Theory of Demand in Incomplete Markets

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Abstract

We develop the theory of demand for commodities and assets facing incompletely insurable uncertainty. First, a Slutsky matrix decomposes into substitution and income effects the derivative of demand with respect to prices and yield structure. Next, we identify the Slutsky matrix's properties.

The Slutsky matrix can be perturbed arbitrarily, subject only to preserving these properties, by perturbing the underlying utility's Hessian, while fixing point demand and marginal utility. The key result identifies these Slutsky perturbations. For arguing genericity, it is an alternative to Citanna, Kajii and Villanacci's (1998) first-order conditions approach.

The latter results extend to incomplete markets Geanakoplos and Polemarchakis (1980), who introduced Slutsky perturbations.

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1 Introduction

We develop the theory of demand for commodities and assets facing incompletely insurable uncertainty, given commodity prices, arbitrage-free yield structures, and contingent incomes.

First, a Slutsky matrix decomposes into substitution and income effects the derivative of demand with respect to prices and yield structure, extending Fischer (1972) to multiple commodities.

Next, we identify the Slutsky matrix's properties.

The Slutsky matrix can be perturbed arbitrarily, subject only to preserving these properties, by perturbing the underlying utility's Hessian, while fixing point demand and marginal utility.

The key result identifies these Slutsky perturbations, via linear constraints defined by prices and the yield structure (theorem 2). We also spell out the identification after omission of markets due to Walras' law and price normalization.

The latter two results extend to incomplete markets Geanakoplos and Polemarchakis (1980), who introduced Slutsky perturbations.

Finally, we include an algorithm that speeds up the computation of Slutsky matrices.

For arguing genericity, Slutsky perturbations are an alternative to Citanna, Kajii and Villanacci's (1998) first-order conditions approach. Geanakoplos and Polemarchakis (1986) were the first to apply Slutsky perturbations to the study of generic Pareto improvements with incomplete markets. Since they allowed the central planner to decide the agents' asset portfolios, they did not need to go beyond perturbations to the Slutsky matrices of demand in spot markets. To show why weaker interventions may improve welfare, such as anonymous taxes and changes in asset payoffs, it became necessary to take into account how agents' portfolio adjustments cause a further price adjustment. Naturally, this required perturbing demand in asset markets as well, not just spot markets. The lack of an extension of Slutsky perturbations to incomplete markets remained an obstacle for over a decade[1], until a breakthrough by Citanna, Kajii, and Villanacci (1998), who circumvented it by analyzing the agents' first order conditions. Researchers have extended the theory of generic Pareto improvements with incomplete markets to many policies by applying this first-order

[1] The exception is Elul (1995).

approach; see Cass and Citanna (1998), Citanna, Polemarchakis, and Tirelli (2003), and Bisin et al. (2001).

We extend Slutsky perturbations to incomplete markets, to recover the advantages of the original demand-based approach to generic improvements. First, analyzing generic welfare requires drastically fewer equations when exploiting the envelope formula, instead of the (more numerous) first-order conditions and budget identities generating it. Perturbations are to the objects in the envelope formula; first order conditions and budget identities completely vanish. Genericity arguments need only invoke the identification of Slutsky perturbations, rather than literally perturb the utilities' Hessians–utilities no longer explicitly appear in the paper. Second, computing the welfare impact of interventions requires knowledge by the policymaker of the derivative of aggregate, instead of individual, demand. In the first-order approach, he needs to know the derivative of every individual's demand, i.e. the second derivative of every individual's utility. Third, explanations become more intuitive with the familiar language of demand theory than with the language of submersions.

The paper continues as follows. Section 2 defines demand for commodities and assets in incomplete markets, and lists the basic properties of neoclassical demand. Section 3 defines the Slutsky matrix. Section 4 focuses on a fixed demand, presenting the properties every Slutsky matrix must satisfy. It also decomposes the derivative of demand into income and substitution effects, records the envelope property, and speeds up the computation of the Slutsky matrix by a recursion. Section 5 focuses on generic demand, defining Slutsky perturbations, identifying them by linear constraints, and discussing the implications of Walras' law and price normalization. Section 6 contains the proofs.

2 Demand

The household knows the present state of nature, denoted 0, but is uncertain as to which among $s = 1, ..., S$ nature will reveal in period 1. It consumes commodities $c = 1, ..., C$ in the present and future, and invests in assets $j = 1, ..., J$ in the present only. Markets assign to the household an income $w \in R^{S+1}_{++}$, to commodity c a price $p_{\cdot c} \in R^{S+1}_{++}$, and to asset j a yield $W^j \in R^{S+1}$. We call $(p_c)_1^C = p = (p_s)$ the spot prices and $(W^j)_1^J = W = (W_s)$ the yield structure. The set of *budget* variables

$$b \equiv (p, W, w) \in B \equiv R_{++}^{C^*} \times R^{J\times S+1} \times R_{++}^{S+1}$$

has some nonempty, open $B' \subset B$ as a distinguished subset, $C^* = C(S+1)$.

Demand for commodities and assets is a function $d = (x, y) : B' \to R_+^{C^*} \times R^J$. It satisfies *Walras' relation* if it makes the following an identity throughout B':

$$p_s' x_s - W_s' y = w_s$$

Alternatively, $[p]' x - W' y = w$ with the useful notation

$$[p] \equiv \begin{bmatrix} \cdot & & & & \\ & p_{s-1} & & 0 & \\ & & p_s & & \\ & 0 & & p_{s+1} & \\ & & & & \cdot \end{bmatrix}_{C^* \times S+1}$$

The interpretation is that, faced with spot prices p and yield structure W, the household modifies its income w to $w + W'y(p, W, w)$ by investing in portfolio $y(p, W, w)$, ultimately financing its state contingent consumption $x(p, W, w)$. Here, a yield structure specifies for each asset j that a buyer is to collect, a seller to deliver, a value W_s^j in state s, and a portfolio $y \in R^J$ specifies how much of each asset to buy ($y_j \geq 0$) or sell ($y_j \leq 0$), hence yielding $W'y$. For a different emphasis, we may view the assets as having present price $q \equiv -W_0$ and future yield $W_{\mathbf{1}} \equiv (W_s)_{s>0}$.

2.1 Neoclassical demand

For $b = (p, W, w) \in B$, the *financeable* bundles are

$$X(b) = \{x \in R_+^{C*} \mid [p]' x - w \in span W'\}$$

Each $x \in X(b)$ implies a *financing* y, $[p]' x - w = W'y$, which is unique if W has linearly independent rows: $y = y(x, b)$. Given a utility function $u : R_+^{C*} \to R$ and $b \in B'$, suppose the problem

$$\max_{x \in X(b)} \quad u(x)$$

has a unique solution $x(b)$. Then *neoclassical demand* at $b \in B'$ is defined to be $d(b) \equiv (x(b), y(x(b), b))$. The following remark hinges on $X(b)$ depending on W' only through its span, and on w only through the component that is orthogonal to $spanW'$.

Remark 1 *Suppose* B' *is* X*-closed:* $b \in B', b \in B, X(b) = X(b') \Rightarrow b' \in B'$.

- **Walras' relation** $[p]' x(p, W, w) - W'y(p, W, w) = w$
- **Revealed yield preference** If $\Delta \in spanW'$ with $w + \Delta \gg 0$, then

$$\begin{array}{ll} \text{i)} & x(p, W, w + \Delta) = x(p, W, w) \\ \text{ii)} & \lambda(p, W, w + \Delta) = \lambda(p, W, w) \end{array}$$

  where $Du'(x(p, W, w)) = [p]\lambda$, should it have a solution, uniquely defines $\lambda(p, W, w) \in R^{S+1}$.

- **Homogeneity** $x(p, W, w) = x(p, \tilde{W}, w)$ if $spanW' = span\tilde{W}'$.

We now recall a subset $B' \subset B$ for which $x(b)$ exists, is unique and interior. Existence obtains if utility is continuous and $X(b)$ compact; it is well known that $X(p, W, w)$ is compact if and only if W is *arbitrage-free*, $W\lambda = 0$ for some $\lambda \in R^{S+1}_{++}$. Uniqueness and interiority obtain if utility is strictly quasiconcave in R^{C*}_{++} and *boundary averse*, $u(x) > u(\tilde{x})$ whenever $x \in R^{C^*}_{++}, \tilde{x} \in \partial R^{C^*}_{+}$, thanks to the convexity of $X(b)$. In sum, neoclassical demand $d = (x, y) : B' \to R^{C*}_{++} \times R^J$ is defined on

$$B' \equiv \{(p, W, w) \in B \mid W \text{ has linearly independent rows, is arbitrage-free}\}$$

given the hypotheses on utility of continuity, strict quasiconcavity in R^{C*}_{++}, and boundary aversion.

