A Comparison of the Average Prekernel and the Prekernel*

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Abstract

We propose positive and normative foundations for the average prekernel of NTU games, and compare them with the existing ones for the prekernel. In our non-cooperative analysis, the average prekernel is understood as the equilibrium payoffs of a game where each player faces the possibility of bargaining at random against any other player. In the cooperative analysis, we characterize the average prekernel as the unique solution that satisfies a set of Nash-like axioms for two-person games, and versions of average consistency and its converse for multilateral settings.

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1 Introduction

The prekernel or Nash set of a non-transferable (NTU) coalitional game consists of those payoffs in which each player is in a situation of "bilateral equilibrium" with any other player.[1] The prekernel was introduced for the class of transferable utility (TU) games in Davis and Maschler (1965) and generalized to the class of NTU games in Serrano (1997). The latter paper, as part of the Nash program for coalitional games (Nash (1950, 1953)), also contained a non-cooperative model of negotiations to support the payoffs in the prekernel. The prekernel was characterized in Peleg (1986) for the class of TU games and in Serrano and Shimomura (1998) for the class of smooth NTU games.

It is important at the outset to clarify our terminology and reconcile it with that found in the literature. While the name *prekernel* is certainly justified for historical reasons, there are good arguments to also adopt the name *Nash set* for this solution concept.[2] Indeed, a payoff x is in the prekernel or Nash set if for every pair of players i and j, the point (x_i, x_j) is a point of the Pareto frontier of the bilateral problem for i and j where the utility elasticity is unity. That is, (x_i, x_j) is a critical point of the Nash product in the bilateral problem's frontier. Thus, it should not come as a surprise that the prekernel or Nash set reduces to a single point in the class of convex bargaining problems and that this point is the Nash solution. Nonetheless, not to bother the reader with the use of both names and following the tradition of coalitional games, we shall use the name *prekernel* in the rest of this paper.

In multilateral settings, the prekernel is a good description of equilibrium in bilateral bargaining between any pair of players. However, it has an important shortcoming. As pointed out in Moldovanu (1990) and Serrano (1997), it often prescribes the empty set. Following an idea of Maschler and Owen's (1992) for the NTU Shapley value and Dagan and Volij's (1997) for bankruptcy rules, this existence problem has been recently solved in Orshan and Zarzuelo (2000), who propose as an alternative solution concept the *average prekernel*.[3] The average prekernel is

[1] See Maschler (1992) for a survey.

[2] Maschler, Owen and Peleg (1988) introduced this solution for the class of bargaining problems.

[3] To continue with our remarks on terminology, we shall use this name to refer to their solution. The one they used, that of *bilateral consistent prekernel*, is somewhat confusing: first, the average prekernel is not bilaterally consistent; and second, one of the key axioms used in Serrano and Shimomura (1998) to characterize the prekernel

the set of efficient payoffs where, for each player, the aggregate (or average) difference of surpluses of a player against all the others is zero. Thus, it always contains the prekernel (because in the latter the difference of surpluses is zero for every pair of players). However, its advantage over the prekernel is that, as Orshan and Zarzuelo (2000) demonstrate, it is non-empty very generally, over a large significant class of NTU games. This result has been refined even further for the intersection of the average prekernel and the core in Orshan, Valenciano and Zarzuelo (2000), who show this intersection is non-empty when the game is "boundary separating."

The purpose of this paper is to clarify the distinction between prekernel and average prekernel, by proposing strategic and axiomatic analyses of the average prekernel. These are to be compared with those for the prekernel, contained in Serrano (1997) and Serrano and Shimomura (1998).

In our non-cooperative analysis, given a status quo payoff x, a player is chosen at random every period and asked whether he accepts or rejects the status quo. If he accepts, the status quo is unaltered and a new player will be randomly selected the next period and asked the same question. If he rejects, he will bargain with another player for a redistribution of payoffs, but at the time he has to respond he does not know who will be his opponent in the bilateral bargaining round. In this round, upon rejection of a proposal, the status quo is unaltered with high probability, while with the rest of probability the rejector has the option of hiring a coalition that includes him but excludes the proposer. Whatever the outcome of this bilateral bargaining round, a new status quo is so determined, followed by the random choice of a new player the next period, and so on. We show that the stationary equilibrium payoffs of this model lead to the average prekernel payoffs as the probability of cooperating with coalitions vanishes. In contrast, the model in Serrano (1997) that yields the prekernel is such that every period a pair of players is chosen at random to bargain if they want to modify the status quo. Thus, while the rules of negotiation in Serrano (1997) lead to a situation of bilateral equilibrium for every pair, a story of "equilibrium in average" is told here.

Likewise, the only differences between the axioms in Serrano and Shimomura (1998) and those used in the present paper are found in the different versions of consistency and its converse. Instead of employing consistency and its converse in the sense of the Davis-Maschler

is precisely that of bilateral consistency.

reduced game, we need average consistency and average converse consistency with respect to certain reduced hyperplane games to characterize the average prekernel over the class of smooth NTU games.[4] The basis of our characterization is the class of two-player games, where we use the same axioms as in Serrano and Shimomura (1998): non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence. As it turns out, one needs to make both changes (consistency versus average consistency, and Davis-Maschler reduced game versus reduced hyperplane games) to go from the axioms of the prekernel to those of the average prekernel. Indeed, we show an impossibility result if one works with average consistency using Davis-Maschler reduced games. Finally, also as the parallel result in Serrano and Shimomura (1998) states, the intersection of the core and the average prekernel is characterized using the same axioms as for the average prekernel, but for the class of smooth games with non-empty cores. In all our characterizations, the axioms utilized are logically independent.

