Equity, Envy and Efficiency under Asymmetric Information

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Abstract

The set of fair (i.e. envy free and efficient) allocation rules may be empty in well-behaved pure exchange economies if the agents are asymmetrically informed at the time of contracting. In addition, there may exist efficient allocation rules such that every agent envies another.

1 Introduction

The objective of the paper is to study to what extent some of the main results of Varian (1974) may be generalized to exchange economies under asymmetric information. Although the agents are asymmetrically informed about the future state of the economy at the time of contracting, I assume for simplicity that the state is verifiable when the contracts are implemented. Incentive and measurability constraints are therefore irrelevant.

Agent i envies agent j if i prefers to be treated as j. Envy freeness, the absence of envy, is an appealing concept of equity in resource allocation problems. Envy freeness combined with efficiency leads to a natural notion of fairness (Foley, 1967; Varian, 1974). The set of fair allocations is non-empty for classical exchange economies (Varian, 1974). Indeed, any competitive equilibrium allocation resulting from an equal sharing of the aggregate endowment is fair.

Efficiency under asymmetric information has been studied by Wilson (1978). I apply his notion of interim efficiency, simply called efficiency in the present

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paper. Further, I will say that an allocation rule is envy free if there is zero probability of an agent envying another. I show in section 3 that these notions of efficiency and of envy freeness may be incompatible in pure[2] exchange economies if the agents are asymmetrically informed at the time of contracting. However, any constrained market equilibrium allocation rule (Wilson, 1978; de Clippel, 2004) resulting from an equal sharing of the aggregate endowment in each state of the economy is efficient and satisfies a weak form of envy-freeness: there does not exist two agents i and j such that it is common knowledge that i envies j.

There is unanimous envy if every agent envies another. It is impossible to have unanimous envy at an efficient allocation in classical exchange economies (Varian, 1974). Indeed, if there were unanimous envy, then one could achieve a Pareto improvement by swapping the bundles of some agents. I show in section 4 that there may be unanimous envy (even with probability one) at some efficient allocation rules when the agents are asymmetrically informed. However, if an allocation rule is efficient, then it is impossible to find a subset $\{i_1, \ldots, i_K\}$ of agents such that is common knowledge that i_k envies $i_{(k+1)modK}$ for each $k \in \{1, \ldots, K\}$.

2 The Model

I start with some general definitions and notations. The finite set of agents is denoted N. The finite set of goods is denoted L. The future state of the economy is uncertain. Let Ω be the finite set of possible states. Let π be the common prior that describes the relative probability of those states. I assume without loss of generality that $\pi(\omega) > 0$ for each $\omega \in \Omega$. The agents may have some private information. The information of agent i is summarized by a partition $\mathcal{P}_i$ of Ω. For each $\omega \in \Omega$, let $P_i(\omega)$ be the atom of the partition that contains ω. The interpretation goes as follows. When the future state of the economy is ω, i knows and only knows that it will be an element of $P_i(\omega)$. His beliefs are derived from π by Bayesian updating. The true state of the economy is common knowledge among the agents at some future date. It determines their preferences and the aggregate endowment. Let $e : \Omega \rightarrow \mathbb{R}^L_{++}$ be the function that specifies the aggregate endowment of the economy. The agents evaluate the lotteries according to the expected utility criterion, given the concave, continuous and strongly increasing state-dependent utility functions denoted $u_i : \mathbb{R}^L_+ \times \Omega \rightarrow \mathbb{R}$ for each $i \in N$. Decisions are taken today about the way to redistribute the endowments when the state will be common knowledge. Hence the agents agree on allocation rules that specify a way to divide the

[2] Pazner and Schmeidler (1974) show that envy freeness and efficiency may be incompatible in economies with production, even under complete information.

total endowment of the economy in each state: $a : \Omega \rightarrow \mathbb{R}_{+}^{L\times N}$ such that $\sum_{i\in N} a_i(\omega) \leq e(\omega)$ for each $\omega \in \Omega$.

An allocation rule a' *Pareto dominates* an allocation rule a if every agent weakly prefers (given his private information) a' over a in each state of the economy and at least one agent strictly prefers a' over a in at least one state, i.e.

$$\sum_{\omega'\in P_i(\omega)} \pi(\omega')u_i(a_i(\omega'),\omega') \leq \sum_{\omega'\in P_i(\omega)} \pi(\omega')u_i(a_i'(\omega'),\omega')$$

for each $i \in N$ and each $\omega \in \Omega$, one of the inequalities being strict. An allocation rule is *efficient* (see the notion of interim efficiency in Wilson, 1978) if it is not Pareto dominated by any other allocation rule.