3 Slutsky matrices

Assumption 1 ***Debreu's setting*** *for* u*:*

$$\begin{array}{cc} i & u \text{ is continuous, } u \text{ is } C^{r\geq 2} \text{ in } R^{C*}_{++} \\ ii & Du(x) \gg 0 \text{ for } x \gg 0 \\ iii & D^2u(x) \text{ is negative definite on } Du(x)^{\perp} \text{ for } x \gg 0 \\ iv & u(x) > \sup_{\partial R^{C*}_{+}} u \text{ for } x \gg 0 \end{array}$$

Debreu's special setting *means the above strengthened to* "$D^2u(x)$ *is negative definite for* $x \gg 0$."

All three hypotheses assumed to define interior neoclassical demand are present, save for strict quasi-concavity in $R^{C^*}_{++}$, which is implied by the first and third ones in Debreu's setting.

Proposition 1 *Debreu's setting implies* $d = (x, y) : B' \to R^{C*}_{++} \times R^J$ *is* C^{r-1}.

Proof. By definition neoclassical demand is the solution to

$$\max \quad u(x) \;\text{ subject to }\; x \geq 0, [p]'\, x - W'y = w \tag{max}$$

which exists, is unique, and interior. For now suppose $(x, y) \in R^{C*}_{++} \times R^J$ is neoclassical demand at $b \in B'$ iff there is $\lambda \in R^{S+1}_{++}$ (necessarily unique) such that

$$F(x, y, \lambda; b) \equiv \begin{bmatrix} Du' - [p]\,\lambda \\ W\lambda \\ -[p]'\,x + W'y + w \end{bmatrix} = 0 \tag{F}$$

Then (x, y, λ) is a C^{r-1} implicit function of $b \in B'$[2], if $H \equiv D_{x,y,\lambda}F$ is surjective:

$$H = \begin{bmatrix} D^2u & 0 & -[p] \\ 0 & 0 & W \\ -[p]' & W' & 0 \end{bmatrix} = \begin{bmatrix} M & -\rho \\ -\rho' & 0 \end{bmatrix} \qquad \text{where } M \equiv \begin{bmatrix} D^2u & 0 \\ 0 & 0 \end{bmatrix}, \rho \equiv \begin{bmatrix} [p] \\ -W \end{bmatrix} \tag{H}$$

Invertibility follows easily from (F), Debreu's third condition, and $W's$ linearly independent rows.[3]

We verify the above equivalence for $(x, y) \in R^{C*}_{++} \times R^J$. If it solves (max), there is $\lambda \in R^{S+1}_{+}$ such that (F), since the constraint qualification holds with linear constraints. (Independently of concavity!) So $\lambda \in R^{S+1}_{++}$ by Debreu's second condition. Conversely, if (F) with $\lambda \gg 0$ then (x, y) solves (max):

If it did not there would be $\tilde{x}, \tilde{y}$ with $u(\tilde{x}) > u(x)$ (so $\tilde{x} \gg 0$ by boundary aversion and $x \gg 0$) and $[p]'\,\tilde{x} - W'\tilde{y} = w$. By the strict quasiconcavity in R^{C*}_{++} $u(\tilde{x}(t)) > u(x)$ for all $t \in (0, 1]$, where

[2] B' is open in B with the product topology. For suppose W has linearly independent rows and $W\lambda = 0 \in R^J, \lambda \in R^{\mathbf{s}+1}_{++}$. Then some open neighborhood O of W preserves the linear independence and admits, by the implicit function theorem, a smooth function $\lambda : O \to R^{\mathbf{s}+1}_{++}$ solving $\tilde{W}\lambda(\tilde{W}) = 0$.

[3] The letter "H" alludes to the Hessian $D^2_{x,y,\lambda}L$ of $L = u(x) - \lambda'([p]'\,x - W'y - w)$.

$\tilde{x}(t) \equiv t\tilde{x} + (1-t)x$, while still $\tilde{x}(t) \gg 0, [p]'\,\tilde{x}(t) - W'\tilde{y}(t) = w$ with $\tilde{y}(t)$ obviously defined. Writing $\Delta_t \equiv \tilde{x}(t) - x$ in a second order Taylor expansion about $t = 0$,

$$u(\tilde{x}(t)) - u(x) = Du(x)\Delta_t + \frac{1}{2}\Delta_t' D^2 u(x)\Delta_t + o(\|\Delta_t\|^2)$$

The orthogonality $Du(x)\Delta_t = \lambda'[p]'\Delta_t = \lambda' W'(\tilde{y}(t) - y) = 0$ implies $\Delta_t' D^2 u(x)\Delta_t < 0$ by assumption on $D^2 u$, so $u(\tilde{x}(t)) - u(x) < 0$ for all $t \approx 0$, a contradiction. ■

Since H is symmetric, so is H^{-1} :

$$\boxed{H^{-1} = \begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix}, \text{ the } \textbf{Slutsky matrices}} \tag{Slutsky}$$

To keep track, S, c are symmetric of dimensions $C^* + J, S + 1$, and m is $C^* + J \times S + 1$. We view ρ as playing the role of prices, since $\rho = p_0 = p$ if $J = S = 0$ (sole budget constraint).

Having defined Slutsky matrices, we develop neoclassical demand theory in two parts. First is demand for a fixed utility: the Slutsky decomposition, the properties of Slutsky matrices, their computation, and the envelope property. Next is demand for a generic utility: identifying the range of perturbations of Slutsky matrices that arise from perturbations of the Hessian of utility.

4 Fixed neoclassical demand

4.1 Slutsky decomposition

We decompose demand into substitution and income effects, generalizing Gottardi and Hens (1999) to multiple commodities and to including the derivative with respect to asset payoffs. Differentiating the identity $F(d(b), \lambda(b); b) \equiv 0$ with respect to $b = (p, W, w)$,

$$D_{p,W,w}\begin{bmatrix} d \\ \lambda \end{bmatrix} = -H^{-1} \cdot D_{p,W,w}F = -\begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix}\begin{bmatrix} -L & 0 & 0 \\ 0 & \Lambda & 0 \\ -[x]' & \Psi & I \end{bmatrix}$$

where

$$L \equiv \begin{bmatrix} \cdot & & 0 \\ & \lambda_s I_C & \\ 0 & & \cdot \end{bmatrix}_{C^* \times C^*} \qquad \Lambda \equiv [\lambda_0 I_J : \ldots : \lambda_s I_J]_{J \times J(S+1)} \qquad \Psi \equiv \begin{bmatrix} y' & & 0 \\ & \cdot & \\ 0 & & y' \end{bmatrix}_{S+1 \times (S+1)J}$$

In differentiating, we vectorized p, W as

$$\begin{bmatrix} \cdot \\ p_s \\ \cdot \end{bmatrix}_{C(S+1)} \qquad \begin{bmatrix} \cdot \\ W_s \\ \cdot \end{bmatrix}_{J(S+1)}$$

Multiplying this out,

$$\boxed{D_p d = S \begin{bmatrix} L \\ 0 \end{bmatrix} - m[x]' \qquad D_W d = -S \begin{bmatrix} 0 \\ \Lambda \end{bmatrix} + m\Psi \qquad \text{with} \quad D_w d = m} \qquad \text{(decomposition)}$$

so that m is the marginal propensity to demand; also, $D_w \lambda = c$.

Let us interpret the decomposition as substitution and income effects. The second summands are clearly income effects: for $D_p d$, the value of demanding x is $[x]'p$, so a change in price of $\dot{p}$ implies a change in relative income of $-[x]'\dot{p}$, which implies a change in demand of $-m[x]'\dot{p}$; for $D_W d$, the value of demanding y is $W'y = \Psi W$, and likewise. The first summands are substitution effects in this sense. Suppose, given a small change in p, W, that we compensate the household so it can just finance the (x, y) it is demanding. Then its compensated income and demand would be $w(p, W) \equiv [p]'x - W'y, d^{com}(p, W) \equiv d(p, W, w(p, W))$, and the *substitution effects* be $D_p d^{com}, D_W d^{com}$. Computing them,

$$D_p d^{com} = D_p d + D_w d[x]' = S \begin{bmatrix} L \\ 0 \end{bmatrix} \qquad D_W d^{com} = D_W d - D_w d\Psi = -S \begin{bmatrix} 0 \\ \Lambda \end{bmatrix}$$

using the chain rule, (decomposition), and $D_w d = m$. Hence the substitution effects are the first summands.