2 Preliminaries

Denote by $\mathbb{R}$ the set of the real numbers. If we use an upper case letter to denote a set, its lower case counterpart denotes its cardinality. Let N be a finite set containing at least two elements, and $n = |N|$. Denote by $\mathbb{R}^N$ the set of all functions from N to $\mathbb{R}$. We identify an element $x \in \mathbb{R}^N$ with an n-dimensional vector whose components are indexed by members of N; thus we write x_i for $x(i)$. If $x \in \mathbb{R}^N$ and $S \subseteq N$, we write x_S for the restriction of x to S, which is the element of $\mathbb{R}^S$ that associates x_i with each $i \in S$. We also write x_{-S} to denote $x_{N \setminus S}$ and x_{-i} to denote $x_{N \setminus \{i\}}$.

Let $S \subseteq N$ and $Y \subseteq \mathbb{R}^S$. A *representation* for Y is a function $g : \mathbb{R}^S \to \mathbb{R}$ such that

$$Y = \{x \in \mathbb{R}^S | \ \ g(x) \leq 0\}$$

and the interior of Y is the set

$$\text{Int}Y = \{x \in \mathbb{R}^S | \ \ g(x) < 0\}.$$

[4]See Thomson (forthcoming) for a comprehensive survey on consistency. The reader will also find there a useful discussion on consistency and average consistency.

We also write the *Pareto frontier* of Y:

$$\partial Y = \{y \in Y| \;\; x_i > y_i \quad \forall i \in S \quad \text{implies} \quad x \notin Y\}.$$

Definition: The pair (N, V) is a *coalitional game*, or simply a *game*, if V is a correspondence that associates with every non-empty $S \subseteq N$ a non-empty subset $V(S) \subseteq I\!R^S$ such that

(1) $V(S)$ is closed;

(2) For each $x_S \in I\!R^S$,

$$\partial V(S) \cap (\{x_S\} + I\!R^S_+)$$

and

$$\partial V(S) \cap (\{x_S\} - I\!R^S_+)$$

are bounded;

(3) If $(x_S, y_S) \in V(S) \times \partial V(S)$ and $x_S \geq y_S$, $x_S = y_S$.[5]

(4) $V(N)$ is convex.

(5) $V(S) \subseteq \{x_S \in I\!R^S | x \in V(T)\}$ whenever $S \subseteq T$.

By Assumption (3), the Pareto frontier of $V(N)$ is

$$\partial V(N) \subseteq \{x \in I\!R^N| \;\; g(x) = 0\},$$

where g is a representation for $V(N)$. Let $g_i(x)$ denote the partial derivative of g at $x \in \partial V(N)$ with respect to component $i \in N$, and let $\nabla g(x)$ denote the gradient vector of g at $x \in \partial V(N)$. Finally, for each $i \in N$ we write $g^i : I\!R^{N\setminus\{i\}} \rightarrow I\!R$ to denote the Pareto projection $g^i(x_{-i})$ along the i-th coordinate of x_{-i}.

A *player* is an element of N, and a non-empty subset of N is a *coalition*. A *payoff* to player i is a point of $I\!R^{\{i\}}$, and a *payoff profile* on coalition S is a point of $I\!R^S$.

The game (N, V) is a *smooth game* if there is a differentiable representation g for $V(N)$ with positive gradients on $\partial V(N)$; namely for each $i \in N$, $g_i(x) > 0$ at any $x \in \partial V(N)$.

[5]For each pair of vectors (x, y), we write $x \geq y$ if $x_i \geq y_i \quad \forall i$.

A *transferable utility game*, or a TU game, is a smooth game (N, V) which is defined by a function v that associates with every coalition S a real number $v(S)$ such that

$$V(S) = \{x_S \in I\!R^S | \sum_{i \in S} x_i \leq v(S)\}.$$

Abusing notation, we use (N, v) to denote the associated coalitional game.

A *hyperplane game* is a smooth NTU game such that the boundary of each $V(S)$ is a hyperplane in $I\!R^S$. A *bargaining problem* is a NTU game where for all $S \subset N$, $V(S) \subseteq \Pi_{i \in S} V(\{i\}) - I\!R^S_+$.

Let Γ be a non-empty class of games. A *solution* on Γ is a relation σ which associates with every $(N, V) \in \Gamma$ a (possibly empty) subset $\sigma(N, V)$ of $V(N)$ for every $(N, V) \in \Gamma$.

Let $\Pi^N = \{P \subseteq N : p = 2\}$, which is the set of two-person coalitions in N.

