Pareto Improving Financial Innovation in Incomplete Markets

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Abstract

Financial innovation in an existing asset generically supports a Pareto improvement, targeting the income effect.

This result, as several on taxation, owes to one unifying notion: that an intervention generically supports Pareto improvements if the implied price adjustment is sufficiently sensitive to the economy's risk aversion.

Elul (1995) and Cass and Citanna (1998) introduce financial innovation in a new unwanted asset, targeting the substitution effect.

Our result requires an initial position of greater asset completeness, but not the addition of a new asset market.

The existence argument relies on recent developments in demand theory with incomplete markets.

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1 Introduction

When asset markets are incomplete, there are almost always many Pareto improving policy interventions, if there are multiple commodities and households. When they are complete, the First Welfare Theorem implies there never exist any. While the Pareto improvements vanish with the completion of asset markets, the process of completion itself can be Pareto worsening, as shown by Hart (1975) in an example and by Elul (1995) and Cass and Citanna (1998) generically.

Focusing on financial innovation policy, we create a framework for proving the existence of Pareto improving financial innovations, for computing them, and for estimating the size of the improvements. We apply the framework to prove the existence of Pareto improving financial innovations targeting the income effect. In contrast, Cass and Citanna's, and Elul's financial innovation policies target the substitution effect. Our result requires an initial position of greater asset completeness than theirs, but not the ad

dition of a new asset market.

The protagonist in Pareto improvements is the price adjustment following an intervention. Its role is to improve on asset insurance by redistributing endowment wealth across states, as anticipated by Stiglitz (1982). The price adjustment is determined by how innovations and prices affect aggregate, not individual, demand.

If financial innovation targeting current incomes is Pareto improving, then it must cause an equilibrium price adjustment, Grossman (1975). Conversely, we prove that if the price adjustment is sufficiently sensitive to risk aversion, then for almost all risk aversions and endowments, Pareto improving financial innovations exist. We show how to verify this sensitivity test with standard demand theory, which Turner (2003a) extends from complete to incomplete markets.

Financial innovation policy targeting only the income effect generically supports a Pareto improvement, because it passes this sensitivity test.

To numerically identify the Pareto improving financial innovations, we give a formula for the welfare impact of financial innovations. It requires information on the individual marginal utilities and net trades, and on the derivative of aggregate, but not individual, demand with respect to innovations and prices.

To bound the rate of Pareto improvement, we define an equilibrium's insurance deficit. Pareto optimality obtains exactly when the insurance deficit is zero. If the financial innovation policy targets only current incomes, then the implied price adjustment determines the best rate, by integration against the covariance of insurance deficit and net trades across agents. The equilibrium's insurance deficit arises from the agents' component of marginal utility for contingent income standing orthogonally to the asset span.

Geanakoplos and Polemarchakis (1986) began the study of generic improvements with incomplete markets, and introduced the idea of quadratically perturbing commodity demands. Since they allowed the central planner to decide the agents' asset demands, they did not need to go beyond perturbing commodity demand. To show why weaker interventions may improve welfare, such as anonymous taxes and financial innovations, it became necessary to take into account how adjustments in agents' asset demands caused a further price adjustment. Naturally, this required perturbing asset demand as well as commodity demand. Missing was an extension, quadratically perturbing both commodity *and* asset demands. This lacuna blocked contributions for over ten years[1], until a breakthrough by Citanna, Kajii, and Villanacci (1998), who analyzed first order conditions instead of Slutsky matrices. Researchers have extended the theory of generic improvements with incomplete markets to many policies by applying this first order approach; on financial innovation, Cass and Citanna (1998), and on taxes, Citanna, Polemarchakis, and Tirelli (2001), Bisin et al. (2001), and Mandler (2003). Recently, Turner (2003a) has supplied the missing extension, and Turner (2004a) used it to unify and extend the literature on Pareto improving taxation, marking a return to the original demand-based approach to generic improvements from the latest first order-based approach.

Our existence result here on financial innovation is based on this recent extension of demand theory with incomplete markets. We believe in the computational and expositional advantages of the original demand-based approach to generic improvements. First, an argument about generic welfare can drastically reduce the number of equations, targeting the envelope formula instead of the first order conditions and budget identities generating it. First order conditions and budget identities completely vanish; perturbations are to the objects in the envelope formula. Second, to compute the welfare impact of interventions, the policy maker needs to know the derivative of aggregate, not individual, demand. In the first order approach, he needs to know the derivative of every individual's demand, i.e. the second derivative of every individual's utility. Third, the economist can express intuitions with the familiar language of demand theory, and avoid the less familiar language of submersions.