We paraphrase the decomposition to stress the parallel with the traditional one, and to obtain a convenient version for general equilibrium analysis. It says about $D_q = D_{-W_0}$ that

$$D_q d = S \begin{bmatrix} 0 \\ \lambda_0 I_J \end{bmatrix} - m\Psi_0 \qquad \text{where} \qquad \Psi_0 = \begin{bmatrix} y' \\ 0 \end{bmatrix}_{S+1 \times J}$$

Concatenating the expressions for $D_p d, D_q d$,

$$D_{p,q}d = SL_+ - m[[x]' : \Psi_0] \qquad \text{where} \qquad L_+ \equiv \begin{bmatrix} L & 0 \\ 0 & \lambda_0 I_J \end{bmatrix}$$

That is,

$$\boxed{D_{p,q}d = SL_+ - m\tilde{d}'} \qquad \text{where} \qquad \tilde{d} \equiv [[x]' : \Psi_0]' \tag{GE}$$

The effect on demand of price changes splits into substitution and income effects, the latter being the product of the marginal propensity to demand with demand itself. (The notation "$\tilde{d}$" expresses that $d, [[x]' : \Psi_0]'$ contain the same information, differing only in its display.)

4.2 Envelope property

Indirect utility $v : B' \to R, v(b) \equiv u(x(b))$ is derived from demand; inversely, according to the envelope property, neoclassical demand is derived from indirect utility:

Proposition 2 *Indirect utility is C^{r-1} in Debreu's setting, and its gradient $D_b v$ equals*

$$\boxed{D_p v = -\lambda'[x]' \qquad D_W v = \lambda' \Psi \qquad D_w v = \lambda'}$$

Thus $D_{p_s} v = -\lambda_s x_s', D_{W_s} v = \lambda_s y'$.

Proof. v is C^{r-1} since u, x are, in Debreu's setting. By the chain rule and (F) $D_b v = Du \cdot D_b x = \lambda'[p]' \cdot [D_p x : D_W x : D_w x] = *$. Differentiating Walras' relation $[p]'x = W'y + w$ with respect to

$$p : \quad [p]' D_p x + [x]' = W' D_p y$$
$$W : \quad [p]' D_W x = W' D_W y + \Psi$$
$$w : \quad [p]' D_w x = W' D_w y + I$$

Inserting this and $\lambda' W' = 0$ from (F), $* = \lambda'[-[x] : \Psi : I]$. ■

4.3 The Slutsky list of properties

What properties do the Slutsky matrices H^{-1} have? Convenient notations are

$$m = D_w d = \begin{bmatrix} X \\ Y \end{bmatrix} \quad \text{with} \quad \begin{matrix} X_{C^* \times S+1} \\ Y_{J \times S+1} \end{matrix} \qquad \rho \equiv \begin{bmatrix} [p] \\ -W \end{bmatrix}$$

X, Y are the marginal propensities to demand commodities, assets. (H) suggests defining functions

$$H(M) \equiv \begin{bmatrix} M & -\rho \\ -\rho' & 0 \end{bmatrix} \qquad M(D) \equiv \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \tilde{H}(D) \equiv H(M(D)) \tag{functions}$$

Of course, in (H) we have $H = \tilde{H}(D)\mid_{D=D^2u}$, where $D \in R^{C^* \times C^*}$. The purpose of $\tilde{H}$ is to study how the Slutsky matrices H^{-1} depend on D^2u.

Toward the properties of H^{-1}, we take as given some $\mu \in R^{S+1}$ with $W\mu = 0$. In Debreu's setting, it corresponds to the $\mu = \lambda$ in (F); in Debreu's special setting, to $\mu = 0$. The point is that μ is unrelated to the second derivative $D = D^2u$.

Theorem 1 *If D is negative definite on $([p]\mu)^\perp$ and symmetric, then $\tilde{H}(D)$ is invertible, with inverse*

$$Smc \equiv \begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix}$$

for some S, m, c satisfying the

S	$\rho' S = 0, S$ *is negative definite on* $\rho^\perp$*, symmetric*	
m	$\rho' m = I \qquad XW' = 0$	(Slutsky list)
c	$cW' = 0, c$ *is negative definite on* $\ker X^\perp \cap \mu^\perp$*, symmetric*[4]	

Conversely, if (Slutsky list), then Smc is invertible, with inverse $\tilde{H}(D)$, for some D that is negative definite on $([p]\mu)^\perp$ and symmetric.

We stress that the Slutsky list of properties is exhaustive, in that it recovers all properties of the one object (D^2u) defining the Slutsky matrices $H^{-1} = \tilde{H}(D^2u)^{-1}$, namely (iii) in Debreu's setting. Any other property of Slutsky matrices must follow from this list; for example, $YW' = -I$ from $\rho' m = I, XW' = 0$.

[4] It is easy to show that $\ker X^\perp \cap \mu^\perp = W^\perp \cap \mu^\perp$.

Note that **revealed yield preference** is manifested infinitesimally in $XW' = 0, cW' = 0$, since this results from differentiating (i,ii) in remark 1 with respect to $\Delta \in spanW'$.

4.4 Computation of Slutsky matrices

We can compute Slutsky matrices H^{-1} faster by exploiting the symmetry and sparseness of H.

Express H^{-1} as

$$S = \begin{bmatrix} A & P \\ P' & B \end{bmatrix} \qquad m = \begin{bmatrix} X \\ Y \end{bmatrix} \qquad \Rightarrow \qquad H^{-1} = \begin{bmatrix} A & P & -X \\ P' & B & -Y \\ -X' & -Y' & -c \end{bmatrix} \tag{*}$$

To keep track, the square A, B, c are symmetric of dimensions $C^*, J, S+1$, and $P_{C^* \times J}, X_{C^* \times S+1}, Y_{J \times S+1}$.

Algorithm 1 H^{-1} *exists if* D *is negative definite, and is recursively computable if* D *is symmetric:*

$$\begin{aligned}
& D^{-1} \\
\Phi &\equiv [p]'D^{-1}[p] && \textit{auxiliary matrix} \\
B &= (W\Phi^{-1}W')^{-1} \\
Y &= -BW\Phi^{-1} && c = \Phi^{-1} - \Phi^{-1}W'BW\Phi^{-1} \\
P &= -D^{-1}[p]Y' && X = D^{-1}[p]c \\
A &= (I - X[p]')D^{-1}
\end{aligned}$$

Computing D^{-1} is the most expensive step, which is cheaper with *state separable* utility,

$$u(x) = a(u_0(x_0), ..., u_S(x_S)) \text{ for some } a, (u_s)_s$$

because then D is block diagonal and its inverse too

$$D = \begin{bmatrix} D_0 & & 0 \\ & . & \\ 0 & & D_S \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} D_0^{-1} & & 0 \\ & . & \\ 0 & & D_S^{-1} \end{bmatrix} \qquad \Rightarrow \Phi \text{ is diagonal}$$

The next properties of the Slutsky matrices must, by theorem 1, already follow from the Slutsky list.

Corollary 1 *Fix* D *as above. Then* B *is negative definite,* $\ker P = \ker Y'$ *and* $rank Y = J$, $\ker c = \ker X$, c *is negative definite on* $\left(W\Phi^{-1}\right)^{\perp}$*, has rank* $S+1-J$*, and* $c = c\Phi c$*. Lastly,*

$$\begin{bmatrix} X \\ Y \end{bmatrix} \Phi c = \begin{bmatrix} X \\ 0 \end{bmatrix}$$

meaning that, for marginal income $\dot{w} \in span\Phi c$*, marginal demand* $m\dot{w}$ *is as if asset markets were absent.*

5 Generic neoclassical demand

Equilibrium theory goes beyond demand theory by adding market clearing. Policy theory then adds some policy parameters. By implicitly differentiating market clearing, the envelope property at once gives a formula for the derivative of equilibrium welfare with respect to the policy parameters. Inevitably, this formula contains the Slutsky matrices. So the generic welfare impact of policy is inevitably tied to the generic Slutsky matrices, which we therefore seek to identify.