Definition: Let (N, V) be a game, $x \in V(N)$, and $P \in \Pi^N$. The two-person (Davis-Maschler) *reduced game* of (N, V) with respect to P given x_{-P} is the pair $(P, V_{x,P})$, consisting of the set P and the correspondence $V_{x,P}$ that associates with every $S \subseteq P$ a subset $V_{x,P}(S)$ of $I\!R^P$, where

$$V_{x,P}(\{i\}) = \{y_i \in I\!R^{\{i\}} | (y_i, x_Q) \in V(\{i\} \cup Q),\ Q \subseteq N \backslash P\}$$

for each $i \in P$, and

$$V_{x,P}(P) = \{y_P \in I\!R^P | (y_P, x_{-P}) \in V(N)\}.$$

Thus, given a payoff profile x for the grand coalition N, the feasible set for the pair P in the Davis-Maschler reduced game is what remains of $V(N)$ after the players not in P are paid according to x. In addition, each player in P expects to be able to cooperate with any of the players not in P provided they are paid their components of x. This is the way each player in P finds his "threat utility" against the other player in P.

Definition: Let (N, V) be a game, $x \in V(N)$, and $P \in \Pi^N$. The two-person *reduced hyperplane game* of (N, V) with respect to P given x_{-P} is the pair $(P, W_{x,P})$, consisting of the set P and the correspondence $W_{x,P}$ that associates with every $S \subseteq P$ a subset $W_{x,P}(S)$ of $I\!R^P$, where

$$W_{x,P}(\{i\}) = V_{x,P}(\{i\})$$

for each $i \in P$, and

$$W_{x,P}(P) = \{(y_i, y_j) \in I\!R^P | g_i(x)(y_i - x_i) + g_j(x)(y_j - x_j) \leq 0\}.$$

Thus, the reduced hyperplane game has the same feasible sets for the two individuals, but it prescribes the hyperplane tangent to the frontier at x_P for P: both players make the fictitious assumption that utility is transferable at the rates prescribed by the gradient of the frontier at x.

The *surplus* of player i against player j at the payoff vector x is defined as follows:

$$s_{i,j}(x) = v_i(x_{-\{i,j\}}) - x_i,$$

where

$$v_i(x_{-\{i,j\}}) = \max V_{x,\{i,j\}}(\{i\}).$$

That is, the surplus of player i against player j at the payoff profile x is the difference between the highest utility that player i could get without cooperating with j (when paying other players for their resources at the rate prescribed by x) and the utility that i receives at x.

Definition: Let (N, V) be a game. The *prekernel* of (N, V) is:

$$\mathcal{P}(N, V) = \{x \in \partial V(N) | \quad g_i(x)s_{i,j}(x) - g_j(x)s_{j,i}(x) = 0 \quad \forall i, j \in N\}.$$

The prekernel is the set of efficient payoff profiles where each player's surplus (when weighted by the corresponding partial derivative of g at x) against every other player is the same. This is the sense in which each prekernel payoff has the flavor of a "bilateral equilibrium." See Serrano (1997) for an elaboration of this point in a non-cooperative setup, and Peleg (1986) and Serrano and Shimomura (1998) for characterizations of the prekernel. Serrano (1997) also explains how the surplus equations of the prekernel are justified, in spite of the apparent interpersonal utility comparisons.

Definition: Let (N, V) be a game. The *average prekernel* of (N, V) is:

$$\mathcal{AP}(N, V) = \{x \in \partial V(N) | \quad \sum_{j \neq i} [g_i(x)s_{i,j}(x) - g_j(x)s_{j,i}(x)] = 0 \quad \forall i \in N\}.$$

The average prekernel was introduced in Orshan and Zarzuelo (2000). It consists of those payoff profiles in which each player is in a bilateral equilibrium situation only in average: it is possible that player i's surplus against player j exceeds that of player j against i, but this is offset by exactly the opposite situation with the players other than j.

3 Strategic Bargaining and the Average Prekernel

In this section we introduce a modification of the non-cooperative model proposed in Serrano (1997). The result will be a strategic bargaining model that yields the payoff profiles in the average prekernel as stationary equilibrium payoffs.

Description of the non-cooperative model. Time runs discretely from $-\infty$ to $+\infty$. In period t, let x_t denote the status quo payoff vector determined by play in period $t-1$. In period t one of the players $i \in N$ is called at random. He is asked whether he accepts the status quo x_t or he rejects it. If he accepts it, play goes to period $t+1$ and $x_{t+1} = x_t$. If he rejects it, each of the players $j \in N \backslash \{i\}$ is selected at random with equal probability to bargain against player i.

The bargaining procedure in period t is as follows. Let player j be the one selected to bargain with player i in period t.

1. The proposer is chosen at random with equal probability. Denote the proposer by $p = i, j$ and the responder by $r = i, j$, $r \neq p$. The proposer p makes a feasible proposal (z_p, z_r), i.e., $(z_p, z_r, x_{-\{i,j\}}) \in V(N)$.

2. The responder r may accept or reject the proposal.

   - If r accepts it, play goes to period $t+1$, where
   $$x_{t+1} = (z_p, z_r, x_{-\{i,j\}}).$$
   - If r rejects the proposal,
     - With probability $1 - \rho > 0$, play goes to period $t+1$, where
     $$x_{t+1} = (v_r(x_{-\{i,j\}}), g^p(v_r(x_{-\{i,j\}}), x_{-\{i,j\}}), x_{-\{i,j\}}).$$
     - With probability ρ, play goes to period $t+1$ with $x_{t+1} = x_t$.