We continue as follows. Section 2 presents a model of financial innovation policy. Section 3 has the formula for the welfare impact of financial innovations. Section 4 obtains the generic existence of Pareto improving financial innovations from the sensitivity condition on price adjustment, which it then reinterprets in terms of the Reaction of Demand to Prices and to Policy. Section 5 summarizes the demand theory in incomplete markets necessary to apply the sensitivity test, then section 6 applies it to financial innovations targeting the income effect. Section 7 estimates the rate of Pareto improvement. Section 8 derives the welfare impact formula, and spells out the notation and the parameterization of economies.

2 GEI model

Households $h = 1, ..., H$ know the present state of nature, denoted 0, but are uncertain as to which among $s = 1, ..., S$ nature will reveal in period 1. They consume commodities $c = 1, ..., C$ in the present and future, and invest in assets $j = 1, ..., J$ in the present only. Each state has commodity C as unit of account, in terms of which all value is quoted. Markets assign to household h an income $w^h \in R^{S+1}_{++}$, to commodity $c < C$ a price $p_{\cdot c} \in R^{S+1}_{++}$, to asset j a price $q^j \in R$ and future yield $a^j \in R^S$. We call $(p_{\cdot c})_1^C = p = (p_{s\cdot})$ the spot prices, $q = (q^j)$ the asset prices, $(a^j) = a = (a_s)$ the asset structure, and $w = (w^h)$ the income distribution, $\mathbf{P} \equiv R^{(C-1)(S+1)}_{++} \times R^J$.[2] The set of **budget variables** is

$$b \equiv (P, a, w) \in B \equiv \mathbf{P} \times R^{J \times S} \times R^{(S+1)H}_{++}$$

and has some distinguished nonempty relatively open subset $B' \subset B$.

[1] The sole one is Elul (1995).

[2] The numeraire convention is that unity is the price of $sC, s \geq 0$, which for this reason is omitted from the description of $\mathbf{P}$. The addition of the $sC, s \geq 0$ coordinates, bearing value unity, is denoted $\overline{p}$. We use the notation $P = (p, q) \in \mathbf{P}$.

Demand for commodities and assets $d = (x, y) : B' \to R^{C(S+1)}_{++} \times R^J$ is a function on B'. The demand $d^h = (x^h, y^h)$ of household h depends on own income only, $(x^h, y^h)(P, a, w, t) = (x^h, y^h)(P, a, w', t)$ if $w^h = w'^h$.

An **economy** (a, e, d) consists of an asset structure a, endowments e, and demands d. For each household h, **endowments** specify a certain number $e^h_{sc} > 0$ of each commodity c in each state s, and **demands** specify a demand d^h. Let Ω be the set of (a, e, d).[3]

A list $(P; a, e) \in \mathbf{P} \times \Omega$ is a **GEI** $\leftrightarrow$

$$\begin{array}{ll} \sum(x^h(b) - e^h) = 0 \qquad \sum y^h(b) = 0 \\ \text{with } \; b \equiv (P, a, (w^h_s = e^{h\prime}_s \overline{p}_s)^h_s) \in B' \end{array}$$

We say $(a, e) \in \Omega$ has **equilibrium** $P \in \mathbf{P}$. Under neoclassical assumptions $(a, e) \in \Omega$ has an equilibrium[4].

2.1 Neoclassical demand

Consider the **budget** function $\beta^h : B \times R^{C(S+1)} \times R^J \to R^{S+1}$

$$\beta^h(b, x, y) \equiv (\overline{p}'_s x_s - w^h_s)^S_{s=0} - \begin{bmatrix} -q' \\ a' \end{bmatrix} y$$

Demand $d^h = (x^h, y^h)$ is **neoclassical** if there is a **utility** function $u : R^{C(S+1)}_{+} \to R$ with

$$u(x^h(b)) = \max_{X^h(b)} u \;\; \text{throughout} \;\; B' \qquad X^h(b) \equiv \{x \in R^{C(S+1)}_{+} \mid \beta^h(b, x, y) = 0, \text{ some } \; y \in R^J\}$$

Neoclassical welfare is $v : B' \to R^H, v(b) = (v^h(b)) \equiv (u^h(x^h(b)))$. The **neoclassical domain** is

$$B' = \{(P, a, w) \in B \mid q \in aR^S_{++}, a \;\text{ has linearly independent rows}\}$$

Debreu's smooth preferences imply neoclassical demand exists and is smooth.