5.1 Slutsky perturbations

A **perturbation** of the Slutsky matrices is a point $\nabla \in R^{\dim H}$. A **Slutsky perturbation** is one arising from a perturbation of the Hessian of utility. More exactly, recall Slutsky matrices are H^{-1} with $H = \tilde{H}(D)\mid_{D=D^2u}$. By continuity of $\tilde{H}$, $\tilde{H}(D)^{-1}$ exists for all close enough $D \approx D^2u$. If D is symmetric then we call the difference $\nabla = \tilde{H}(D)^{-1} - H^{-1}$ a **Slutsky perturbation**. With this perturbation Slutsky matrices go from H^{-1} to $H^{-1} + \nabla$, and ∇ arises from a perturbation from D^2u to D.

Being symmetric, we write

$$\nabla = \begin{bmatrix} \dot{S} & -\dot{m} \\ -\dot{m}' & -\dot{c} \end{bmatrix}$$

and identify a Slutsky perturbation with a triple $\dot{S}, \dot{m}, \dot{c}$. Our main goal is to identify Slutsky perturbations, without reference to the inversion defining them, in terms of individual constraints on ∇ :

$$\boxed{\begin{array}{ll} \text{on } \dot{S} & \rho'\dot{S} = 0 \text{ and } \dot{S} \text{ is symmetric} \\ \text{on } \dot{m} & \rho'\dot{m} = 0 \text{ and } \dot{X}W' = 0 \\ \text{on } \dot{c} & \dot{c}W' = 0 \text{ and } \dot{c} \text{ is symmetric} \end{array}} \qquad \text{(constraints)}$$

Each of these three independent linear constraints is satisfied by zero, $0 = \dot{S}, \dot{m}, \dot{c}$.

Theorem 2 (Slutsky perturbations identified) *Given u in Debreu's setting and b in B', consider the Slutsky matrices H^{-1}. Every small enough Slutsky perturbation ∇ satisfies (constraints). Conversely, every small enough perturbation ∇ that satisfies (constraints) is Slutsky: $H^{-1} + \nabla$ is the inverse of $\tilde{H}(D)$ for some D that is negative definite on $Du(x(b))^{\perp}$ and symmetric. (Negative definite, given u in Debreu's special setting.)*

Thus Slutsky perturbations are characterized as those that satisfy (constraints), affecting S, m, c simultaneously or separately. A proof of theorem from theorem 1 is trivial , if we appeal to

Lemma 1 (Stability) *Fix a dimension $0 < \dim \leq C^*$ for the Grassmanian $G_{C^*,\dim}$. Suppose continuous functions $D : K \to R^{C^* \times C^*}, S : K \to G_{C^*,\dim}$. If $D(x)$ is negative definite on $\sigma(x)$, then $D(\tilde{x})$ is negative definite on $\sigma(\tilde{x})$, for all nearby $\tilde{x} \approx x$.*

Proof. A matrix D is negative definite on a nonzero subspace σ iff $\max_{z \in \sigma^*} z'Dz < 0$, by compactness of $\sigma^* \equiv \{z \in \sigma \mid z'z = 1\}$. By hypothesis, $\epsilon(x) \equiv \max_{z \in \sigma^*(x)} z'D(x)z < 0$, and by the maximum principle $\epsilon(\cdot)$ is continuous, so $\epsilon(\tilde{x}) < 0$ is an open neighborhood of x. (To apply the principle, note $\sigma^*(\cdot)$ is a continuous, nonempty, compact valued correspondence and $(x, z) \mapsto z'D(x)z$ a continuous function.) ■

Proof. of theorem 2. Clearly $\nabla = \tilde{H}(D)^{-1} - H^{-1}$ satisfies (constraints) if both $H^{-1}, \tilde{H}(D)^{-1}$ satisfy (Slutsky list). This hypothesis in turn holds, by the first part of theorem 1, if D^2u, D are (1) symmetric and (2) negative definite on $([p]\mu)^{\perp}$. These conditions hold for D^2u in Debreu's setting; by definition of a Slutsky perturbation (1) holds for D, and by stability ($\sigma \equiv ([p]\mu)^{\perp}$) so does (2), if it is small enough.

Conversely, suppose ∇ satisfies (constraints). By the first part of theorem 1 H^{-1} satisfies (Slutsky list), so clearly $H^{-1} + \nabla$ satisfies (Slutsky list)–save perhaps for the definiteness statements, which by stability $(\sigma = \rho^{\perp}, \ker(X+\dot{X})^{\perp} \cap \mu^{\perp})$ *still* hold if ∇ is small enough. Therefore $H^{-1}+\nabla$ is invertible, by the converse part of theorem 1, with inverse $\tilde{H}(D)$ for some D that is negative definite on $([p]\mu)^{\perp}$ and symmetric. Thus $\tilde{H}(D)^{-1} = H^{-1}+\nabla$ and ∇ is a Slutsky perturbation. ■

In sum, the range of $\nabla = \tilde{H}(D)^{-1} - H^{-1}$ as D varies symmetrically is the ∇ satisfying (constraints).

5.2 Quadratic perturbations

A **Hessian perturbation** is a symmetric point $\Delta \in R^{C^*}$. We just saw which perturbations $H^{-1} \to H^{-1}+\nabla$ of the Slutsky matrices arise from Hessian perturbations $D^2u \to D = D^2u+\Delta$. Now we recall a well-known "fact" that every such Δ arises from a quadratic perturbation of utility preserving Debreu's setting and $x(b)$ as neoclassical demand.

Definition 1 *A* ***quadratic perturbation of utility at*** $\overline{x} \in R^{C^*}_{++}$ *is a pair* (ω, Δ) *consisting of a* $C^{r\geq 2}$ *weight function* $\omega : R^{C^*}_{+} \to [0,1]$ *that equals unity in a neighborhood of* $\overline{x}$ *and has compact support in* $R^{C^*}_{++}$*, and of a symmetric matrix* Δ *of dimension* C^**. It operates on functions* $R^{C^*}_{+} \to R$ *as* $u \mapsto u_{(\omega,\Delta)}(x) \equiv u(x) + \frac{\omega(x)}{2}(x-\overline{x})'\Delta(x-\overline{x})$*.*

Proposition 3 (Hessian perturbations identified) *If* u *is in Debreu's setting, so is* $u_{(\omega,\Delta t)}$ *for all small enough support(*ω*),t, and then* $\overline{x}$ *is the* $u-$*demand at* b *iff it is the* $u_{(\omega,\Delta t)}-$*demand at* b*. Last but not least,* $D^2u_{(\omega,\Delta t)}(\overline{x}) = D^2u(\overline{x}) + \Delta t$*, so that* $\frac{\partial}{\partial t}|_{t=0}\, D^2u_{(\omega,\Delta t)}(\overline{x}) = \Delta$*.*

Conclusion 1 *Suppose* u *belongs in Debreu's setting and* b *in* B'*, and consider the Slutsky matrices* S, m, c *at* $x(b)$*. Then any small enough perturbation to them that satisfies (constraints), and none other, we can rationalize by a quadratic perturbation* $u_{(\omega,\Delta)}$ *of* u *such that* $u_{(\omega,\Delta)}$ *preserves Debreu's setting and demand* $d_{u_{(\omega,\Delta)}}(b) = d_u(b)$ *at* b*, and has the perturbed* S, m, c *for its Slutsky matrices.*

An argument about generic policy can exchange weighty luggage–quadratic perturbations of utility, first order conditions, and budget identities–for the lighter identification.

5.3 Slutsky perturbations as a transversality tool in equilibrium

Here we describe the range of Slutsky perturbations, identified in theorem 2, once the Slutsky matrices have been trimmed due to Walras' law and to a price normalization.

Walras' identity implies **Walras' law**, that $S+1$ of the market clearing equations are redundant at equilibrium, where household incomes $w^h = [p]'e^h$ arise from endowments $e^h \in R_+^{C^*}$. For asset market clearing $\Sigma y^h = 0$ and Walras' identity $[p]'x^h = w^h + W'y^h$ imply

$$
\begin{aligned}
& [p]'\Sigma\left(x^h - e^h\right) \\
= \; & \Sigma\left([p]'x^h - [p]'e^h\right) \\
= \; & \Sigma\left(w^h + W'y^h - w^h\right) \\
= \; & W'\Sigma y^h \\
= \; & W'0 = 0
\end{aligned}
$$

But $[p]'\Sigma\left(x^h - e^h\right) = 0$ says $p_s'\Sigma\left(x_s^h - e_s^h\right) = 0$ for every state $s \geq 0$. Since $p_{sC} > 0$, it follows that all commodity markets clear, $\Sigma\left(x_s^h - e_s^h\right) = 0$, if in every state all but the last commodity's market clears, $\Sigma\left(\underline{x}_s^h - \underline{e}_s^h\right) = 0$, the underbar denoting omission of the last coordinate.