Let a be an allocation rule and let ω be a state of the economy. Then, agent i *envies* agent j *at* ω if

$$\sum_{\omega'\in P_i(\omega)} \pi(\omega')u_i(a_i(\omega'),\omega') < \sum_{\omega'\in P_i(\omega)} \pi(\omega')u_i(a_j(\omega'),\omega').$$

The allocation rule a is *envy free* if there is zero probability of an agent envying another, i.e. there does not exist a state ω at which some agent i envies some agent j. An allocation rule is *fair* if it is both efficient and envy free. Palfrey and Srivastava (1987) suggest the same notion of fairness in a slightly different framework. There is *unanimous envy at* ω if every agent envies another at ω.

Subsets of Ω are called *events*. An event $\mathcal{E}$ is *common knowledge* if it can be written as a union of elements of $\mathcal{P}_i$ for each $i \in N$.

3 On the Impossibility to Achieve Fairness

The allocation rule that equally splits the aggregate endowment in each state of the economy is envy free and therefore the set of envy free allocation rules is not empty. Obviously, the set of efficient allocation rules is not empty as well. The next two examples show that the set of fair allocation rules may be empty though.

Example 1 Consider three agents and one good (money). There are four equally likely states: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\pi(\omega) = 1/4$ for each $\omega \in \Omega$. The following table specifies the aggregate endowment as well as the information and the utility function of the agents.

State	$e(.)$	$P_1(.)$	$P_2(.)$	$P_3(.)$	$u_1(x,.)$	$u_2(x,.)$	$u_3(x,.)$
ω_1	1800	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_1\}$	x	$\sqrt{x}$	x
ω_2	1200	$\{\omega_1,\omega_2\}$	$\{\omega_2,\omega_4\}$	$\{\omega_2\}$	x	$\sqrt{x}$	x
ω_3	1200	$\{\omega_3,\omega_4\}$	$\{\omega_1,\omega_3\}$	$\{\omega_3\}$	x	$\sqrt{x}$	x
ω_4	1800	$\{\omega_3,\omega_4\}$	$\{\omega_2,\omega_4\}$	$\{\omega_4\}$	x	$\sqrt{x}$	x

An allocation rule a is envy free if and only if it satisfies the following inequations:

$$\begin{cases} a_1(\omega_1) + a_1(\omega_2) \geq \max_{i\in\{2,3\}}[a_i(\omega_1) + a_i(\omega_2)] \\ a_1(\omega_3) + a_1(\omega_4) \geq \max_{i\in\{2,3\}}[a_i(\omega_3) + a_i(\omega_4)] \\ \sqrt{a_2(\omega_1)} + \sqrt{a_2(\omega_3)} \geq \max_{i\in\{1,3\}}[\sqrt{a_i(\omega_1)} + \sqrt{a_i(\omega_3)}] \\ \sqrt{a_2(\omega_2)} + \sqrt{a_2(\omega_4)} \geq \max_{i\in\{1,3\}}[\sqrt{a_i(\omega_2)} + \sqrt{a_i(\omega_4)}] \\ (\forall \omega \in \Omega) : a_3(\omega) \geq \max_{i\in\{1,2\}} a_i(\omega) \end{cases}$$

which amounts to $a_1(\omega) = a_2(\omega) = a_3(\omega)$ for each $\omega \in \Omega$. The best among those allocation rules is a where $a(\omega_1) = a(\omega_4) = (600, 600, 600)$ and $a(\omega_2) = a(\omega_3) = (400, 400, 400)$. It is Pareto dominated by the allocation rule a' where $a'(\omega_1) = a'(\omega_4) = (701, 498, 601)$ and $a'(\omega_2) = a'(\omega_3) = (301, 498, 401)$. The no-envy property combining restrictions based on different pieces of information may lead to allocation rules that do not exploit the possibilities of insurance.

I suggest a similar example where, contrarily to the previous example, the aggregate endowment is constant while the utility functions are state dependent.

Example 2 Only the aggregate endowment and the utility functions of the agents differ from the previous example. They are specified as follows.

