How much trade does the transfer paradox require?

The threshold computed*

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Abstract

Samuelson (1947) stated that a regular equilibrium exhibits the transfer paradox if and only if it is unstable. Gale (1974) and many in the early 1980's debunked this equivalence by adding extra countries, reaching an anti consensus.

We reinterpret Samuelson's result as identifying the threshold, i.e. the minimum level of trade beyond which the transfer paradox appears.

This reinterpretation generalizes fully to finitely many countries and commodities, and reaffirms the anti consensus quantitatively.

A by-product is an explicit general example of Donsimoni and Polemarchakis (1994), that whatever the equilibrium prices and incomes, the welfare impact of a transfer is made arbitrary by some compatible economy.

JEL Classification: D51, D64, F11

Keywords: transfer paradox, threshold, Slutsky, instability

*I wish to record my gratitude to Donald Brown, John Geanakoplos, and Stephen Morris for their feedback, guidance, and support in the course of the dissertation, of which this work is part, as well as to the Cowles Foundation for its generous support in the form of a Carl A. Anderson fellowship. Herakles Polemarchakis, Herbert Scarf, and Bjorn Tuypens also provided valuable feedback. All shortcomings are mine.

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1 Introduction

Germany's reparations after World War I provoked a controversy about terms of trade. Did the reparations improve or worsen her terms of trade? Did the new terms of trade exacerbate or mitigate her income loss due to reparations? Leontief (1937) showed by example that a donation could so change terms of trade as to erase the income loss and benefit donor–the transfer paradox.

Samuelson (1947) noted the regular equilibria exhibiting the transfer paradox were those unstable with respect to tatonnement. Others confirmed this beautiful characterization of the transfer paradox, at least with two countries and two commodities; Mundell (1968), Balasko (1978).

Theorem 1 (Samuelson 1947) *With two countries and two goods, suppose a regular equilibrium. Then the local transfer paradox is present if and only if it is unstable.*

Most deemed instability a theoretical curiosity, the situation where demand increases with prices. By Samuelson's equivalence, the transfer paradox too became a theoretical curiosity, and interest in it waned. Accordingly, Samuelson's equivalence remained the big result on the transfer paradox, and became the wisdom on the topic, seemingly even after Scarf's (1960) examples of instability.

Almost thirty years later, Gale (1974) showed by example that Samuelson's equivalence broke down with a third country.

Theorem 2 (Gale 1974) *With three Leontief countries and two goods, there is an example of a stable equilibrium exhibiting the local transfer paradox.*

Yet the example failed to shatter the received wisdom, perhaps because Gale never pointed out its stability, never wrote "transfer paradox."

Chichilnisky (1980) discovered the stability of Gale's example, and further showed its dependence on the preferences of the countries. That it took so long to detect stability evidenced how ingrained Samuelson's wisdom had been–why check, if it must be unstable? Once advertised, this set off a stampede of research in the early eighties, excited by the surprising news, by the renewed plausibility of the transfer paradox, and by the chance to charge at current wisdom.

The stampede mostly split between extending Gale's counterexample and Chichilnisky's analysis, always with two goods. New examples appeared in Polemarchakis (1983), and in Leonard and Manning (1983) with non-Leontief utilities (two Cobb-Douglas, one quasilinear).[1] The analyses (a) relaxed utilities from being Leontief, (b) clarified the role of excess demands, marginal propensities to consume, and elasticities of excess demand, (c) derived formulas for the welfare impact of small donations in terms of these notions. Yano (1983), Ravallion (1983), Bhagwati et al. (1983), Dixit (1983) singly managed all these extensions. Retaining Leontief utilities, Geanakoplos and Heal (1983), Polemarchakis (1983), and Chichilnisky (1983) gave a priori, equilibrium-independent bounds on endowments and utilities guaranteeing the equilibrium to be unique, globally stable, and consistent with the transfer paradox. Consensus settled on

- the donor's trade level being required large enough,

and on this requisite level being increasing in

- 1) the proximity between the donor's and the recipient's marginal propensities to consume
- 2) the substitution effect, explaining the preponderance of Leontief utilities in examples

In particular, emphasis turned toward the notions in (b) and away from stability.

The remainder focused on the existence question. From Dixit's (1983) formula Safra (1984) obtained

Theorem 3 (Safra 1984) *With more than two countries and with two goods, suppose an unstable equilibrium where some trading country's marginal propensity to consume is neither largest nor lowest. Then there is a stable equilibrium exhibiting the transfer paradox, with the same equilibrium prices and incomes but less trade.*

This was another charge, generalizing Gale's example to smooth preferences and multiple countries–curiously, instability did cameo. Earlier, Safra (1983) had argued nonconstructively that for almost any equilibrium prices and incomes, there was a compatible economy exhibiting the transfer paradox. Given any small desired welfare impact and endowment reallocation, Donsimoni and Polemarchakis (1994) showed

[1] Aumann and Peleg (1974) discarded endowments, instead of reallocating them.

more constructively that for almost any equilibrium prices and incomes, there was a compatible economy exhibiting the given welfare impact as the de facto welfare impact of the given reallocation.

Altogether, the stampede set off by Gale and Chichilnisky sidelined Samuelson's equivalence as dramatically as it itself had sidelined the transfer paradox. If not all targeted what was wrong with Samuelson's equivalence, none looked for what was right with it.

