

By A5, the convex combination of $p_{k,n}$ and x_k is in $P_k(x_k)$. On the other hand, for λ_k big enough, the convex combination of ω_k and x_k (which is entirely contained in the non-negative orthant) is arbitrarily close to x_k. Since the preferred sets are open (A2), we can bridge the gap between x_k and its convex combination with ω_k by choosing a suitable $x_k + \epsilon_{k,n} \in P_k(x_k)$ in some neighborhood of $p_{k,n}$. That is,

$$\begin{aligned} y_i &= y_i' + \textstyle\sum_{k\in N}[(\lambda_k+1)[n_k x_k - \sum_{n=1}^{n_k}(\frac{1}{\lambda_k+1}(x_k+\epsilon_{k,n}) + \frac{\lambda_k}{\lambda_k+1}x_k)]] \\ &= y_i' + \textstyle\sum_{k\in N}[(\lambda_k+1)[n_k x_k - (n_k x_k + \sum_{n=1}^{n_k}\frac{1}{\lambda_k+1}\epsilon_{k,n})]] \\ &= y_i' + \textstyle\sum_{k\in N}[n_k(\lambda_k+1)[x_k - (x_k + \sum_{n=1}^{n_k}\frac{1}{n_k(\lambda_k+1)}\epsilon_{k,n})]]. \end{aligned}$$

Define $\kappa_k = n_k(\lambda_k+1) - 1$ and $\alpha_k = \sum_{n=1}^{n_k}\frac{\epsilon_{k,n}}{n_k}$. Then, we can write:

$$y_i = y_i' + \sum_{k\in N}\kappa_k(x_k - x_k) + \sum_{k\in N}[x_k - (x_k + n_k\alpha_k)].$$

Since α_k can be chosen arbitrarily small (by choosing λ_k arbitrarily large, which makes $\epsilon_{k,n}$ arbitrarily small) and $x_k + \alpha_k \in P_k(x_k)$, we know that $x_k + n_k\alpha_k \in P_k(x_k)$. Moreover, by the same argument, we can make α_i arbitrarily small so that, choosing $j \neq i$ we have that: $x_j + n_j\alpha_j + n_i\alpha_i \in P_j(x_j)$.

That is, we can write:

$$y_i = y_i' + \sum_{k \in N\setminus\{i\}} \kappa_k(x_k - x_k) + (\kappa_i + 1)(x_i - x_i) + \sum_{k \in N\setminus\{i,j\}} [x_k - (x_k + n_k\alpha_k)] + [x_j - (x_j + n_j\alpha_j + n_i\alpha_i)].$$

Therefore:

$$y_i \in [C_i + \sum_{k \in N\setminus\{i\}} [\kappa_k(C_k - W_k(x_k)) \cap \mathbb{R}^l_+] + [(\kappa_i + 1)(C_i - W_i(x_i)) \cap \mathbb{R}^l_+] + \sum_{k \in N\setminus\{i\}} (C_k - W_k(x_k))] \cap \mathbb{R}^l_+.$$

But this set is contained in:

$$[C_i + \sum_{k \in N\setminus\{i\}} (C_k - W_k(x_k))] \cap \mathbb{R}^l_+ \subseteq C_i,$$

where the last inclusion follows from the fact that $\langle (x_i)_{i\in N}, (C_i)_{i\in N}\rangle$ is an arbitrage-free equilibrium without short sales. Thus, we have found a contradiction since $y_i \in C_i \cap P_i(x_i)$. □

Thus, in economies without short sales, convexity of preferences is critical for the characterization. That is, while the result in Dagan (1996) and Theorem 2 do not need convexity, they rely on the existence of unlimited short sales. Recall that strict convexity implies equal treatment of core allocations and leads to core convergence. Thanks to convexity, if an agent can find an arbitrage opportunity moving along a particular direction in the set of net trades, there exist infinitesimal net trades that are also arbitrage opportunities.

The next result shows the version of anonymity of Walrasian allocations that can be found in economies without short sales. At a Walrasian allocation, every agent faces the same common set of net trades in his supermarket, although the intersection of the common set of net trades and the consumption set will differ across agents (see also Schmeidler and Vind (1972) on this point). The proposition also shows the essentiality of our construction based on supermarkets.