A word on the interpretation of what happens following a rejection is called for. With probability $1-\rho$, the responder executes his threat against the proposer: he gets to utilize the resources that includes him but excludes the proposer and pays the members of that coalition their components of x. The proposer's payoff is determined by whatever is left of $V(N)$ after this contract between the responder and the outside coalition (note how the grand coalition always forms; other coalition structures are not our focus in this paper).

As in Serrano (1997), one should assume that either players are myopic and care only about present payoffs, or that there are n large populations (each being a "social class") underlying the model, and in each period one representative of some of these "social classes" is given the opportunity of modifying the social status quo.

To analyze this non-cooperative model, we shall use the following solution concept:

Definition: An *equilibrium* of the non-cooperative model is a strategy profile satisfying the following two requirements:

(i) *Steady state* requirement: $x_t = x$ for all t; and

(ii) players i and j play according to *stationary subgame perfect equilibrium* strategies in each bilateral bargaining procedure.

Theorem 1 Let (N, V) be a smooth NTU game satisfying Assumptions (1)-(5). The limit as $\rho \to 1$ of the equilibrium payoff profiles of the non-cooperative model coincides with $\mathcal{AP}(N, V)$.

Proof : The proof follows closely that of Theorem 3 in Serrano (1997). In particular, establishing existence is based on similar fixed point arguments developed there. The stationary strategies supporting the equilibrium payoffs in the bargaining procedure in each period t are also similar to the ones written in that proof, and we omit them.

We elaborate only on the differences. Player i, chosen at random in period t, must consider whether bargaining or not against a randomly chosen opponent. For the status quo x_t to remain unchanged, the following conditions are required:

$$x_i \geq \sum_{j \neq i} \frac{1}{n-1} [\frac{1}{2}((1-\rho)v_i(x_{-\{i,j\}}) + \rho x_i) + \frac{1}{2} g^i((1-\rho)v_j(x_{-\{i,j\}}) + \rho x_j, x_{-\{i,j\}})] \quad \forall i \in N.$$

Note how, by assumption (5), a proposer always wishes to get an acceptance from the responder by leaving him exactly indifferent to his continuation payoff. This explains the last term of the right hand side.

Given our assumptions, standard arguments based on Brouwer's fixed point theorem guarantee that the system above has a solution for any arbitrary $\rho < 1$, where all the inequalities are equalities. (see Serrano (1997) for details.)

It is easy to see that, as $\rho \to 1$, the limit of the sequence of solutions $x(\rho)$ must be efficient: to see this, examine the right hand side of the above inequalities and note that, if $\lim_{\rho\to 1} x(\rho)$ is inefficient, we have that $\lim_{\rho\to 1} g^i(x_{-i}(\rho)) > \lim_{\rho\to 1} x(\rho)$, contradicting the inequalities of the system when ρ is close enough to 1. Therefore, there is no loss of generality in considering sequences along which the above system is one of n equalities.

Continuing to write x instead of the more explicit $x(\rho)$ in the above equalities, we can rewrite them as follows:

$$x_i = \sum_{j\neq i} \frac{1}{n-1}[\frac{1}{2}v_i(x_{-\{i,j\}}) + \frac{1}{2}\frac{g^i((1-\rho)v_j(x_{-\{i,j\}}) + \rho x_j, x_{-\{i,j\}}) - \rho x_i}{1-\rho}] \quad \forall i \in N.$$

Next we take limits as $\rho \to 1$, whose existence follow from the assumptions made on the game. Using L'Hopital's rule and denoting $\lim_{\rho\to 1} x(\rho) = z$, we have that:

$$z_i = \sum_{j\neq i} \frac{1}{n-1}[\frac{1}{2}v_i(z_{-\{i,j\}}) + \frac{1}{2}(z_i - \frac{g_j(z)}{g_i(z)}(z_j - v_j(z_{-\{i,j\}})))] \quad \forall i \in N,$$

which can be easily transformed into the equations of the average prekernel. □

The most substantive difference between this non-cooperative model of bargaining and that found in Serrano (1997) is that each player faces the random possibility of bargaining against any player to affect the status quo. In contrast, in the model of Serrano (1997), the status quo may be modified by the bargaining between a specific pair of players $\{i, j\}$. This brings out quite clearly the distinction between the average prekernel and the prekernel from a strategic stand-point.

Assumption (4) on the coalitional game (convexity of $V(N)$) is used in the model to guarantee feasibility of the outcomes in and out of equilibrium. Note the heavy use of lotteries in the procedure. Neither this assumption nor assumption (5) will be used in the next section.

4 Axiomatic Analysis

In this section we characterize the average prekernel by means of seven logically independent axioms. The results here are to be compared with the closely related results in Serrano and Shimomura (1998). A comparison with the axiomatic result in Orshan and Zarzuelo (2000) will also be given.

Let Γ be a non-empty class of games. Then a solution σ on Γ satisfies *non-emptiness* if $\sigma(N,V) \neq \emptyset$ for all $(N,V) \in \Gamma$, and satisfies *Pareto efficiency* if $\sigma(N,V) \subseteq \partial V(N)$ for each $(N,V) \in \Gamma$.