We propose a point of view that rescues Samuelson's equivalence and reaffirms the anti consensus, even though the latter arose out of attacks on the former. The key idea is that whether an equilibrium is unstable or stable is a precise answer to whether the trade level is or is not large enough relative to

- 1) the proximity between the donor's and the recipient's marginal propensities to consume
- 2) the substitution effect

To see it, we revisit the classical decomposition of the Jacobian J of aggregate demand

$$J = S - \Sigma m^i z^{i\prime}$$

where S is the sum of the countries' substitution effects, m^i is country $i's$ marginal propensity to consume, and z^i its excess demand for the nonnumeraire commodities. With two countries, it reads

$$J = S - \nabla z^{1\prime}$$

where $\nabla = m^1 - m^2$ is the difference between their marginal propensities to consume, thanks to market clearing $z^1 + z^2 = 0$. With two goods, an equilibrium is unstable, by definition, if $J > 0$. Thus Samuelson's equivalence is that the transfer paradox is present or absent according as $J > 0$ or $J < 0$. The threshold is $J = 0$, i.e. the **threshold** trade level z^1 is

$$z^1 = \frac{S}{\nabla} \tag{|}$$

Indeed, this reaffirms the anti consensus, in that the threshold trade level $\frac{S}{\nabla}$ is increasing in the proximity $\frac{1}{\nabla}$ between marginal propensities to consume, and in the substitution effect S. Samuelson's equivalence, once reinterpreted, ironically encapsulates and quantifies the anti consensus.

We show that the threshold reinterpretation generalizes fully to a finite number of countries and commodities. This requires making sense of the ratio $\frac{S}{\nabla}$ with multiple commodities, when ∇ is no longer a scalar. It requires making sense of the trade level $|z|$ with multiple countries, when the equality $\left|z^1\right| = \left|z^2\right|$ as an unambiguous norm is unavailable.

Fixing the price of C commodities and incomes of H countries, implies the aggregate substitution effect $S = \Sigma S^i$ and the marginal propensities to consume (m^i). Discarding the numeraire, S is negative definite and symmetric, hence defines an inner product on net trades $n \in R^{C-1}$ of nonnumeraire commodities, $(n, n) = n'(-S^{-1})n$, and a norm, $\|n\| = \sqrt{(n, n)}$. If $z = (z^h)$ are the equilibrium net trades at the equilibrium prices and incomes, the **trade level** is $\|z\|^* = \sqrt{\frac{1}{H}\Sigma \left\|z^h\right\|^2}$.

What is Samuelson's threshold in this language? Multiplying (|) by $-z^1 S^{-1}$,

$$\begin{aligned} z^1(-S^{-1})z^1 &= \frac{-z^1}{\nabla} = \frac{1}{\nabla(-S^{-1})\nabla} \\ \left\|z^1\right\| &= \frac{1}{\|\nabla\|} \end{aligned}$$

Thanks to market clearing, $\left\|z^2\right\| = \frac{1}{\|\nabla\|}$ and

$$\|z\|^* = \frac{1}{\|\nabla\|} \qquad (|)$$

Theorem 4 (Samuelson reinterpreted) *With two countries and two goods, the threshold for the transfer paradox at regular equilibria is* $\frac{1}{\|\nabla\|}$.

One generalization is to multiple goods $C \geq 2$.

Theorem 5 (Threshold with multiple goods) *With two countries, the threshold for the transfer paradox at regular equilibria is still* $\frac{1}{\|\nabla\|}$.

With multiple countries, the donor can play the welfare of one recipient against another's, unboundedly. With just two countries, this is impossible because there is a sole recipient. For this reason the threshold is no greater than the above. Specifically, for each country let

$$\nabla^h = m^h - \frac{1}{H-1}\Sigma_{i \neq h} m^i$$

With $H = 2$ clearly $\nabla^1 = \nabla$. Then

Theorem 6 (Threshold bounded above) *With* $H, C \geq 2$ *countries and goods, the threshold for* h *to be a protagonist in the transfer paradox at regular equilibria is at most* $\frac{1}{\sqrt{H-1}\left\|\nabla^h\right\|}$. *So the threshold for the transfer paradox at regular equilibria is at most* $\min_h \frac{1}{\sqrt{H-1}\left\|\nabla^h\right\|}$.

We now report the threshold for h to be a protagonist.

Definition 1 *Fix* $\dot{v} \in R^H$ *with* $\dot{v}^h = 1, 1'\dot{v} = 0$, *to be interpreted as the welfare impact of an infinitesimal donation. Then define the numerator*

$$n(\dot{v}) = \sqrt{\frac{\left(\Sigma_S \dot{v}^i + 1\right)^2}{|S| + 1} + \Sigma_{\setminus S} \dot{v}^{i2}} \tag{1}$$

where $S \subset \{1, ..., H\} \backslash \{h\}$ *is as follows. Ordering* $\dot{v}^{-h} : \dot{v}^{i_1} \geq ... \geq \dot{v}^{i_{H-1}}, S = \{i_1, ..., i_n\}$ *for the largest* n *such that*

$$\text{if } \; i \in S \qquad \dot{v}^i > \frac{\Sigma_S \dot{v}^i + 1}{|S| + 1}$$

That is, $S \subset H - h$ *consists of the best off: those better off than the average of the group consisting of the even better off and of* h. *Now define*

$$\nabla^h(\dot{v}) = m^h + \Sigma_{i \neq h} m^i \dot{v}^i$$

Finally, define[2]

$$\boxed{T^h = \inf \frac{n(\dot{v})}{\sqrt{H}\left\|\nabla^h(\dot{v})\right\|} \qquad \text{subject to } \; \dot{v}^h = 1, 1'\dot{v} = 0}$$

Theorem 7 (Threshold computed) *With* $H, C \geq 2$ *countries and goods, the threshold for* h *to be a protagonist in the transfer paradox at regular equilibria is* T^h. *So the threshold for the transfer paradox at regular equilibria is* $\min_h T^h$.

It seems impossible to compute T^h in general; after all, the program is the ratio of two convex functions over a noncompact domain. Of course, given particular equilibrium prices and incomes, a computer would.

[2] This exists by completeness of the reals, because the objective is bounded below by zero.

On the other hand, the upper bound is easily seen to come from $\dot{v}^{-h} = -\frac{1}{H-1}$. For the numerator, note $S = \{1, ..., H\} \backslash \{h\}$ and $n(\dot{v}) = \frac{H}{H-1}$.[3] Noting also $\nabla^h(\dot{v}) = \nabla^h$,

$$T^h \leq \frac{\sqrt{\frac{H}{H-1}}}{\sqrt{H}\left\|\nabla^h\right\|} = \frac{1}{\sqrt{H-1}\left\|\nabla^h\right\|}$$

This gives theorem 6. Further, when $H = 2$ the constraint set $\dot{v}^h = 1, 1'\dot{v} = 0$ is a singleton, the above $\dot{v}^{-h} = -\frac{1}{H-1}$, and this upper bound is the $\inf$. This gives theorem 5.