The interpretation of X is that the cost of consumption x in excess of income w is financed by some portfolio $y \in R^J$ of assets. A **portfolio** specifies how much of each asset to buy or sell ($y_j \gtrless 0$), and a^j_s how much value in state s an asset j buyer is to collect, a seller to deliver.

3 Welfare impact of financial innovation

Financial innovation in an asset structure a is a smooth path $t = t(\xi)$ in $R^{J\times S}$ through $t(0) = 0$, defining $a(\xi) = a + t(\xi)$ as a new asset structure. We think of *infinitesimal financial innovation* as its initial velocity $\dot{t} = \dot{t}(0)$. Suppose the GEI (P, a, e) is regular in that equilibrium prices are locally a smooth function of the economy, so that financial innovation lifts locally to a unique path $(P(\xi), a + t(\xi), e)$ of nearby GEI. Then welfare is $v(b(\xi))$ with $b(\xi) = (P(\xi), a+t(\xi), (w^h_s = e^{h\prime}_s \overline{p}_s(\xi))^h_s)$. Thus financial innovation impacts welfare only via the budget variables it implies. By the fundamental theorem of calculus the welfare

[3] The appendix spells out the parameterization of demand d.

[4] Geanakoplos and Polemarchakis (1986).

impact is the integral of $D_b v^h \cdot \dot{b}$, which by abuse we call the *welfare impact*. We compute this product in the appendix, using the envelope theorem for $D_b v^h$ and the chain rule for $\dot{b}$, where details of the notation appear.

Proposition 1 (Envelope) *The* ***welfare impact*** $\dot{v} \in R^H$ *of infinitesimal innovation* $\dot{t}$ *at a regular GEI is*

$$\dot{v} = (\lambda)'\dot{m} \qquad \dot{m} = \underbrace{\overline{y}_{\mathbf{1}}\dot{t}}_{PRIVATE} \quad \underbrace{-\underline{\overline{z}}\dot{P}}_{PUBLIC}$$

Here $(\lambda)'$ *collects the households' marginal utilities of income across states, and* $\dot{m}$ *the impact on their incomes, private and public. The private one is the impact* $\overline{y}_{\mathbf{1}}\dot{a}$ *on portfolio payoffs, and the public one is the impact on the value of their excess demands* $\underline{\overline{z}}$ *in all nonnumeraire markets, that implied by the impact* $\dot{P}$ *on prices.*

Policy targeting welfare must account for the equilibrium price adjustment it causes.

At a regular GEI there is a **price adjustment** matrix dP, smooth in a neighborhood of it, such that $\dot{P} = dP\dot{t}$. Thus the welfare impact is a differential $\dot{t} \to \dot{v}$,

$$\boxed{dv = (\lambda)' \left(\overline{y}_{\mathbf{1}} - \underline{\overline{z}} dP\right)} \tag{1}$$

Note $dv = dv(b)$ is a function of the budget variables, since v itself is.

We consider two types of financial policy, perturbing an existing asset in a substitution-free way, and perturbing a new unwanted asset, as in Elul (1995) and Cass and Citanna (1998). Aggregate demand is provoked by the income effect of one policy, and by the substitution effect of the other. In either case, financial innovation is parameterized by a vector subspace $\dot{t} \in T = T(b)$ associated with the equilibrium budget variables b :

$$dv : T(b) \to R^H$$

4 Framework for generic existence of Pareto improving innovation

We prove the generic existence of Pareto improving innovations, stressing the role of changing commodity prices over the role of the particular financial policy. Existence follows directly from a hypothesis on price adjustment. Thus the financial policy is relevant only insofar as it meets the hypothesis on price adjustment. Then we reinterpret this hypothesis on dP in terms of primitives, the Reaction of Demand to Prices and the Reaction of Demand to Policy.