Let (N,v) be a TU game, and i,j be two distinct players in N. Then i and j are *substitutes* in (N,v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i,j\}$.

Let Γ be a class of games. A solution σ on Γ satisfies *equal treatment for TU games* if $x_i = x_j$ for each $x \in \sigma(N,v)$ whenever (N,v) is a TU game in Γ and i and j are substitutes in (N,v).

Let (N,V) be a game, $\alpha \in I\!R^N_{++}$, and $\beta \in I\!R^N$. For each coalition S, we define the function $\lambda^{\alpha,\beta}_S$ from $I\!R^S$ to itself by

$$\lambda^{\alpha,\beta}_S(x_S) = (\alpha_i x_i + \beta_i)_{i \in S}$$

for each $x_S \in I\!R^S$. We then define $\lambda^{\alpha,\beta}(V)$ as the correspondence that associates with every coalition S a set

$$\lambda^{\alpha,\beta}(V)(S) = \{y \in I\!R^N : \text{ there exists} x_S \in V(S) | \;\; y_S = \lambda^{\alpha,\beta}_S(x_S)\}.$$

That is, these two definitions simply describe positive affine transformations of the utility scales.

Let Γ be a class of games. A solution σ on Γ satisfies *scale invariance* if for each $(N,V) \in \Gamma$, for each $\alpha \in I\!R^N_{++}$ and each $\beta \in I\!R^N$, $\sigma(N, \lambda^{\alpha,\beta}(V)) = \lambda^{\alpha,\beta}_N(\sigma(N,V))$.

Let Γ be a non-empty class of two-person smooth games. A solution σ on Γ satisfies *local independence* if for each $(\{i,j\},V) \in \Gamma$, and each $x \in \sigma(\{i,j\},V)$, we have that $x \in \sigma(N,V')$ whenever

1. $x \in \partial V(\{i,j\}) \cap \partial V'(\{i,j\})$,
2. $(v_i, v_j) = (v'_i, v'_j)$ and
3. $\nabla g(x)$ is proportional to $\nabla g'(x)$,

where $(v_i, v_j) = (\max V(\{i\}), \max V(\{j\}))$, $(v'_i, v'_j) = (\max V'(\{i\}), \max V'(\{j\}))$, and g and g' are representations for $V(\{i,j\})$ and $V'(\{i,j\})$, respectively.

For a discussion of each of these axioms, see Serrano and Shimomura (1998).

Proposition 1 Let $\Gamma^{\{i,j\}}$ be the class of two-person smooth games $(\{i,j\}, V)$ satisfying Assumptions (1)-(3). Then a solution on $\Gamma^{\{i,j\}}$ satisfies non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence if and only if it is $\mathcal{AP}$.

Proof : The proof is identical to that of Proposition 1 in Serrano and Shimomura (1998) after one observes that, over the considered class of games, $\mathcal{AP} = \mathcal{P}$. □

Thus, over the class of two-player games, the average prekernel coincides with the prekernel. If the set $V(\{i,j\})$ is convex and we have a bargaining problem, the average prekernel consists of a unique payoff profile, the nash solution.

Next, we shall present the axioms that will be operative in multilateral settings. To facilitate the comparison with the prekernel, we also state the axioms used in Serrano and Shimomura (1998).

Let Γ be a non-empty class of games. Then a solution σ on Γ satisfies *bilateral consistency* if for each $(N, V) \in \Gamma$, each $x \in \sigma(N, V)$, and each $P \in \Pi^N$, $(P, V_{x,P}) \in \Gamma$ and $x_P \in \sigma(P, V_{x,P})$.

Bilateral consistency says that the solution should be invariant to projections to two-player games, provided players have the expectations embodied in the bilateral Davis-Maschler reduced game.[6]

Let Γ be a non-empty class of games. Then a solution σ on Γ satisfies *converse consistency* if for each $(N, V) \in \Gamma$, and each $x \in \partial V(N)$, $x \in \sigma(N, V)$ whenever $x_P \in \sigma(P, V_{x,P})$ for every $P \in \Pi^N$.

If a solution is converse consistent, then in order to impose it on a society, an arbitrator should simply make sure that it is imposed on every pair of agents.

[6] A strengthening of this property is consistency. A solution σ on a class Γ satisfies *consistency* if the same condition as above holds, but for all subsets P, that is, not only restricted to those subsets P of N of cardinality 2.

Serrano and Shimomura (1998) show that, over the class of smooth games, the prekernel $\mathcal{P}$ is the only solution satisfying bilateral consistency, converse consistency, and the five axioms for two-player games used in Proposition 1.

We next formulate concepts of average consistency.

Let Γ be a non-empty class of games. Then a solution σ on Γ satisfies *average bilateral consistency* if for each $(N,V) \in \Gamma$, each $x \in \sigma(N,V)$, and each $P \in \Pi^N$, $(P, V_{x,P}) \in \Gamma$, we have that for each $i \in N$ there exist $n-1$ vectors $(y_i^j, y_j^j) \in \sigma(\{i,j\}, V_{x,\{i,j\}})$ for all $j \neq i$ and $x_i = \frac{\sum_{j \neq i} y_i^j}{n-1}$.