Our notion of threshold with multiple commodities is Samuelson's if $H = 2$, but weaker if $H > 2$. It is equal in that no equilibria are paradoxical with trade levels below the threshold. It is different in that not all equilibria with trade levels beyond the threshold need be paradoxical, but there exists a sequence of paradoxical equilibria with trade levels converging from above to the threshold.

Curiously, the definition of this threshold that rescues Samuelson's equivalence hinges on Samuelson's very consumer theory: the substitution effect is symmetric and negative semidefinite. Even more, Samuelson's equivalence first appears in the same place as his consumer theory, a footnote in Samuelson (1947).

2 Model

Countries $h = 1, ..., H$ consume commodities $c = 1, ..., C$, C being the unit of account, in terms of which all value is quoted. Markets assign prices $p \in P \equiv R_{++}^{(C-1)}$ to commodities $c < C$, and incomes $w \in R_{++}^H$ to all countries.[4] The set of **budget variables** is

$$b \equiv (p, w) \in B \equiv P \times R_{++}^H$$

and commodity **demands** $x^h : B \to R_{++}^C$ depend on own income only, $x^h(p, w) = x^h(p, w')$ if $w^h = w'^h$.

The **price-income equilibria** for total **resources** $r \in R_{++}^C$ are

$$B(r) = \left\{b \in B \mid \Sigma x^h(b) = r\right\}$$

[3] $n(\dot{v}) = \frac{\left(\Sigma_{i \neq h} - \frac{1}{H-1}\right)^2}{1} + \Sigma_{i \neq h}\left(-\frac{1}{H-1}\right)^2 = 1 + (H-1)\frac{1}{(H-1)^2} = \frac{H}{H-1}$

[4] Unity is the price of C, which P omits. The addition to p of the C coordinate with value unity is denoted $\overline{p}$.

In an **economy**, countries' endowments of commodities make up total resources,

$$\Omega(r) = \left\{e \in R_{++}^{C\times H} \mid \Sigma e^h = r\right\}$$

The **equilibria** are

$$E(r) = \{(p,e) \in P \times \Omega(r) \mid (p, e'\overline{p}) \in B(r)\}$$

There is a natural projection $\pi : E(r) \to B(r), \pi(p,e) = (p, e'\overline{p})$ and a $b-$**equilibrium** is one in $\pi^{-1}(b)$.

Demand is neoclassical if there is a utility $u^h : R_+^C \to R$ solving $u^h(x(b)) = \max_{\beta^h(b)} u$ throughout $b \in B$, where $\beta^h(b) = \left\{x \in R_+^C \mid \overline{p}'x = w^h\right\}$. In this case **welfare** is $v(b) = (v^h(b)) = (u^h(x^h(b)))$. The point of separating budget variables from the economy is that welfare is determined by the budget variables, and in turn these are determined by the economy in equilibrium. We assume Debreu's smooth preferences.

3 Welfare impact of reallocation

We think of a smooth path $e(\xi)$ through a given economy $e = e(0)$, and of an **infinitesimal reallocation** as its velocity $\dot{e}$. Suppose the equilibrium (p,e) is regular in that equilibrium prices are locally a smooth function of the economy. Then welfare is $v(b(\xi))$ with $b(\xi) = (p(\xi), e(\xi)'\overline{p}(\xi))$. Thus a reallocation impacts welfare only via the budget variables it implies. By the fundamental theorem of calculus the welfare impact is the integral of $\dot{v} = D_b v \cdot \dot{b}$, which by abuse we call the **welfare impact**. We prefer to quote it not as $\dot{v}^h$, in individual utils, but in the numeraire, as $\dot{v}^{*h} = \frac{\dot{v}^h}{\lambda^h}$, where $\lambda^h = D_{w^h} v^h$ is the marginal utility of the numeraire. Roy's identity gives $D_b v^h$:

Proposition 1 (Envelope) *The welfare impact $\dot{v} \in R^H$ of $\dot{e}$ at a regular equilibrium is*

$$\dot{v}^* = \dot{t} - \underline{z}'\dot{p}$$

where $\dot{t} \equiv \dot{e}'\overline{p}$ is its value, and $z \in R^{C\times H}$ the countries' ***excess demands***.[5]

As we show next, at a regular equilibrium there is a unique **price adjustment** matrix dp, smooth in a neighborhood of it, such that $\dot{p} = dp\dot{t}$. Thus the welfare impact differential is

$$dv^* = I - \underline{z}'dp \tag{2}$$

[5] Throughout, an underscore denotes the omission of the numeraire coordinate C.

This implies that the welfare impact depends on the infinitesimal reallocation only through its value $\dot{t}$, not its identity $\dot{e}$.

Remark 1 *dv^* is an operator $\dot{t} \mapsto \dot{v}^*$ in $1^{\perp} \subset R^H$.*

Indeed, $1'\dot{t} = 1'\dot{e}'\overline{p} = \dot{r}'\overline{p} = 0$ given that aggregate resources are fixed, and $1'dv^*\dot{t} = (1' - \underline{0}'dp)\dot{t} = 0$ given that total excess demand is zero in equilibrium.

To compute dp, we totally differentiate total **nonnumeraire demand**

$$x^{\sigma}(b) \equiv \Sigma \underline{x}^h(b)$$

Write

$$J \equiv D_p x^{\sigma}((p, e'\overline{p}))$$

and suppose a path $(p(\xi), e(\xi))$ of equilibria. Then

$$x^{\sigma}((p, e'\overline{p})) = \underline{r}$$

is an identity. Differentiating it,

$$J\dot{p} + D_w x^{\sigma}\dot{t} = 0$$

An equilibrium is **regular** if J is invertible. By the implicit function theorem and Walras' law, at a regular equilibrium (p, e) equilibrium prices are locally a smooth function of the economy.