Proposition 4 Let $\mathcal{E}$ be an economy satisfying assumptions A1-A5. If $\langle (x_i)_{i\in N}, (C_i)_{i\in N}\rangle$ is an arbitrage-free equilibrium without short sales, then so is $\langle (x_i)_{i\in N}, (Z_i^{x+})_{i\in N}\rangle$, where $Z_i^{x+} = Z_i^x \cap \mathbb{R}^l_+ = (\omega_i + Z) \cap \mathbb{R}^l_+$ for $Z = \sum_{k\in N}[Z_k^x - W_k(x_k)]$.

Proof : Suppose that $\langle (x_i)_{i\in N}, (C_i)_{i\in N}\rangle$ is an arbitrage-free equilibrium without short sales. Then, by Theorems 3 and 2, we know that $\langle (x_i)_{i\in N}, (Z_i^x)_{i\in N}\rangle$ is an arbitrage-free equilibrium. It is then straightforward to see that $\langle (x_i)_{i\in N}, (Z_i^x \cap \mathbb{R}_+^l)_{i\in N}\rangle$ is an arbitrage-free equilibrium without short sales. The rest of the statement follows from Proposition 2 and its proof. □

Thus, we can state the following characterization of Walrasian allocations in terms of arbitrage-free equilibria without short sales based on supermarkets.

Theorem 4 Let $\mathcal{E}$ be an economy satisfying assumptions A1-A5. An allocation $(x_i)_{i\in N}$ is Walrasian if and only if $\langle (x_i)_{i\in N}, (Z_i^x \cap \mathbb{R}_+^l)_{i\in N}\rangle$ is an arbitrage-free equilibrium without short sales.

Proof : If $(x_i)_{i\in N}$ is Walrasian, by Theorem 3 and Proposition 4, $\langle (x_i)_{i\in N}, (Z_i^x \cap \mathbb{R}_+^l)_{i\in N}\rangle$ is an arbitrage-free equilibrium without short sales. The other direction follows from Theorem 3.

□

7 Concluding Remarks

We have presented an introspective approach to the characterization of Walrasian allocations by means of abstract sets of choice. Walrasian allocations arise as the only ones that survive after all arbitrage opportunities have been eliminated. Our model, based on the introspective construction of supermarkets, dispenses with the services of the auctioneer, thereby departing from earlier work.

Makowski and Ostroy (1995) also present an approach to Walrasian allocations based on arbitrage. Unlike our model, theirs consists of an exchange economy with a continuum of agents. They show that arbitrage in their context leads to sets of net trades for each agent that are convex cones. These need not be flat cones, though, when the economy is not differentiable. Thus, without differentiability, additional allocations that are not Walrasian can also be supported by their arbitrage-based equilibria. We have shown that in finite economies arbitrage need not yield convex cones as the agents' sets of net trades. Supermarkets (the smallest sets that can appear as sets of choice in an arbitrage-free equilibrium) may be neither convex sets nor cones. However, all allocations supported by arbitrage-free equilibria are Walrasian.

Schmeidler and Vind (1972) impose a condition of "strong fairness" on the sets of net trades available to agents. Each agent must prefer his net trade to any linear combination (taking into account only integer multiples) of the individual net trades in the economy. This is weaker than the property of anonymity, whereby each agent faces the same set of net trades. It turns out that "strong fairness" only imposes that all net trades must lie on the same price hyperplane, and it is compatible with some non-Walrasian allocations. Vind (1977) requires that the set of net trades be anonymous and additive, and obtains the same results as Schmeidler and Vind (1972).

McLennan and Sonnenschein (1991) impose in their Theorem A an extra condition that amounts to requiring that the set of net trades is the same half-space for every agent, so characterizing Walrasian allocations. Finally, Dagan (1996) deduces anonymity and additivity of the sets of net trades from the more primitive arbitrage-free condition. One of our contributions is to construct explicitly (through the agents' introspection) the smallest sets of net trades that satisfy the condition in Dagan (1996).

All this literature utilizes sets of net trades that are not contained in the consumption set. Our paper, in addition, has extended the analysis by precluding short sales. There, we have found that convexity of preferences is needed in order to characterize Walrasian allocations by means of supermarkets. Also, the anonymity of the sets of net trades is reinterpreted: the intersection of the common set of net trades with the non- negative orthant will in general differ across agents.

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