This is weaker than bilateral consistency: given a payoff profile x in the solution, it is required only that for every player $i \in N$ x_i be the average of all the possible projections to solution points in bilateral reduced games. Note how single-valuedness of the solution in reduced games is not required.

Let Γ be a non-empty class of games. Then a solution σ on Γ satisfies *average converse consistency* if for each $(N,V) \in \Gamma$, and each $x \in \partial V(N)$, $x \in \sigma(N,V)$ whenever for each $i \in N$ there exist $n-1$ vectors $(y_i^j, y_j^j) \in \sigma(\{i,j\}, V_{x,\{i,j\}})$ for all $j \neq i$ and $x_i = \frac{\sum_{j \neq i} y_i^j}{n-1}$.

Contrary to the relationship between bilateral consistency and its average counterpart, average converse consistency is stronger than converse consistency. The reason is that the requirement for the decentralization of x is now much weaker: we require from an efficient payoff x that, if for all $i \in N$ x_i is the average of solution payoffs to player i over all the bilateral reduced games, x must be recommended to the whole society as a solution point.

A class of games is rich if it contains all two-person smooth games.

We study now the logical implications of the five axioms in Proposition 1 for two-player games, and average bilateral consistency and average converse consistency in our next impossibility result.

Theorem 2 Let Γ_0 be a rich class of smooth games satisfying Assumptions (1)-(3). There is no solution on Γ_0 satisfying average bilateral consistency and average converse consistency, as well as the following five axioms for two player games (non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence).

Proof : For $n = 2$, average bilateral consistency reduces to bilateral consistency and average converse consistency to converse consistency. Therefore, by Proposition 1, the only solution satisfying the seven axioms considered is $\mathcal{P}(N,V) = \mathcal{AP}(N,V)$ when $n = 2$.

Let $n \geq 3$ and consider a smooth NTU game (N,V) such that $\partial V(N)$ does not contain flat segments. Suppose a solution σ satisfies all seven axioms. Let $x \in \sigma(N,V)$. By Pareto efficiency and average bilateral consistency of σ and by our choice of the game excluding flat segments in $\partial V(N)$, we have that σ must satisfy bilateral consistency. That is, for all $P \in \Pi^N$, $x_P \in \sigma(P, V_{x,P})$. Since for all two-player games, we know that σ coincides with the prekernel, we have that for all $P \in \Pi^N$, $x_P \in \mathcal{P}(P, V_{x,P})$. By the converse consistency of the prekernel, we get that $x \in \mathcal{P}(N,V)$. Therefore, for every game (N,V), $\sigma(N,V) \subseteq \mathcal{P}(N,V)$.

Now consider $x \in \mathcal{P}(N,V)$. By the bilateral consistency of the prekernel, we know that for all $P \in \Pi^N$, $x_P \in \mathcal{P}(P, V_{x,P})$, and given that $\sigma = \mathcal{P}$ for two-player games, we get that for all $P \in \Pi^N$, $x_P \in \sigma(P, V_{x,P})$. Therefore, by the average converse consistency of σ, $x \in \sigma(N,V)$. Hence, $\mathcal{P}(N,V) \subseteq \sigma(N,V)$.

Thus, the only possibility is that σ is the prekernel. However, it is easy to see that $\mathcal{P}$ does not satisfy average converse consistency. $\square$

As the proof of Theorem 2 shows, efficiency and average consistency imply consistency when one considers NTU games. This fact suggests the following alternative definitions.

Let Γ be a non-empty class of smooth games. Then a solution σ on Γ satisfies *average bilateral consistency with respect to reduced hyperplane games* if for each $(N,V) \in \Gamma$, each $x \in \sigma(N,V) \cap \partial V(N)$, and each $P \in \Pi^N$, $(P, W_{x,P}) \in \Gamma$, we have that for each $i \in N$ there exist $n-1$ vectors $(y_i^j, y_j^j) \in \sigma(\{i,j\}, W_{x,\{i,j\}})$ for all $j \neq i$ and $x_i = \frac{\sum_{j \neq i} y_i^j}{n-1}$.

Using as utility weights the gradient of the frontier of $V(N)$ at x, we can make the thought experiment that utility is transferable at those rates. Then, this version of average bilateral consistency has the same interpretation as the previous one: each payoff in the solution is the expectation of solution points to bilateral games where the threat points are determined by the Davis-Maschler logic and the feasible set is the half-space generated by the utility weights.

Let Γ be a non-empty class of smooth games. Then a solution σ on Γ satisfies *average converse* *consistency with respect to reduced hyperplane games* if for each $(N, V) \in \Gamma$, and each $x \in \partial V(N)$, $x \in \sigma(N, V)$ whenever for each $i \in N$ there exist $n-1$ vectors $(y_i^j, y_j^j) \in \sigma(\{i,j\}, W_{x,\{i,j\}})$ for all $j \neq i$ and $x_i = \frac{\sum_{j \neq i} y_i^j}{n-1}$.

Remark 1. On a rich class of smooth games, $\mathcal{AP}$ satisfies average bilateral consistency and average converse consistency with respect to reduced hyperplane games.

Our main result in this section follows:

Theorem 3 Let Γ_0 be a rich class of smooth games satisfying Assumptions (1)-(3). A solution on Γ_0 satisfies average bilateral consistency and average converse consistency with respect to reduced hyperplane games, as well as the following five axioms for two-player games (non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence) if and only if it is $\mathcal{AP}$.