Proposition 2 (Price Adjustment) *At a regular equilibrium the Price Adjustment is*[6]

$$dp = -J^{-1} D_w x^{\sigma} \qquad (dp)$$

This implies that a reallocation matters for prices only through its value, not its identity.

Substituting into (2),

$$\boxed{dv^* = I + \underline{z}' J^{-1} D_w x^{\sigma}} \qquad (dv^*)$$

This formula generalizes Dixit (1981) from $C = 2$ and appears in Donsimoni and Polemarchakis (1994). Note, the welfare impact $\dot{v}^*$ of a reallocation equals its value $\dot{t}$ if there is no trade $\underline{z} = 0$ or if all marginal propensities to consume $D_{w^h}\underline{x}^h$ agree. (For then $\dot{t} \in 1^{\perp}$ implies $D_w x^{\sigma}\dot{t} = 0$.)

[6] Since demands depend on own income only, $D_w x^{\sigma} = [D_{w^1}\underline{x}^1 : \ldots : D_{w^H}\underline{x}^H]$.

If demand is neoclassical, then the Slutsky decomposition $D_p\underline{x}^h = S^h\lambda^h - D_{w^h}\underline{x}^h \cdot \underline{x}^{h\prime}$ and the equilibrium incomes $e'\overline{p}$ imply that $D_p\underline{x}^h((p, e'\overline{p})) = S^h\lambda^h - D_{w^h}\underline{x}^h \cdot \underline{z}^{h\prime}$. Adding,

$$\boxed{J = S - D_w x^\sigma \cdot \underline{z}'} \tag{3}$$

Here the sum $S \equiv \Sigma S^h\lambda^h$ is symmetric and negative definite, since each summand $S^h\lambda^h$ is.

4 Definition of threshold

We reinterpret Samuelson's condition for general C, H, in terms of the requisite **trade level** $L \in R$.

Definition 2 (Trade levels for a protagonist: Necessary and Sufficient) *Fix $b \in B(r)$ and the associated $S(b) \in R^{C-1\times C-1}$ in (3). The **norm at** b is defined on R^{C-1} as $\|a\| = \sqrt{a \cdot a}$ from the inner product $a \cdot b = a(-S^{-1})b$.*[7] *At a $b-$equilibrium, the **trade level** is $\|\underline{z}\| \equiv \sqrt{\frac{1}{H}\Sigma\,\|\underline{z}^k\|^2}$. L is $b-$**necessary for** h if every regular $b-$equilibrium with h a protagonist in the transfer paradox has $\|\underline{z}\| \geq L$. L is $b-$**sufficient for** h if for every $\epsilon > 0$ there is a regular $b-$equilibrium with h a protagonist in the transfer paradox and $\|\underline{z}\| \leq L + \epsilon$.*

Whenever L_n is necessary and L_s is sufficient, $L_n \leq L_s$, so there is at most one threshold:

Definition 3 *Call $L^h \in R$ **the** $b-$**threshold for** h **to be a protagonist in the transfer paradox** if it is both $b-$sufficient and necessary for h.*

Definition 4 *Call $L \in R$ **the** $b-$**threshold for the transfer paradox** if $L = \min_h L^h$.*

Remark 2 *As shown in the introduction, Samuelson's result with $C = H = 2$ means that a threshold exists and equals $\frac{1}{\|\nabla\|}$ –for both to be protagonists and for the transfer paradox. To fully generalize this, we need to explicitly compute the inverse of the welfare impact differential.*

[7] Recall, an inner product is the root of a symmetric, positive definite quadratic form, and indeed $-S^{-1}$ is such according to the consumer theory of Samuelson.

5 The inverse of the welfare impact differential dv^*

Remarkably, the inverse of dv^* exists and admits an explicit description!

Theorem 8 (The inverse of the welfare impact differential dv^*) *Suppose the equilibrium is regular, so that dv^* is defined. Then it is invertible, with inverse*

$$\boxed{dv^{*-1} = I - \underline{z}' S^{-1} D_w x^\sigma} \tag{dv^{*-1}}$$

Proof. We use the decomposition $J = S - D_w x^\sigma \cdot \underline{z}'$. By definition, the inverse of dv^*, should it exist, is a solution (necessarily unique) to the equations $dv^* s = I, s dv^* = I$. We show that $I - \underline{z}' S^{-1} D_w x^\sigma$ is such a solution:

$$\begin{aligned}
& dv^* \left(I - \underline{z}' S^{-1} D_w x^\sigma\right) \\
= & \left(I + \underline{z}' J^{-1} D_w x^\sigma\right) \left(I - \underline{z}' S^{-1} D_w x^\sigma\right) \\
= & I - \underline{z}' S^{-1} D_w x^\sigma + \underline{z}' J^{-1} D_w x^\sigma - \underline{z}' J^{-1} \left(D_w x^\sigma \underline{z}'\right) S^{-1} D_w x^\sigma \\
= & I - \underline{z}' S^{-1} D_w x^\sigma + \underline{z}' J^{-1} D_w x^\sigma - \underline{z}' J^{-1} (S - J) S^{-1} D_w x^\sigma \\
= & I - \underline{z}' S^{-1} D_w x^\sigma + \underline{z}' J^{-1} D_w x^\sigma - \underline{z}' \left(J^{-1} - S^{-1}\right) D_w x^\sigma \\
= & I
\end{aligned}$$

Likewise, the equation $\left(I - \underline{z}' S^{-1} D_w x^\sigma\right) dv^* = I$ holds. ■

Remark 3 dv^{-1*} *is an operator* $\dot{t} \mapsto \dot{v}^*$ *in* $1^\perp \subset R^H$.

This follows from remark 1.