State	$e(.)$	$u_1(x,.)$	$u_2(x,.)$	$u_3(x,.)$
ω_1	1200	x	$2x$	x
ω_2	1200	x	x	x
ω_3	1200	x	x	x
ω_4	1200	x	$2x$	x

As before, it is easy to check that an allocation rule a is envy free if and only if $a_1(\omega) = a_2(\omega) = a_3(\omega)$ for each $\omega \in \Omega$. The best among those allocation rules is a where $a(\omega) = (400, 400, 400)$ for each $\omega \in \Omega$. It is Pareto dominated though by the allocation rule a' where $a'(\omega_1) = a'(\omega_4) = (3, 796, 401)$ and $a'(\omega_2) = a'(\omega_3) = (799, 0, 401)$.

I now discuss a related paper by Gajdos and Tallon (2002). The *ex-ante* stage represents the situation that prevails before the agents learn their private information. There is uncertainty but the information is symmetrically distributed: $\mathcal{P}_i = \{\Omega\}$ for each $i \in N$. The *ex-post* stage represents the situation that prevails once the state of the economy is common knowledge. I studied the resource allocation problem at the *interim stage*: the agents know their private information while the state of the economy is not yet common knowledge. Gajdos and Tallon study allocation rules that are both ex-ante efficient and ex-post envy free. These are called *intertemporally fair*. Intertemporal fairness is a strong requirement that is hard to satisfy. In particular, it is stronger than my notion of fairness. Indeed, ex-post envy freeness implies envy freeness and ex-ante efficiency implies efficiency. Results establishing the existence of some intertemporally fair allocation rules are powerful.[3] On the contrary, examples showing the non-existence of such rules are weak. Indeed, it amounts to state that strong ex-post criteria are incompatible with insurance. My two examples are more interesting as they show the possible incompatibility of efficiency with envy freeness at the interim stage. Observe finally that if the third agent is dropped from the first example then the allocation rule a where $a(\omega_1) = a(\omega_4) = (105, 75)$ and $a(\omega_2) = a(\omega_3) = (45, 75)$ is fair but not intertemporally fair.

I conclude the section by showing that efficiency may be compatible with a weaker notion of envy freeness. A constrained market equilibrium is a notion of price equilibrium introduced by Wilson (1978, footnote 6) as a technical tool to prove the non-emptiness of the coarse core. It can be justified through a convergence result, as the type-agent core shrinks towards the set of constrained market equilibria when the economy is replicated (see de Clippel, 2004). An allocation rule a is a *constrained market equilibrium* resulting from an equal split of the aggregate endowment in each state of the economy if it is feasible and there exists a price system $p : \Omega \to \mathbb{R}^L_+$ such that

$$\sum_{\omega' \in P_i(\omega)} \pi(\omega') u_i(a'_i(\omega'), \omega') \leq \sum_{\omega' \in P_i(\omega)} \pi(\omega') u_i(a_i(\omega'), \omega')$$

[3] There always exists some intertemporally fair allocation rule in any well-behaved sunspot economy. Indeed, the allocation rule that specifies in each state of the economy the same competitive equilibrium allocation resulting from an equal sharing of the (state-independent) aggregate endowment is both ex-ante efficient and ex-post envy free. This is the only non-emptiness result that Gajdos and Tallon obtain under the common prior assumption.

for each $a'_i \in \mathbb{R}_+^{L \times P_i(\omega)}$ with $\sum_{\omega' \in P_i(\omega)} p(\omega').a'_i(\omega') \leq \sum_{\omega' \in P_i(\omega)} p(\omega').\frac{e(\omega')}{N}$, each $\omega \in \Omega$ and each $i \in N$.