If the future yields $s > 0$ are **real**, say $W_s^j = p_s' a_s^j$ for some real asset $a_s^j \in R^C$, then no equilibrium allocation is lost by the **price normalization** $p_{sC} = 1$ for every $s \geq 0$. For in equilibrium no-arbitrage state prices $\mu \in R_{++}^S$ exist, $W_0 = -q = -W_{\mathbf{1}}\mu$, making the set $X(b)$ of financeable bundles homogeneous of degree zero in price levels $\gamma \in R_{++}^{S+1} \to \tilde{p}_s = \gamma p_s$, in that $X(b) = X(\tilde{b})$ for $\tilde{b} = \left(\tilde{p}, (\tilde{W}_0, \tilde{W}_{\mathbf{1}}), \tilde{w}\right) = \left(\tilde{p}, (-\tilde{W}_{\mathbf{1}}\mu, \tilde{W}_{\mathbf{1}}), \left([\tilde{p}]'e^h\right)_h\right)$ where $\tilde{W}_{\mathbf{1}}$ has $\tilde{W}_s^j = \tilde{p}_s' a_s^j$. So the normalization $\gamma_s = \frac{1}{p_{sC}}$ leaves neoclassical demand, hence the equilibrium allocation, intact.

The relevance of Walras' law and the price normalization is that equilibrium with real assets is described by as many equations of demand and supply as equilibrating prices, $C^* - (S+1) = (S+1)(C-1) + J$. The differential $(d\underline{p}, dq)$ of equilibrium prices with respect to perturbations of the economy, say, arising from policy, depends on the Jacobian of **aggregate demand** $\sigma' =_{def} \Sigma(\underline{x}^{h\prime}, y^{h\prime})(b(\underline{p}, q))$, according to the implicit function theorem:

$$
\begin{aligned}
0 &= \Delta \begin{bmatrix} d\underline{p} \\ dq \end{bmatrix} + D_{(w^h)}\Sigma \begin{bmatrix} \underline{x}^h \\ y^h \end{bmatrix} D_{\underline{p},q}\left([e^h]'p\right)_h = 0 \\
\begin{bmatrix} d\underline{p} \\ dq \end{bmatrix} &= -\Delta^{-1}\left[D_{(w^h)}\Sigma \begin{bmatrix} \underline{x}^h \\ y^h \end{bmatrix} D_{\underline{p},q}\left([e^h]'p\right)_h \right]
\end{aligned}
$$

provided Δ is invertible. We can compute the Jacobian Δ from the above Slutsky decompositions once we realize how budget variables $b = b(\underline{p}, q) = \left((\underline{p}, 1), (-q, W_{\mathbf{1}}), \left([e^h]'p\right)_h\right)$ depend on $(\underline{p}, q)$. In turn, the Jacobian depends on the Slutsky matrices $\underline{S}^h, \underline{m}^h, c^h$ through the Slutsky decompositions, the dimensions being trimmed by Walras' law and the price normalization: $(S+1)(C-1)+J$ squared, $(S+1)(C-1)+J \times (S+1)$, and $(S+1)\times(S+1)$. The values of the missing coordinates are recoverable from the Slutsky properties $\rho' S = 0, S$ is symmetric, $\rho' m = I$. Thus at an equilibrium where Δ is invertible, $(d\underline{p}, dq)$ exists and depends on the trimmed Slutsky matrices.

Any question about the equilibrium welfare impact of perturbations of the economy involves the differential $(d\underline{p}, dq)$, owing to the envelope property, proposition 2. Often, the answer to such a question is true only generically in the economy's utility parameters. Arguing such an answer involves the transversality theorem (see Mas-Colell I.2.2). Verifying the rank hypothesis of this theorem then involves the derivative of $(d\underline{p}, dq)$ with respect to utility parameters; recalling how $(d\underline{p}, dq)$ depends on the Slutsky matrices, this involves the derivative of the Slutsky matrices S, m, c with respect to utility parameters. If the utility parameters index quadratic perturbations, as in definition 1, then the *range* of this latter derivative (and this is all that is needed to verify the rank hypothesis) is identified by theorem 2 as the Slutsky perturbations. In sum, Slutsky perturbations allows us to argue properties of equilibrium that are generic with respect to utilities, without having to specify which quadratic perturbations of utility lead to the Slutsky perturbations.

Since verifying the transversality hypothesis involves trimmed Slutsky perturbations $\dot{\underline{S}}, \dot{\underline{m}}, \dot{c}$, we describe the correspondence between trimmed Slutsky perturbations $\dot{\underline{S}}, \dot{\underline{m}}, \dot{c} \approx 0$ and Slutsky perturbations. The usefulness of this correspondence is that in proofs we can invoke such $\dot{\underline{S}}, \dot{\underline{m}}, \dot{c}$ and know that they correspond to quadratic perturbations of utility. The point is that all the information lost by trimming is recoverable from the Slutsky properties (constraints).

Proposition 4 *Fix a small enough square matrix* $\underline{\dot{S}} \in R^{[(S+1)(C-1)+J]^2}$. *Then it is the trimmed* $\dot{S}$ *from a Slutsky perturbation iff* $\underline{\dot{S}}$ *is symmetric.*

Proof. Suppose $\dot{S}$ is from a Slutsky perturbation. Then by theorem 2 it satisfies the constraints (constraints), hence is symmetric, so the trimmed submatrix $\underline{\dot{S}}$ is symmetric.

Conversely, suppose $\underline{\dot{S}}$ is symmetric. We now show how the constraints (constraints), namely $\rho'\dot{S} = 0$ and $\dot{S}$ is symmetric, imply a unique $\dot{S}$ from a Slutsky perturbation. First uniqueness. Use symmetry to write

$$\dot{S} = \begin{bmatrix} \dot{A} & \dot{P} \\ \dot{P}' & \dot{B} \end{bmatrix}$$

with dimensions $\dot{A}_{C^* \times C^*}, \dot{B}_{J \times J}$ (symmetric), $\dot{P}_{C^* \times J}$. Note

$$\underline{\dot{S}} = \begin{bmatrix} \underline{\dot{A}} & \underline{\dot{P}} \\ \underline{\dot{P}}' & \dot{B} \end{bmatrix}$$

Then

$$\begin{aligned} \dot{S}\rho &= \begin{bmatrix} \dot{A}[p] - \dot{P}W \\ \dot{P}'[p] - \dot{B}W \end{bmatrix} = 0 \\ &\Leftrightarrow \begin{matrix} \dot{A}[p] = \dot{P}W \\ [p]'\dot{P} = W'\dot{B} \end{matrix} \end{aligned}$$

We see that $[p]'\dot{P} = W'\dot{B}$ determines $S+1$ rows of $\dot{P}$, for

$$[p]'\dot{P} = [p]' \begin{bmatrix} \dot{P}^0 \\ ... \\ \dot{P}^S \end{bmatrix} = \begin{bmatrix} p_0'\dot{P}^0 \\ ... \\ p_S'\dot{P}^S \end{bmatrix}$$

(Superscripts index sets of rows.) That is, the last row of each $\dot{P}^s$ is determined by the equation $p_s'\dot{P}^s = 1_s'W'\dot{B}$, and recoverable from it in conjunction with the other rows, i.e. with $\underline{\dot{P}}$. Further, we see that

$\dot{A}[p] = \dot{P}W$ determines $S+1$ columns of $\dot{A}$, for

$$\dot{A}[p] = \begin{bmatrix} \dot{A}_0 & ... & \dot{A}_S \end{bmatrix} [p] = \begin{bmatrix} \dot{A}_0 p_0 \\ ... \\ \dot{A}_S p_S \end{bmatrix}$$