5.1 A universal example of the arbitrariness of the welfare impact

Donsimoni and Polemarchakis (1994) in the case of general C, H conclude that given any $\dot{t}, \dot{v} \in 1^\perp$ satisfying $\dot{t}^h, \dot{v}^h \neq 0$ for some h, there exist marginal propensities to consume $D_{w^h} \underline{x}^h$ and net trades for which $\dot{v} = dv^* \dot{t}$. Save for Pareto optimality, the welfare impact of reallocations is arbitrary without knowledge of marginal propensities to consume and of net trades. Here we sharpen this result: the welfare impact is arbitrary without knowledge of net trades, even granting knowledge of the marginal propensities to consume. Both in their construction and in ours, equilibrium prices and incomes are known, but endowments may be nonpositive.

Our construction is explicit.

Theorem 9 (Universal example of arbitrariness) *Fix* $b = (p, w) \in B(r)$, *plus the desired welfare impact* $\dot{v} \in 1^{\perp}$ *and the desired value* $\dot{t} \in 1^{\perp}$ *of the reallocation. Then except if* $\nabla(b) =_{def} D_w \underline{x}^{\sigma} \dot{v} = 0$ *and for finitely many values of* $\dot{t}$, *the economy* $\underline{e} \equiv \underline{x}(b) - \underline{z}$ *with*

$$\underline{z} = \frac{1}{\|\nabla\|^2} \nabla (\dot{t} - \dot{v})'$$

and numeraire endowments set by the budget identity $e'\overline{p} = w$, *defines a regular* $b-$*equilibrium where the de facto welfare impact of* $\dot{t}$ *is* $\dot{v}$, *nonnumeraire excess demand is* $\underline{z}$, *and the trade level* $\|\underline{z}\|$ *is*

$$\frac{\|\dot{t} - \dot{v}\|_2}{\sqrt{H}\,\|\nabla\|} \tag{*}$$

Conversely, any regular $b-$*equilibrium where the welfare impact of* $\dot{t}$ *is* $\dot{v}$ *has trade level* $\|\underline{z}\|$ *at least (*).*

Proof. Nonnumeraire markets do clear: $\underline{z}1 = \frac{-1}{\lambda\|\nabla\|^2} \nabla (0 - 0) = 0$. So does the numeraire market: the numeraire endowments were defined by Walras' law. Suppose a regular $b-$equilibrium. Then $\dot{v}$ is the welfare impact of $\dot{t}$ iff $dv^{*-1}\dot{v} = \dot{t}$, which by theorem 8 means

$$\begin{aligned} \left(I - \underline{z}' S^{-1} D_w x^{\sigma}\right) \dot{v} &= \dot{t} \\ -\underline{z}' S^{-1} \nabla &= \dot{t} - \dot{v} \\ -\underline{z}^{k\prime} S^{-1} \nabla &= \dot{t}^k - \dot{v}^k \end{aligned} \tag{4}$$

The Cauchy-Schwarz inequality implies $\left\|\underline{z}^k\right\| \|\nabla\| \geq \underline{z}^k \cdot \nabla = -\underline{z}^{k\prime} S^{-1} \nabla$ hence

$$\left\|\underline{z}^k\right\| \geq \frac{\dot{t}^k - \dot{v}^k}{\|\nabla\|}$$

with equality only if $\underline{z}^k = \nabla \alpha^k$ for some scalar α^k. Applying definition 2 of $\|\underline{z}\|$,

$$\|\underline{z}\| \geq \sqrt{\frac{1}{H} \Sigma \left(\frac{\dot{t}^k - \dot{v}^k}{\|\nabla\|}\right)^2} = \frac{\|\dot{t} - \dot{v}\|_2}{\sqrt{H}\,\|\nabla\|}$$

To find the $\alpha = (\alpha^k)$ achieving equality, substitute $\underline{z}' \equiv \alpha \nabla'$ in (4) to get

$$\alpha = \frac{-1}{\nabla' S^{-1} \nabla}(\dot{t} - \dot{v}) = \frac{1}{\|\nabla\|^2}(\dot{t} - \dot{v})$$

Thus $dv^{*-1}\dot{v} = \dot{t}$ holds with

$$\underline{z}' \equiv \frac{1}{\|\nabla\|^2}(\dot{t} - \dot{v})\nabla'$$

provided this $\dot{t}$ makes the equilibrium regular, i.e., $\left|J(\dot{t})\right|$ invertible:

Now $\left|J(\dot{t})\right|$ is polynomial in $J(\dot{t})$, which is linear in $\dot{t}$ (writing $\Delta = D_w x^\sigma \dot{t}$) :

$$\begin{aligned} J(\dot{t}) &= S - D_w x^\sigma \cdot \underline{z}' \\ &= S - \frac{1}{\|\nabla\|^2}(D_w x^\sigma \dot{t} - \nabla)\nabla' \end{aligned} \tag{5}$$

So $\left|J(\dot{t})\right|$ is polynomial in $\dot{t}$, hence zero for all but finitely many values–unless it is the zero polynomial, which the choice $\dot{t} = \dot{v}$ rules out: $J(\dot{v}) = S$ is negative definite, invertible, making $|J(\dot{v})|$ nonzero. ■

5.2 A universal example of the transfer paradox

For each price-income equilibrium, we construct a compatible equilibrium with the transfer paradox.