Proposition 1 *Let a be a constrained market equilibrium resulting from an equal split of the aggregate endowment in each state of the economy. Then, it is impossible to find two agents (i, j) and a common knowledge event $\mathcal{E}$ such that i envies j at each $\omega \in \mathcal{E}$.*

<u>Proof</u>: Suppose on the contrary that there exists a common knowledge event $\mathcal{E}$ such that i envies j at each $\omega \in \mathcal{E}$. Let $p : \Omega \to \mathbb{R}_+^L$ be the price vector associated to a. We have:

$$\begin{aligned}
\textstyle\sum_{\omega \in \mathcal{E}} p(\omega).a_j(\omega) &= \textstyle\sum_{P_i \in \mathcal{P}_i \text{ s.t. } P_i \subseteq \mathcal{E}} \sum_{\omega \in P_i} p(\omega).a_j(\omega) \\
&> \textstyle\sum_{P_i \in \mathcal{P}_i \text{ s.t. } P_i \subseteq \mathcal{E}} \sum_{\omega \in P_i} p(\omega).\frac{e(\omega)}{N} \\
&= \textstyle\sum_{\omega \in \mathcal{E}} p(\omega).\frac{e(\omega)}{N} \\
&= \textstyle\sum_{P_j \in \mathcal{P}_j \text{ s.t. } P_j \subseteq \mathcal{E}} \sum_{\omega \in P_j} p(\omega).\frac{e(\omega)}{N} \\
&\geq \textstyle\sum_{P_j \in \mathcal{P}_j \text{ s.t. } P_j \subseteq \mathcal{E}} \sum_{\omega \in P_j} p(\omega).a_j(\omega) \\
&= \textstyle\sum_{\omega \in \mathcal{E}} p(\omega).a_j(\omega).
\end{aligned}$$

This is absurd. The strict inequality follows from the fact that a_j must be out of the budget set of agent i for each $\omega \in \mathcal{E}$ as he prefers a_j over a_i at those states. The weak inequality follows from the fact that a_j satisfies the budget constraint of agent j for each $\omega \in \mathcal{E}$. ■

Notice that the set of constrained market equilibrium resulting from an equal split of the aggregate endowment in each state of the economy is non-empty (de Clippel, 2004, theorem 4) and is included in the set of efficient allocation rules, as it is a subset of the coarse core.

4 Efficiency with Unanimous Envy

The following example shows that there may be unanimous envy at some $\omega \in \Omega$ even if the allocation rule is efficient.

Example 3 Consider the following allocation rule in the economy described in example 2: $a(\omega) = (0, 700, 500)$ if $\omega \in \{\omega_1, \omega_4\}$ and $a(\omega) = (700, 0, 500)$ if $\omega \in \{\omega_2, \omega_3\}$. It is easy to check that a is efficient. On the other hand, every agent envies another, whatever the future state of the economy: agents 1 and 2 both envy agent 3 whatever the future state, while agent 3 envies either agent 1 or agent 2 as a function of the future state.

The previous example shows that it may even be common knowledge that there is unanimous envy at some efficient allocation rule. A necessary condition for this result is that the improving cycle varies with the future state of the economy, as the following proposition highlights.

Proposition 2 *Let a be an efficient allocation rule. Then there do not exist a subset $\{i_1, \ldots, i_K\}$ of agents and a common knowledge event $\mathcal{E}$ such that i_k envies $i_{(k+1)modK}$ for each $k \in \{1, \ldots, K\}$ and at each $\omega \in \mathcal{E}$.*

Proof: Otherwise, the allocation rule a' defined as follows Pareto dominates a: $a'(\omega) := a(\omega)$ for each $\omega \in \Omega \setminus \mathcal{E}$, $a'_i(\omega) := a_i(\omega)$ for each $\omega \in \mathcal{E}$ and each $i \in N \setminus \{i_1, \ldots, i_K\}$, and $a'_{i_k}(\omega) := a_{i_{(k+1)modK}}(\omega)$ for each $\omega \in \mathcal{E}$ and each $k \in \{1, \ldots, K\}$. ■

5 Conclusion

There may exist a fundamental trade-off between the objectives of efficiency and equity (envy freeness) when the agents are asymmetrically informed. As a second best, one may search for the allocation rules that are efficient within the class of envy-free allocation rules. Another option is to develop a criterion that allows to compare the equity properties of different allocation rules. Let for instance $n(\omega) \in \{0, \ldots, n\}$ be the number of agents envying at least one other agent at ω. Section 3 shows that it may be impossible to find an efficient allocation rule with $n(\omega) = 0$ for each $\omega \in \Omega$. Section 4 shows that there may exist efficient allocation rules with $n(\omega) = n$ for each $\omega \in \Omega$. One could search for efficient allocation rules that minimize some envy index such as $\max_{\omega\in\Omega} n(\omega)$ or $\sum_{\omega\in\Omega} p(\omega)n(\omega)$.

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