(Subscripts index sets of columns.)That is, the last column of each $\dot{A}_s$ is determined by the equation $\dot{A}_s p_s = \dot{P}W_s$, and recoverable from it in conjunction with the other columns, i.e. with $\underline{\dot{A}}$. This shows uniqueness, in that $\dot{S} \equiv \dot{A}, \dot{P}, \dot{B}$ is the image of at most one $\underline{\dot{S}} \equiv \underline{\dot{A}}, \underline{\dot{P}}, \dot{B}$. Now we show existence. Given $\underline{\dot{S}} \equiv \underline{\dot{A}}, \underline{\dot{P}}, \dot{B}$, use the equations (1) $p'_s \dot{P}^s = 1'_s W' \dot{B}$ to define $S+1$ extra rows for $\underline{\dot{P}}$ and hence to define $\dot{P}$, and (2) $\underline{\dot{A}}_s p_{s,-C} + a_s p_{sC} = \underline{\dot{P}} W_s$ to define $S+1$ extra columns $a_0, ..., a_S \in R^{C^*-(S+1)}$. Then for each s extend $\underline{\dot{A}}_s^s \in R^{C-1 \times C-1}$ to $\dot{A}_s^s \in R^{C \times C}$ as follows:

$$\dot{A}_s^s = \begin{bmatrix} \underline{\dot{A}}_s^s & a_s \\ a'_s & x_s \end{bmatrix}$$

where $x_s \in R$ is to be determined. This is symmetric because by assumption $\underline{\dot{S}}$ hence $\underline{\dot{A}}$ hence $\underline{\dot{A}}_s^s$ is. Since already $\underline{\dot{A}}_s p_{s,-C} + a_s p_{sC} = \underline{\dot{P}} W_s$, to show $\dot{A}_s p_s = \dot{P} W_s$ it suffices to pick x_s as the unique solution of $a'_s p_{s,-C} + x_s p_{sC} = 1'_{sC} \dot{P} W_s$. This extension of $\underline{\dot{S}} \equiv \underline{\dot{A}}, \underline{\dot{P}}, \dot{B}$ to $\dot{S} \equiv \dot{A}, \dot{P}, \dot{B}$ by construction satisfies the constraints (constraints) and so by theorem 2 is a Slutsky perturbation. ■

Proposition 5 *Fix a small enough matrix* $\underline{\dot{m}} = \begin{bmatrix} \underline{\dot{X}} \\ \dot{Y} \end{bmatrix} \in R^{(S+1)(C-1)+J \times (S+1)}$. *Then it is the trimmed* $\dot{m}$ *from a Slutsky perturbation iff* $\underline{\dot{X}} W' = 0$.

Proof. Suppose $\dot{m}$ is from a Slutsky perturbation. Then by theorem 2 it satisfies the constraints (constraints), in particular $\dot{X} W' = 0$, hence $\underline{\dot{X}} W' = 0$.

Conversely, suppose $\underline{\dot{X}} W' = 0$. We now show how the constraints (constraints), namely $\rho' \dot{m} = 0$ and $\dot{X} W' = 0$, imply a unique $\dot{m}$ from a Slutsky perturbation. Indeed, $0 = \rho' \dot{m} = [p]' \dot{X} - W' \dot{Y}$ states

$$[p]' \dot{X} = [p]' \begin{bmatrix} \dot{X}^0 \\ ... \\ \dot{X}^S \end{bmatrix} = \begin{bmatrix} p'_0 \dot{X}^0 \\ ... \\ p'_S \dot{X}^S \end{bmatrix} = W' \dot{Y}$$

so that the last row of each $\dot{X}^s$ is determined by the equation $p_s'\dot{X}^s = 1_s'W'\dot{Y}$, and recoverable from it in conjunction with the other rows, i.e. with $\underline{\dot{X}}$. This shows uniqueness, in that $\dot{m} \equiv \dot{X}, \dot{Y}$ is the image of at most one $\underline{\dot{m}} \equiv \underline{\dot{X}}, \dot{Y}$. Now we show existence. Given $\underline{\dot{m}} \equiv \underline{\dot{X}}, \dot{Y}$, use the equation $p_s'\dot{X}^s = 1_s'W'\dot{Y}$ to define $S+1$ extra rows for $\underline{\dot{X}}$ and hence to define $\dot{X}$. ■

To conclude, it is possible to argue transversality with Slutsky perturbations, which appear naturally in verifying the rank hypothesis of the transversality theorem whenever the underlying system of equations involves the differential $(d\underline{p}, dq)$, such as in equilibrium welfare analysis. In the case of real assets, the computations facing us involve trimmed Slutsky matrices. The previous two propositions show how to perturb these trimmed Slutsky matrices in a way compatible with unique Slutsky perturbations, hence with local Hessian perturbations of utility.

6 Proofs

6.1 The Slutsky properties

We tie the Slutsky properties to each of three increasingly stringent descriptions of H in (H):

$$H = \begin{bmatrix} M & -\rho \\ -\rho' & 0 \end{bmatrix} \qquad \begin{array}{cc} \text{(I)} & \text{the relationship between } M, \rho \\ \text{(II)} & M = M(D) \text{ for some } D \\ \text{(III)} & D \text{ is negative definite on } ([p]\mu)^{\perp} \end{array}$$[5]

Equivalence 1 *Fix a matrix* ρ.[6] *Suppose*

$$M \text{ is negative definite on } \rho^{\perp} \text{ and symmetric, and } \rho \text{ has no kernel} \tag{I}$$

Then

$$\begin{bmatrix} M & -\rho \\ -\rho' & 0 \end{bmatrix} \tag{1}$$

[5] We will take $\mu = \lambda$ or 0, according as we are in Debreu's setting or Debreu's special setting. (III) says "D is negative definite" if $\mu = 0$.

[6] This does not have to be the particular one in (H).

is invertible, with inverse

$$\begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix} \tag{1'}$$

for some S, m, c *satisfying*

$$\begin{gathered} \rho' S = 0, S \text{ is negative definite on } \rho^{\perp}, \text{ symmetric} \\ \rho' m = I \\ c \text{ is symmetric} \end{gathered} \tag{I'}$$

Conversely, suppose (I'). Then (1') is invertible, with inverse (1), for some M *satisfying (I).*

We use the convenient notation

$$\rho \equiv \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \quad \text{where} \quad \begin{aligned} \rho_1 &= \text{first } C^* \text{ rows of } \rho \\ \rho_2 &= \text{last } J \text{ rows of } \rho \end{aligned}$$

Equivalence 2 *Fix* ρ *with no kernel. Suppose (I) and consider the* m, c *implied by Equivalence 1. If*

$$M(D)'s \text{ last } J \text{ rows and columns are zero} \tag{II}$$

then

$$X\rho_2' = 0 \qquad Y\rho_2' = I \qquad c\rho_2' = 0 \tag{II'}$$

Conversely, suppose (I') and consider the M *implied by Equivalence 1. If (II') then (II) for some* D*. Lastly,* $Y\rho_2' = I$ *is redundant in (II') if* ρ_2 *has linearly independent rows.*

Equivalence 3 *Fix* ρ *with no kernel and* $\rho_2\mu = 0$*. Suppose (I) and consider the* m, c *implied by Equivalence 1; suppose (II). If*

$$D \text{ is negative definite on } (\rho_1\mu)^{\perp} \tag{III}$$

then

$$c \text{ is negative definite on } \ker X^{\perp} \cap \mu^{\perp} \tag{III'}$$

Conversely, suppose (I') and consider the M *implied by Equivalence 1; suppose (II') and consider the solution to* $M = M(D)$ *implied by Equivalence 2. If (III') then (III).*

We now apply the Equivalences to our particular case:

$$M = M(D^2u) \qquad \rho_1 = [p] \qquad \rho_2 = -W \tag{particular}$$

Lemma 2 *Suppose* $M = M(D)$ *with* D *negative definite on* $(\rho_1\mu)^\perp$*, where* $\rho_2\mu = 0$ *and* ρ_2 *has linearly independent rows. Then* M *is negative definite on* $\rho^\perp$*.*

Proof.

$$[a' : b']M \begin{bmatrix} a \\ b \end{bmatrix} = a'Da$$

Suppose $[a' : b'] \in \rho^\perp$, that is, $a'\rho_1 = -b'\rho_2$. *Claim*: $a \in (\rho_1\mu)^\perp$. For $a'\rho_1\mu = -b'\rho_2\mu = 0\rfloor$ So $a'Da < 0$ unless $a = 0 \Rightarrow b'\rho_2 = 0 \Rightarrow b = 0$ given the linearly independent rows. ■

Proof. of theorem 1. By hypothesis and the lemma, $M(D)$ is negative definite on $\rho^\perp$, and ρ has no kernel because $\rho_1 = [p]$ has none. So by Equivalence 1 (I') holds for $S, m, c \approxeq \tilde{H}(D)^{-1}$. Obviously $M(D)$ satisfies (II), so by Equivalence 2 (II') holds, with $-YW' = I$ redundant since $\rho_2 = -W$ has linearly independent rows. Lastly, by Equivalence 3 (III') holds. That is, (Slutsky list) = (I', II', III') holds.