Corollary 1 (Universal example of the transfer paradox) *Fix $b = (p, w) \in B(r)$ with $\nabla(b) \equiv D_{w^h}\underline{x}^h - D_{w^i}\underline{x}^i \neq 0$. Then for all $\lambda > 0$ but for finitely many values, the λ-donation from h to $i, \dot{t} = \lambda(1^i - 1^h)$, benefits h and hurts i and fixes all others' welfare, $\dot{v}^* = 1^h - 1^i$, at the regular $b-$equilibrium defined by the economy $\underline{e} \equiv \underline{x}(b) - \underline{z}$ with excess demands*

$$\underline{z} = \frac{1+\lambda}{\|\nabla\|^2}\nabla(1^i - 1^h)'$$

and numeraire endowments set by the budget identity $e'\overline{p} = w$

Proof. This follows from theorem 9 since $\dot{t} - \dot{v}^* = -\lambda\dot{v}^* - \dot{v}^* = (1+\lambda)(1^i - 1^h)$. ■

Remark 4 (Sufficient level of trade) *Fix $b \in B(r)$ where all marginal propensities to consume $D_{w^h}\underline{x}^h$ are distinct. Then $\frac{\sqrt{2}}{\sqrt{H}\|\nabla\|}$ is a $b-$sufficient trade level for the transfer paradox.*

Proof. In example 1 $\|\underline{z}\| = \frac{\|\lambda\dot{t}-\dot{v}\|_2}{\sqrt{H}\|\nabla\|}$ and $\lambda\dot{t} - \dot{v} = (1+\lambda)(1^i - 1^h)$, so $\|\underline{z}\| = \frac{\sqrt{2}(1+\lambda)}{\sqrt{H}\|\nabla\|}$.Let $\lambda \searrow 0$. ■

Safra (1983) is a predecessor, concluding nonconstructively that ∞ is $b-$sufficient. Note, with $H = 2$ this says that $\frac{1}{\|\nabla\|}$ is sufficient, giving half of Samuelson's result. In this example everyone's welfare is fixed other than the donor and the recipient's; in contrast, with $H > 2$ there are paradoxical equilibria with even less trade, where the donor affects everyone's welfare. This is not the threshold with $H > 2$.

6 The threshold for the transfer paradox

Theorem 9 states that the trade level at any regular equilibrium where $\dot{v}$ is the welfare impact of $\dot{t}$ is at least $\|\underline{z}\| \geq \frac{\|\dot{t}-\dot{v}\|_2}{\sqrt{H}\|D_w\underline{x}^\sigma\dot{v}\|}$, with equality achieved. This suggests that the threshold trade level for the transfer paradox is the infimum of $\frac{\|\dot{t}-\dot{v}\|_2}{\sqrt{H}\|D_w\underline{x}^\sigma\dot{v}\|}$ "subject to the transfer paradox." To formalize this we consider a problem for each possible protagonist h ::

$$T^h(b) =_{def} \inf_{\dot{t},\dot{v}} \frac{\|\dot{t}-\dot{v}\|_2}{\sqrt{H}\,\|D_w\underline{x}^\sigma\dot{v}\|} \qquad \text{subject to } \dot{t}^h \leq 0, \dot{v}^h = 1, \dot{t}^{-h} \geq 0, 1'\dot{t} = 0 = 1'\dot{v} \qquad (P^h)$$

The constraints $1'\dot{t} = 0 = 1'\dot{v}$ reflect remark 1 on the welfare impact dv^*. The constraints $\dot{t}^h \leq 0, \dot{v}^h = 1, \dot{t}^{-h} \geq 0$ state that h is a protagonist. There is no loss of generality in setting $\dot{v}^h$ to unity instead of some positive scalar, because the objective and the other constraints and the equation $\dot{t} = dv^{-1}\dot{v}$ are invariant to rescalings of $(\dot{t}, \dot{v})$. The problem is defined only if $D_w\underline{x}^\sigma\dot{v} \neq 0$ for some $0 = 1'\dot{v}$, which is equivalent to

Assumption 1 *Not all* ***marginal propensities to consume*** $m^i = D_{w^i}\underline{x}^i$ *are equal* $m^1 = ... = m^H$.

Theorem 10 (Protagonist's threshold) *Fix* $b \in B(r)$ *and assumption 1. Then the threshold trade level for* h *to be a protagonist in the transfer paradox at regular equilibria is the value* $T^h(b)$ *of problem* (P^h).

Corollary 2 (Threshold for paradox) *Fix* $b \in B(r)$ *and assumption 1. Then the threshold for the transfer paradox is* $\min_h T^h(b)$.

This is true by definition 4.

Proof. The last sentence of theorem 9 states that the trade level at any regular equilibrium where $\dot{v}$ is the welfare impact of $\dot{t}$ is at least $\frac{\|\dot{t}-\dot{v}\|_2}{\sqrt{H}\|D_w \underline{x}^\sigma \dot{v}\|}$, which is the objective of problem (P^h), whose constraints state in addition that h is a protagonist. Therefore the value of problem (P^h) is $b-$necessary for h.

Conversely, we show the value of problem (P^h) is $b-$sufficient for h. We want a sequence of regular equilibria where h is a protagonist and trade levels converge to $T^h(b)$. Let $\dot{t}_n, \dot{v}_n$ be a feasible sequence making $\frac{\|\dot{t}_n-\dot{v}_n\|_2}{\sqrt{H}\|D_w \underline{x}^\sigma \dot{v}_n\|}$ converge to the value of problem (P^h). We verify the hypothesis of theorem 9. Clearly, for all large enough n, $\nabla_n(b) =_{def} D_w \underline{x}^\sigma \dot{v}_n \neq 0$, else convergence to the finite infimum would fail. If the $\dot{t}_n$ are not all distinct, then by continuity of the objective we can perturb them so that they are and still convergence obtains. Because they are all distinct, for all large enough n, $\dot{t}_n$ is none of the finitely many $\dot{t}'s$ appearing in the hypothesis. Thus theorem 9 applies to yield a sequence (b, e_n) of regular $b-$equilibria where $\dot{v}_n$ is the welfare impact of $\dot{t}_n$ and the trade level is exactly $\frac{\|\dot{t}_n-\dot{v}_n\|_2}{\sqrt{H}\|D_w \underline{x}^\sigma \dot{v}_n\|}$. ■

We make $T^h(b)$ more explicit by considering the auxiliary $\dot{v}$-problem

$$n(\dot{v}) =_{def} \inf_{\dot{t}} \left\|\dot{t} - \dot{v}\right\|_2 \qquad \text{subject to } \dot{t}^h \leq 0, \dot{t}^{-h} \geq 0, 1'\dot{t} = 0 \tag{n}$$

The value of this problem is uniquely achieved. Uniqueness owes to the strict convexity of the objective and the convexity of the feasible set. Existence of a minimizer owes to the continuity of the objective and to the fact, which we show next, that the closed feasible set can be intersected with a compact ball without affecting the problem's value. Indeed, $\dot{t} = 0$ is feasible and makes $\left\|\dot{t} - \dot{v}\right\|_2 = \|\dot{v}\|_2$, so the infimum is necessarily the limit of some sequence of $\dot{t}'s$ inside the compact ball $\left\|\dot{t} - \dot{v}\right\|_2 \leq \|\dot{v}\|_2$. Let $\dot{t}(\dot{v})$ be the unique minimizer.