Proof : By Proposition 1 and Remark 1, the solution $\mathcal{AP}$ on Γ_0 satisfies the seven axioms listed. Now we prove uniqueness.

Let $(N, V) \in \Gamma_0$, and let σ be a solution on Γ_0 that also satisfies the seven axioms of the Theorem. We prove that $\sigma(N, V) = \mathcal{AP}(N, V)$. The proof for $n = 1$ is trivial. We have already proven the case of $n = 2$ (Proposition 1 and Remark 1). Then consider the case of $n \geq 3$.

Suppose that $\mathcal{AP}(N, V) \neq \emptyset$. Let $x \in \mathcal{AP}(N, V)$. By the average bilateral consistency (with respect to reduced hyperplane games) of $\mathcal{AP}$, for each $i \in N$ there exist $n-1$ vectors $(y_i^j, y_j^j) \in \mathcal{AP}(\{i,j\}, W_{x,\{i,j\}})$ for all $j \neq i$ and $x_i = \frac{\sum_{j \neq i} y_i^j}{n-1}$. But for every two-player smooth game (P, W), if σ satisfies all seven axioms, $\sigma(P, W) = \mathcal{AP}(P, W)$. Hence, for each $i \in N$ there exist $n-1$ vectors $(y_i^j, y_j^j \in \sigma(\{i,j\}, W_{x,\{i,j\}})$ for all $j \neq i$ and $x_i = \frac{\sum_{j \neq i} y_i^j}{n-1}$. By the average converse consistency (with respect to reduced hyperplane games) of σ, $x \in \sigma(N, V)$. Hence, $\mathcal{AP}(N, V) \subseteq \sigma(N, V)$. Note in particular that $\sigma(N, V)$ is non-empty. Then, following similar steps, we can show that $\sigma(N, V) \subseteq \mathcal{AP}(N, V)$. Thus, $\sigma(N, V) = \mathcal{AP}(N, V)$.

Suppose that $\mathcal{AP}(N, V) = \emptyset$. Let $x \in \partial V(N)$. Then by the average converse consistency (with respect to reduced hyperplane games) of $\mathcal{AP}$, there exists at least one player $i \in N$ such that $x_i \neq \frac{\sum_{j \neq i} y_i^j}{n-1}$ for any collection of $n-1$ vectors $(y_i^j, y_j^j \in \sigma(\{i,j\}, W_{x,\{i,j\}})$ for all $j \neq i$.

Since for every two-person smooth game (P, W), $\sigma(P, W) = \mathcal{AP}(P, W)$, it follows from the average bilateral consistency (with respect to reduced hyperplane games) of σ that $x \notin \sigma(N, V)$. Therefore, there is no payoff profile in $\sigma(N, V)$, so that $\sigma(N, V) = \emptyset$. Thus, $\sigma(N, V) = \mathcal{AP}(N, V)$.

□

Remark 2: The basis of the characterization in Theorem 3 is to pin down the solution on two-person games with the five axioms imposed there, and to extend it to multilateral settings with the versions of average consistency and its converse for reduced hyperplane games. In contrast, the axiomatic result in Orshan and Zarzuelo (2000) is based on characterizing the solution first on the class of hyperplane games. They then require their versions of consistency and its converse only over this class and use local independence for the extension of the result to general smooth NTU games. Note, for example, how their version of consistency (2-CO) is violated by $\mathcal{AP}$ outside of the class of hyperplane games because it is not generally single-valued.

Next we show that the seven axioms used in the characterization are logically independent. In each example, the axiom in brackets is the one violated by the solution proposed.

Example 1 [non-emptiness]: For every $(N, V) \in \Gamma_0$, let $\sigma(N, V) = \emptyset$. Then σ vacuously satisfies all the conditions of Theorem 3 except non-emptiness for two-person games.

Example 2 [Pareto efficiency]: For every two-person game (P, V), define $b(P, V) = (v_i)_{i \in P}$ if $(v_i)_{i \in P} \in \text{Int}V(P)$ and $b(P, V) = \mathcal{AP}(P, V)$ otherwise. For every $(N, V) \in \Gamma_0$, let

$$\sigma(N, V) = \{x \in V(N) | \forall i \in N, x_i = \frac{\sum_{j \neq i} y_i^j}{n-1} \quad \text{for some} \quad (y_i^j, y_j^j) \in b(\{i, j\}, W_{x,\{i,j\}})\}.$$

Then σ satisfies all the conditions of Theorem 3 except Pareto efficiency for two-person games.

Example 3 [equal treatment for TU games]: For every $(N, V) \in \Gamma_0$, let

$$\sigma(N, V) = \partial V(N).$$

Then σ satisfies all the conditions of Theorem 3 except equal treatment for two-person TU games.

Example 4 [scale invariance]: For every $(N, V) \in \Gamma_0$, let

$$\sigma(N, V) = \{x \in \partial V(N) | \quad \sum_{j \neq i} [s_{i,j}(x) - s_{j,i}(x)] = 0 \quad \forall i \in N\}.$$

Then σ satisfies all the conditions of Theorem 3 except scale invariance for two-person games.