Conversely, if (Slutsky list) = (I', II', III') holds, then we apply the converse part of the Equivalences. By Equivalence (1) (Smc)=(1') is invertible, and the symmetric M appearing in (1) must by Equivalence 2 be $M = M(D)$ for some (necessarily symmetric) D (recall $Y\rho_2' = I$ is redundant), and by Equivalence (3) D must satisfy (III). ■

6.2 Equivalence lemmas

Equivalence 1

Proof. Invertibility: Suppose $[x', y']'$ is in the kernel of (1). Then $Mx - \rho y = 0$ and $\rho' x = 0 \Rightarrow x'Mx = 0$ and $x \in \rho^\perp \Rightarrow x = 0 \Rightarrow \rho y = 0 \Rightarrow y = 0$ since ρ has no kernel, hence (1) is invertible. Since (1) is symmetric, so is its inverse, making S, c symmetric. By definition of inverse,

$$MS + \rho m' = I \quad -Mm + \rho c = 0$$
$$\rho' S = 0 \qquad\qquad \rho' m = I$$

Hence $\rho' S = 0, \rho' m = I$. Turning to $S's$ semidefiniteness, fix γ and consider $\gamma' S \gamma$. Solve

$$\begin{bmatrix} M & -\rho \\ -\rho' & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \qquad \equiv \qquad \begin{matrix} Ma - \rho b = \gamma \\ \rho' a = 0 \end{matrix}$$

which is possible by invertibility. Then

$$\begin{aligned} \gamma' S \gamma &= \\ (a'M - b'\rho') S \gamma &= a' M S \gamma = a'(I - \rho m')\gamma = \\ a'\gamma &= a'(Ma - \rho b) = a' M a \end{aligned}$$

Since $a \in \rho^{\perp}$, by hypothesis on M $\gamma' S \gamma = a' M a < 0$ unless $a = 0 \Rightarrow -\rho b = \gamma$ or $\gamma \in span\rho$. So if $\gamma \in \rho^{\perp}$, then $\gamma = 0$. That is, S is negative definite on $\rho^{\perp}$.

Conversely, suppose (I'). Then the invertibility of (1') is established similarly as above. Since (1') is symmetric, so is its inverse

$$\begin{bmatrix} M & -\alpha \\ -\alpha' & \beta \end{bmatrix}$$

for some symmetric M, β. Claim: $\alpha = \rho, \beta = 0$. By definition of inverse, $MS + \alpha m' = I$ and $\alpha' S + \beta m' = 0$; postmultiplying by ρ and invoking (I') establishes the claim.⌋ Clearly $\rho' m = I$ implies ρ has no kernel. Lastly, M is negative definite on $\rho^{\perp}$: Fix γ and consider $\gamma' M \gamma$. Solve

$$\begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \qquad \equiv \qquad \begin{matrix} Sa - mb = \gamma \\ m'a + cb = 0 \end{matrix}$$

and suppose $\gamma \in \rho^{\perp} \equiv 0 = \rho'(Sa - mb) = -b$. That is, $Sa = \gamma$ and $m'a = 0$. Since $M\gamma = MSa = (I - \rho m')a = a, \gamma' M \gamma = a' S a$. Invoking (I'), we see $\gamma' M \gamma < 0$ unless $a = \rho\alpha$ for some $\alpha \Rightarrow 0 = m'a = m'\rho\alpha = \alpha \Rightarrow a = 0 \Rightarrow \gamma = Sa = 0$. Hence M is negative definite on $\rho^{\perp}$. ■

Equivalence 2

Proof. By hypothesis, write

$$M = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

Focusing on the bottom part of $MS + \rho m' = I$,

$$0 + \rho_2 m' = [0 : I]$$

which says $X\rho_2 = 0, Y\rho_2 = I$. As for $c\rho_2' = 0$: Using $-Mm + \rho c = 0$, $0 = M[0 : I]' = Mm\rho_2' = \rho c\rho_2'$. Since ρ has no kernel, $c\rho_2' = 0$.

Conversely, applying (II') to $-Mm + \rho c = 0$:

$$M \begin{bmatrix} 0 \\ I \end{bmatrix} = Mm\rho_2' = \rho c\rho_2' = 0$$

This and the symmetry of M imply that M is zero off the northwestern corner.

Lastly, $I = \rho' m = \rho_1' X + \rho_2' Y$, so $X\rho_2' = 0$ implies $\rho_2' = 0 + \rho_2' Y\rho_2'$ or $\rho_2'(I - Y\rho_2') = 0$. If ρ_2 has linearly independent rows, $I - Y\rho_2' = 0$. ■

Equivalence 3

Expressing H^{-1} as in (*), by definition of inverse we have:

$$\begin{bmatrix} A & P & -X \\ P' & B & -Y \\ -X' & -Y' & -c \end{bmatrix} \begin{bmatrix} D & 0 & -\rho_1 \\ 0 & 0 & -\rho_2 \\ -\rho_1' & -\rho_2' & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\begin{array}{lll} AD + X\rho_1' = I & X\rho_2' = 0 & A\rho_1 + P\rho_2 = 0 \\ P'D + Y\rho_1' = 0 & Y\rho_2' = I & P'\rho_1 + B\rho_2 = 0 \\ -X'D + c\rho_1' = 0 & c\rho_2' = 0 & X'\rho_1 + Y'\rho_2 = I \end{array}$$

Lemma 3 *Every* $z \in R^{C^*}$ *can be expressed as* $z = Aa + Xb$ *for some* $a \in X^\perp, b \in \ker X^\perp$.

Proof. Set $b = \rho_1' z, a = Dz - \rho_1 cb$. Then $Aa + Xb = A(Dz - \rho_1 cb) + Xb = (AD)z - (A\rho_1)cb + Xb = (I - X\rho_1')z - (-P\rho_2)cb + Xb = z - X(\rho_1' z - b) + P(\rho_2 c)b = z$ since from the equations $\rho_2 c = 0$. Now $a \in X^\perp : X'a = X'(Dz - \rho_1 cb) = c\rho_1' z - (I - Y'\rho_2)cb = c(\rho_1' z - b) = 0$. To get $b \in \ker X^\perp$, redefine $b = (\rho_1' z)^*$ where "*" denotes the orthogonal projection to $\ker X^\perp$, but keep a as before. ■

Lemma 4 *If* $S\rho = 0, S$ *is negative definite on* $\rho^\perp$*, symmetric, then* A *is negative definite on* $X^\perp$.

Proof. Fix a and write

$$\begin{bmatrix} a \\ 0 \end{bmatrix} = x + y \in \rho^{\perp} + span\rho$$

Since $\rho' S = 0, S\rho = 0,$

$$a'Aa = [a' : 0]S \begin{bmatrix} a \\ 0 \end{bmatrix} = x'Sx$$

By hypothesis on S, $a'Aa < 0$ unless $x = 0 \Rightarrow [a' : 0]' = y = \rho\gamma$ some $\gamma \Rightarrow 0 = \rho_2\gamma$. If $a \in X^{\perp}$ then $0 = X'a = X'\rho_1\gamma = (I - Y'\rho_2)\gamma = \gamma \Rightarrow y = 0 \Rightarrow a = 0$. That is, A is negative definite on $X^{\perp}$. ■

Proof. of Equivalence 3. Suppose throughout $\rho_2\mu = 0$. We will appeal twice to the string

$$(X\delta)'(\rho_1\mu) = \delta'(X'\rho_1)\mu = \delta'(I - Y'\rho_2)\mu = \delta'\mu \tag{string}$$

The third row implies $X'DX = c$: $X'DX = c\rho_1'X = c(I - \rho_2'Y) = c$. For every δ, $\delta'c\delta = (X\delta)'D(X\delta) = *$, and $X\delta \in (\rho_1\mu)^{\perp}$ if $\delta \in \mu^{\perp}$ by the (string), so $* < 0$ by hypothesis on D, unless $X\delta = 0$ or $\delta \in \ker X$. If $\delta \in \ker X^{\perp}$ then $\delta = 0$. That is, c is negative definite on $\delta \in \ker X^{\perp} \cap \mu^{\perp}$.