Thus if $\dot{t}_n, \dot{v}_n$ is a feasible sequence making $\frac{\|\dot{t}_n-\dot{v}_n\|_2}{\sqrt{H}\|D_w \underline{x}^\sigma \dot{v}_n\|}$ converging to $T^h(b)$, so does $\frac{\|\dot{t}(\dot{v}_n)-\dot{v}_n\|_2}{\sqrt{H}\|D_w \underline{x}^\sigma \dot{v}_n\|} = \frac{n(\dot{v})}{\sqrt{H}\|D_w \underline{x}^\sigma \dot{v}_n\|}$ because for each n the latter is no larger than the former but still at least $T^h(b)$. We conclude

$$\boxed{T^h(b) = \inf_{\dot{v}} \frac{n(\dot{v})}{\sqrt{H}\|D_w \underline{x}^\sigma \dot{v}\|} \qquad \text{subject to } \dot{v}^h = 1, 0 = 1'\dot{v}} \tag{P^h}$$

We now report $n(\dot{v})$.

Lemma 1 (Best donation given welfare impact) *Fix* $\dot{v}^h = 1, 0 = 1'\dot{v}$. *Problem* n*'s value is*

$$n(\dot{v}) = \frac{\left(\Sigma_S \dot{v}^i + 1\right)^2}{|S| + 1} + \Sigma_{\backslash S} \dot{v}^{i2} \tag{6}$$

where, on ordering $\dot{v}^{-h} : \dot{v}^{i_1} \geq ... \geq \dot{v}^{i_{H-1}}, S = \{i_1, ..., i_k\}$ *for the largest* n *such that*

$$\text{if } \ i \in S \qquad \dot{v}^i > \frac{\Sigma_S \dot{v}^i + 1}{|S| + 1}$$

That is, $S \subset H - h$ *is the best off: those better off than the average of those even better off joined by* h*.*

Proof. See appendix. ■

For example, if $\dot{v}^{i \neq h} = -\frac{1}{H-1}$ then $S = \{1, ..., H\} \backslash \{h\}$ and $n(\dot{v})^2 = \frac{1}{1} + \Sigma_{i \neq h} \left(-\frac{1}{H-1}\right)^2 = \frac{H}{H-1}$.

Corollary 3 (Protagonist's threshold bounded above) *An explicit upper bound is*

$$T^h(b) \leq \frac{1}{\sqrt{H-1} \left\| \nabla^h \right\|}$$

$\nabla^h =_{def} m^h - \frac{1}{H-1} \Sigma_{i \neq h} m^i$ *being the difference from the mean of all others' marginal propensities to consume.*

Proof. We know $\dot{v}^{i \neq h} = -\frac{1}{H-1}$ gives $n(\dot{v})^2 = \frac{H}{H-1}$, and clearly $D_w \underline{x}^\sigma \dot{v} = \nabla^h$, so

$$T^h \leq \frac{\sqrt{\frac{H}{H-1}}}{\sqrt{H} \left\| D_w \underline{x}^\sigma \dot{v} \right\|} = \frac{1}{\sqrt{H-1} \left\| \nabla^h \right\|}$$

■

Corollary 4 (Appearance of protagonist) *Fix* $b \in B(r)$ *and assumption 1. Then* h *is a protagonist in the transfer paradox at some equilibrium with any trade level above* $\frac{1}{\sqrt{H-1} \left\| \nabla^h \right\|}$.

Corollary 5 (Threshold with multiple goods) *Suppose* $H = 2$*. Fix* $b \in B(r)$ *and assumption 1. Then the threshold trade level* $T^h(b)$ *for* h *to be a protagonist in the transfer paradox is exactly* $\frac{1}{\left\| \nabla^h \right\|}$.

Proof. When $H = 2$, the constraint set $\dot{v}^h = 1, 0 = 1'\dot{v}$ in (**??**) is a singleton, namely $\dot{v}^{i \neq h} = -\frac{1}{H-1}$, so the upper bound is the infimum. ■

Samuelson (1947) is the special case $H = 2 = C$ of this.

Remark 5 *It is hard to make the infimum more explicit, since the objective is the ratio of two nonconcave functions and the constraint set not compact with* $H > 2$*.*

7 Appendix

7.1 Proof of Envelope proposition

By Roy's identity with $\lambda^h = D_{w^h} v^h$,

$$dv^h = \lambda^h(-\underline{x}^h dp + dw^h)$$

In equilibrium $w^h = e^{h\prime}\overline{p}$ so

$$dw^h = \underline{e}^{h\prime} dp + \overline{p}' de^h$$

Letting $dt^h = \overline{p}' de^h$ and substituting,

$$dv^{*h} \equiv \frac{dv^h}{\lambda^h} = -\underline{x}^h dp + \underline{e}^{h\prime} dp + dt^h = dt^h - \underline{z}^{h\prime} dp$$

7.2 Minimizing $\|t - v\|_2^2$

Fix $\dot{v}^h = s, 0 = 1'\dot{v}$.

$$\min \left\|\dot{t} - \dot{v}\right\|_2^2 \qquad \text{subject to } \dot{t}^h \leq 0, \dot{t}^{-h} \geq 0, 1'\dot{t} = 0 \tag{7}$$

Using the constraints,

$$\begin{aligned} \left\|\dot{t} - \dot{v}\right\|_2^2 &= \left(\dot{t}^h - \dot{v}^h\right)^2 + \left\|\dot{t}^{-h} - \dot{v}^{-h}\right\|_2^2 \\ &= \left(-1'\dot{t}^{-h} - s\right)^2 + \left\|\dot{t}^{-h} - \dot{v}^{-h}\right\|_2^2 \end{aligned}$$

Write x for $\dot{t}^{-h}, y$ for $\dot{v}^{-h}$, so that

$$\|x - y\|_2^2 = (1'x + s)^2 + (x - y)'(x - y)$$

with constraint $-x \leq 0$.