Example 5 [local independence]: For every two-person game (P, V), define $a(P, V) = (a_i(P, V))_{i \in P}$ by

$$a_i(P, V) = \max\{x_i \in I\!R^{\{i\}} | (x_i, v_j(x_{-P})) \in V(P), \quad P = \{i, j\}\}.$$

For every $(N, V) \in \Gamma_0$, let

$$\sigma(N, V) = \{x \in \partial V(N) | \quad \forall i \in N, x_i = \frac{\sum_{j \neq i} y_i^j}{n-1}\}$$

for some

$$(y_i^j, y_j) \in \partial W_{x,\{i,j\}}(\{i, j\}) \cap [(v_k(x_{-\{i,j\}}))_{k=i,j}, a(\{i, j\}, W_{x,\{i,j\}})]\},$$

where $[c, d] = \{(1-t)c + td | 0 \leq t \leq 1\}$ for each $c, d \in I\!R^{\{i,j\}}$. That is, for every player $i \in N$, x_i can be expressed as the average of maximal points of the feasible set $W_{x,\{i,j\}}(\{i, j\})$ on the segment connecting $(v_k(x_{-\{i,j\}}))_{k=i,j}$ to $a(\{i, j\}, W_{x,\{i,j\}})$ if $x \in \sigma(N, V)$. Note that σ is a sort of average of Kalai- Smorodinsky bargaining solutions, and it satisfies all the conditions of Theorem 3 except local independence for two-person games.

Example 6 [average bilateral consistency with respect to reduced hyperplane games]: Let $(N, V) \in \Gamma_0$. Let $\sigma(N, V) = \mathcal{AP}(N, V)$ when $n = 1, 2$, and $\sigma(N, V) = \partial V(N)$. It is easy to see that σ satisfies the five axioms imposed on two-person games and average converse consistency with respect to reduced hyperplane games. Therefore, by Theorem 3, it viollates average bilateral consistency with respect to reduced hyperplane games.

Example 7 [average converse consistency with respect to reduced hyperplane games]: For every $(N, V) \in \Gamma_0$. Let $\sigma(N, V) = \mathcal{P}(N, V)$. Then, σ satisfies all the conditions of Theorem 3 except average converse consistency with respect to reduced hyperplane games.

As in Serrano and Shimomura (1998), we next investigate the implications of the same sets of seven axioms on the class of smooth games with non-empty cores.

Let (N, V) be a game, S a subset of N, and $x \in I\!R^N$. Then we say that S can *improve upon* x if there is $y \in V(S)$ such that $y_i > x_i$ for all $i \in S$. The *core* of (N, V) is:

$$\mathcal{C}(N, V) = \{x \in V(N) | \text{ There is no coalition that can improve upon } \quad x\}.$$

Let (N,V) be a smooth game. The *intersection of the core and the prekernel* of (N,V) is:

$$\mathcal{P}^*(N,V) = \mathcal{C}(N,V) \cap \mathcal{P}(N,V).$$

Let (N,V) be a smooth game. The *intersection of the core and the average prekernel* of (N,V) is:

$$\mathcal{AP}^*(N,V) = \mathcal{C}(N,V) \cap \mathcal{AP}(N,V).$$

We can show the following in exactly the same way as for Proposition1 .

Proposition 2 Let $\Gamma_c^{\{i,j\}}$ be the class of two-person smooth games $(\{i,j\},V)$ satisfying Assumptions (1)-(3) and with non-empty cores. Then a solution on $\Gamma_c^{\{i,j\}}$ satisfies non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence if and only if it is $\mathcal{AP}^*$.

The proof simply follows from the fact that for two-player games $\mathcal{P}^* = \mathcal{AP}^*$. Serrano and Shimomura (1998) show that, over a rich class of smooth n-player games, $\mathcal{P}^*$ is the only solution satisfying bilateral consistency, converse consistency, and the group of five axioms for two-player games in Proposition 2. Similarly, we can prove the theorems below as we have done for Theorems 2 and 3.

Theorem 4 Let Γ_c be the class of smooth games satisfying Assumptions (1)-(3) with non-empty cores containing all the two-person games with the same properties. There is no solution on Γ_c satisfying average bilateral consistency and average converse consistency, as well as the following five axioms for two-player games (non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence).

Theorem 5 Let Γ_c be the class of smooth games satisfying Assumptions (1)-(3) with non-empty cores containing all the two-person games with the same properties. A solution on Γ_c satisfies average bilateral consistency and average converse consistency with respect to reduced hyperplane games, as well as the following five axioms for two-player games (non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence) if and only if it is $\mathcal{AP}^*$.

It can be shown by modifying the seven examples that followed the proof of Theorem 3 that the same seven axioms are also logically independent over the smaller class of games Γ_c.

5 Conclusion

We have obtained positive and normative foundations of the average prekernel, and we have compared them to those for the prekernel. The prekernel captures naturally those payoffs profiles where players find themselves in a situation of "bilateral equilibrium." Given its frequent existence difficulties, it is convenient to settle for the weaker property of "bilateral equilibrium in average" that the average prekernel describes. Both the strategic and axiomatic analyses performed here provide support to this notion. It will be interesting to test the differences between the two solution concepts in specific applications.

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