Conversely, fix $z \in R^{C^*}$ and by lemma 3 write $z = Aa + Xb$ with $a \in X^{\perp}, b \in \ker X^{\perp}$. *Claim*: $z'Dz = a'Aa + b'cb$. $Dz = D(Aa + Xb) = (I - \rho_1 X')a + \rho_1 cb = a + \rho_1 cb$. Thus $z'Dz = (a'A + b'X')(a + \rho_1 cb) = a'Aa + a'A\rho_1 cb + b'X'a + b'X'\rho_1 cb = *$. The second term is zero, since the equations say $A\rho_1 c = -P\rho_2 c$ and $\rho_2 c = 0$, and so is the third one, since $X'a = 0$. So $* = a'Aa + b'(I - Y'\rho_2)cb = a'Aa + b'cb.\rfloor$

By lemma 4 and $a \in X^{\perp}, a'Aa < 0$ unless $a = 0$. By hypothesis on c and $b \in \ker X^{\perp}, b'cb < 0$ unless $b = 0$–so long as $b \in \mu^{\perp}$. So to show D is negative definite on $(\rho_1\mu)^{\perp}$, it suffices that $z \in (\rho_1\mu)^{\perp} \Leftrightarrow b \in \mu^{\perp}$. To see this implication, we take the particular $b = (\rho_1'z)^*$ from the proof of lemma 3, and apply (string) twice, with $\delta = \rho_1'z$ and $\tilde{\delta} = b$: $z'\rho_1\mu = \delta'\mu = (X\delta)'(\rho_1\mu) = (Xb)'(\rho_1\mu) = b'\mu$ (the definition of $b \Rightarrow \delta - b \in \ker X \Rightarrow X\delta = Xb$). ■

6.3 Quadratic perturbations

Proof. of proposition 3. Assuming that $u_{(\omega,\Delta)}$ is also in Debreu's setting, the remainder is easy:

Given its interiority, $\overline{x}$ is the $u-$neoclassical demand at (p, W, w) iff (F) holds at $\overline{x}$ and u iff (F) holds at $\overline{x}$ and $u_{(\omega,\Delta)}$ iff $\overline{x}$ is the $u_{(\omega,\Delta)}-$neoclassical demand at (p, W, w). The first and last equivalences hold because $u, u_{(\omega,\Delta)}$ belong in Debreu's setting, and the middle one because $Du(\overline{x}) = Du_{(\omega,\Delta)}(\overline{x})$.

Last but not least, $\omega \equiv 1$ in a neighborhood $x \approx \overline{x}$, where $u_{(\omega,\Delta t)}(x) \equiv u(x) + \frac{1}{2}(x-\overline{x})'\Delta t(x-\overline{x})$ and $D^2 u_{(\omega,\Delta)}(\overline{x}) = D^2 u(\overline{x}) + \frac{1}{2}(\Delta + \Delta')t = D^2 u(\overline{x}) + \Delta t$, the last equality by $\Delta' s$ symmetry.

To verify for $u_{(\omega,\Delta)}$ the four conditions in Debreu's setting, fix ω and write $K \equiv \text{support}(\omega)$.

(i) Obvious.

(ii, iii) These hold with the proviso $x \in R^{C^*}_{++}\backslash K$, since $R^{C^*}_{++}\backslash K$ is open and $u_{(\omega,\Delta)} \mid_{R^{C^*}_{++}\backslash K} = u \mid_{R^{C^*}_{++}\backslash K}$, so we turn to $x \in K$. Both $\sup_K \left\| Du_{(\omega,\Delta)}(x) - Du(\overline{x})\right\|, \sup_K \left\| D^2 u_{(\omega,\Delta)}(x) - D^2 u(\overline{x})\right\|$ are bounded since $Du_{(\omega,\Delta)}(x), D^2 u_{(\omega,\Delta)}(x)$ are continuous in x and K compact, and homogeneous of degree one in t, hence may be chosen smaller than any given $\delta > 0$ by replacing Δ with Δt for all small enough $t > 0$. Choosing δ small enough to make true the implications $\left\| Du_{(\omega,\Delta)}(x) - Du(\overline{x})\right\| < \delta \Rightarrow Du_{(\omega,\Delta)}(x) \gg 0, \left\| D^2 u_{(\omega,\Delta)}(x) - D^2 u(\overline{x})\right\| < \delta \Rightarrow D^2 u_{(\omega,\Delta)}(x)$ is negative definite on $Du(x)^{\perp}$ (appealing to lemma 1 with $D(x) \equiv D^2 u_{(\omega,\Delta)}(x), \sigma(x) \equiv Du(x)^{\perp}$), these conditions also hold at $x \in K$.

(iv) This holds with the proviso $x \in R^{C^*}_{++}\backslash K$ since $u_{(\omega,\Delta)} \mid_{R^{C^*}_{++}\backslash K} = u \mid_{R^{C^*}_{++}\backslash K}$, so we turn to $x \in K$. Write $\epsilon \equiv u(\overline{x}) - \sup_{\partial R^{C^*}_{+}} u$. Condition (iv) states $\epsilon > 0$. Now suppose that K is small enough (possible by $u's$ continuity), in that $|u(x) - u(\overline{x})| < \frac{\epsilon}{2}$ for $x \in K$, and that the rescaling of Δ is too, in that $|(x-\overline{x})'\Delta(x-\overline{x})| < \epsilon$ for $x \in K$. Then for $x \in K$ $u_{(\omega,\Delta)}(x) = u(x) + \frac{\omega(x)}{2}(x-\overline{x})'\Delta(x-\overline{x}) > u(\overline{x}) - \frac{\epsilon}{2} + \frac{\omega(x)}{2}(-\epsilon) \geq u(\overline{x}) - \epsilon = \sup_{\partial R^{C^*}_{+}} u = \sup_{\partial R^{C^*}_{+}} u_{(\omega,\Delta)}$, the latter since $u = u_{(\omega,\Delta)}$ on $\partial R^{C^*}_{+} \rfloor$ ■

6.4 Computation of Slutsky matrices

As in the proof of Equivalence 3, but substituting $\rho_1 = [p], \rho_2 = -W$,

$$\begin{array}{lll} AD + X[p]' = I & XW' = 0 & A[p] - PW = 0 \\ P'D + Y[p]' = 0 & -YW' = I & P'[p] - BW = 0 \\ -X'D + c[p]' = 0 & cW' = 0 & X'[p] - Y'W = I \end{array} \qquad \text{(system)}$$

Proof. of algorithm 1. Invertibility is easy. We deduce formulas for A, B, c, P, X, Y recursively, while imposing $A, B, c's$ symmetry, which we verify last, and refer to equation ij as that appearing in row i, column j of the (system). Note $\Phi \equiv [p]' D^{-1}[p]$ is symmetric, negative definite since $[p]$ has no kernel.

Equation 21 holds iff $P \equiv -D^{-1}[p]Y'$; equation 31 iff $X \equiv D^{-1}[p]c$; equation 11 iff $A \equiv [(I - X[p]')D^{-1}]'$. With this definition of X, 12 holds if 32 holds. So far P, X, A are in terms of Y, c, which we describe in terms of B.

Given this formula for P, 23 holds iff $-Y\Phi - BW = 0$ iff $Y \equiv -BW\Phi^{-1}$. Given the formulas for X, Y, 33 holds iff $c\Phi + \Phi^{-1}W'BW = I$ iff $c \equiv \Phi^{-1} - \Phi^{-1}W'BW\Phi^{-1}$.

Claim: A, P as defined make 13 true. $A[p] - PW = D^{-1}(I - [p]X')[p] + D^{-1}[p]Y'W = *$. Since 33 holds by definition of c, $* = D^{-1}(I - [p]X')[p] + D^{-1}[p](X'[p] - I) = 0$.

Now define $B \equiv (W\Phi^{-1}W')^{-1}$. Note, $W\Phi^{-1}W'$ is invertible if negative definite, which it is since Φ^{-1} is (as the inverse of a negative definite matrix) and W' has no kernel.

Claim: B as defined makes $22, 32$ true. 22: $-YW' = BW\Phi^{-1}W' = I$. 32: $cW' = (\Phi^{-1} - \Phi^{-1}W'BW\Phi^{-1})W' = \Phi^{-1}W'(I - B \cdot W\Phi^{-1}W') = \Phi^{-1}W'(0) = 0$.

These definitions solve the system modulo $A, B, c's$ symmetry, which does exist: B is symmetric indeed, which implies c is, which implies $A = D^{-1}(I - [p]X') = D^{-1} - D^{-1}[p]c[p]'D^{-1}$ is. ■

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