By Kuhn-Tucker (with constraint qualification holding by linearity of the constraint), $x \geq 0$ solves the problem iff there is a nonnegative multiplier $\mu \geq 0$ satisfying complementary slackness such that x minimizes L,

$$L = (1'x + s)^2 + (x - y)'(x - y) - 2\mu' x$$

This being a convex function, its minimum in R^{H-1} is achieved at $DL = 0$:

$$DL = 2\left(1'x + s\right)1' + 2(x-y)' - 2\mu' = 0$$

That is,

$$x = y + \mu - 1\left(1'x + s\right)$$

Let $S = \{i \neq h : x^i > 0\}$. By complementary slackness, this says

$$\begin{array}{lll} \text{if } i \in S & \mu^i = 0 \text{ and } x^i = y^i - (1'x + s) > 0 \\ \text{if } i \notin S & x^i = 0 \text{ and } \mu^i = -y^i + (1'x + s) \geq 0 \end{array} \tag{8}$$

The above implies

$$\begin{array}{ll} \text{if } i \in S & y^i > 1'x + s \\ \text{if } i \notin S & 1'x + s \geq y^i \end{array}$$

We compute $1'x + s$ now:

$$\begin{aligned} 1'x + s &= \Sigma_S x^i + \Sigma_{\setminus S} x^i + s \\ &= \Sigma_S \left[y^i - (1'x + s)\right] + 0 + s \\ &= \left(\Sigma_S y^i\right) - |S|\left(1'x + s\right) + s \end{aligned}$$

So

$$1'x + s = \frac{\left(\Sigma_S y^i\right) + s}{|S| + 1}$$

Note, since $s = \dot{v}^h$, the right side is the average welfare in $S + h$, so we denote it $\overline{y}_{S+h}$. Therefore $S = S(y)$ satisfies

$$\text{if } i \in S \qquad y^i > \overline{y}_{S+h} \tag{10}$$

$$\text{if } i \notin S \qquad \overline{y}_{S+h} \geq y^i \tag{11}$$

Lemma 2 (identification of S) *Order $\dot{v}^{-h} : \dot{v}^{i_1} \geq ... \geq \dot{v}^{i_{H-1}}$. The above $S = \{i_1, ..., i_n\} \subset \{1, ..., H\}\backslash\{h\}$ for the largest n such that (10). Further, this description is independent of how ties in v^{-h} are ordered.*

Proof. Since $S \subset \{1,...,H\}\backslash\{h\}$ must satisfy both (10), (11) it is clear that it consists of the indices corresponding to the $|S|$ largest elements in v^{-h}. To see it is the largest, it suffices to show $S_+ = S + i_{n+1}$ violates (10):

$$y^{i_{n+1}} \leq_{i_{n+1} \notin S} \overline{y}_{S+h} \Rightarrow y^{i_{n+1}} \leq \overline{y}_{S+i_{n+1}+h}$$

To see this description is independent of how ties in v^{-h} are ordered, it suffices to show S is closed under ties. That is, if $\dot{v}^{i_n} = \dot{v}^{i_{n+1}}$ and $i_n \in S$, we want $i_{n+1} \in S$. This is immediate, because i_{n+1} satisfies (10) since $i_n \in S$ and $\dot{v}^{i_n} = \dot{v}^{i_{n+1}}$, and S as described is *largest* with respect to (10). ■

We make a few observations about $\overline{y}_{S+h}$. It is monotonically decreasing in S. Also, $\overline{y}_S > s$ if $S \neq \emptyset$, and $S = \emptyset$ iff $x = 0$.

Having found $S = S(y)$, the candidate minimizer is described uniquely by substituting (**??**) in (8):

$$\begin{aligned} &\text{if } i \in S \quad x^i = y^i - \overline{y}_{S+h} > 0 \\ &\text{if } i \notin S \qquad\quad x^i = 0 \end{aligned}$$

Then the value $n(\dot{v})^2$ of the problem is given by S :

$$\begin{aligned} \|x-y\|_2^2 &= (1'x+s)^2 + \Sigma_S\left(x^i - y^i\right)^2 + \Sigma_{\backslash S}\left(x^i - y^i\right)^2 \\ &= \left(\overline{y}_{S+h}\right)^2 + \Sigma_S\left(\overline{y}_{S+h}\right)^2 + \Sigma_{\backslash S} y^{i2} \\ &= (|S|+1)\left(\frac{\Sigma_{S+h} y^i}{|S|+1}\right)^2 + \Sigma_{\backslash S} y^{i2} \\ &= \frac{\left(\Sigma_{S+h} y^i\right)^2}{|S|+1} + \Sigma_{\backslash S} y^{i2} \end{aligned}$$

Remark 6 *Property (10) is preserved by reduction. That is, whenever* $T = \{i_1,...,i_{k+1}\}$ *satisfies (10), so does* $T - i_{k+1} = \{i_1,...,i_k\}$.

Too see why, it suffices that $y^{i_k} > \overline{y}_{T-i_{k+1}+h}$, which follows from rewriting $y^{i_{k+1}} > \overline{y}_{T+h}$ as $y^{i_{k+1}} > \overline{y}_{T-i_{k+1}+h}$ and recalling $y^{i_k} \geq y^{i_{k+1}}$.

It follows that $S = \emptyset$ iff $\dot{v}^{i_1} \leq s$ iff $\dot{v}^{-h} \leq 1s$, and then $n(\dot{v})^2 = \|y\|_2^2$.

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