Pareto improving financial innovation exists if there exists a solution to $dv\dot{t} \gg 0$. In turn this exists if $dv \in R^{H \times \dim T(b)}$ has rank H, which in turn forces us to suppose the innovation parameters outnumber household types $\dim T(b) \geq H$. The key idea is that if $dv = (\lambda)'\overline{y}_{\mathbf{1}} - (\lambda)'\underline{\overline{z}}dP$ is rank deficient, then a perturbation of the economy would restore full rank by preserving the first summand but affecting the second one. Namely, if some economy's dP is not appropriate, then almost every nearby economy's dP is.

We have in mind a perturbation of the households' **risk aversion** $(D^2u^h)_h$, which affects nothing but dP in the welfare impact dv. Now, to restore the rank the risk aversion must map into $(\lambda)'\underline{\overline{z}}dP$ richly enough. Since this map keeps $(\lambda)'\underline{\overline{z}}$ fixed, we require that $(\lambda)'\underline{\overline{z}}$ have rank H and that dP be sufficiently sensitive to risk aversion. Cass and Citanna (1998) gift us the first requirement:

Fact 1 (Full Externality of Price Adjustment on Welfare) *Suppose asset incompleteness exceeds household heterogeneity* $S-J \geq H > 1$. *Then generically in endowments every GEI has* $(\lambda_s^h z_{s1}^h)_{s\leq H-1}^{h\leq H}$ *invertible.*

Fact 2 *At a regular active GEI,* dP *is locally a smooth function of risk aversion; the marginal utilities* λ^i *and excess demands* z^i *are locally constant in risk aversion.*

For $k \in R^{(S+1)(C-1)+J}$ we say that a *commodity coordinate* is one of the first $(S+1)(C-1)$.

Definition 1 *At a regular active GEI,* dP *is* k***-Sensitive to risk aversion*** *if for every* $\alpha \in R^{\dim(T)}$ *there is a path of risk aversion that solves* $k'd\dot{P} = \alpha'$.[5] *It is* ***Sensitive to risk aversion*** *if it is* k*-Sensitive to risk aversion for all* k *with a nonzero commodity coordinate.*

Assumption 1 (Generic Sensitivity of dP**)** *If* $H > 1$, *then generically in endowments and utilities, at every GEI* dP *is Sensitive to risk aversion.*

This assumption banishes the particulars of the financial innovation policy, leaving only its imprint on dP. Of course, dP is defined only at regular GEI, so implicitly assumed is that regular GEI are generic in endowments. Lastly, the requirement $\dim T(b) \geq H$ with b arising in equilibrium makes sense only with

Assumption 2 (Innovation has a dimension) *If* $S - J \geq H$, *then there is an integer* $\dim$ *such that generically in utilities, at every GEI the vector subspace* $\dot{t} \in T = T(b)$ *parameterizing financial innovation has dimension* $\dim$. *Call it gen* $\dim$.

Theorem 2 (Logic of Pareto Improvement) *Fix a financial policy and the desired welfare impact* $\dot{v} \in R^H$. *Grant the Generic Sensitivity of* dP *under* $gen\dim, S-J \geq H > 1, C > 1$. *Then generically in utilities and endowments, at every GEI* $\dot{v}$ *is the welfare impact of some* $\dot{t} \in T$. *Hence financial innovation supports a nearby Pareto superior GEI.*

Proof. Fix generic endowments, utilities from the lemma, assumptions, and apply transversality to

$$\begin{array}{ll} 1 & \text{nonnumeraire excess demand equations} \\ 2 & \gamma'(\lambda)'(\overline{y}_{\mathbf{1}} - \underline{\overline{z}}dP) = 0 \\ 3 & \gamma'\gamma - 1 = 0 \end{array}$$

where $dv : T(b) \to R^H$. Suppose endowments and utilities make this transverse to zero and the natural projection is proper. By the transversality theorem, for generic such, the system of $(\dim p + \dim q) + gen\dim + 1$ equations is transverse to zero in the remaining endogenous variables, which number $\dim p + \dim q + \dim \gamma$. By hypothesis $gen\dim \geq H = \dim \gamma$, so for these endowments and utilities the preimage theorem implies that no endogenous variables solve this system–every GEI has dv with rank H.

[5] The appendix spells out a path of risk aversion. Here the dot denotes differentiation with respect to the path's parameter.

This is transverse to zero. As is well known, we can control the first equations by perturbing one household's endowment. For a moment, say that we can control the second equations and preserve the top ones. We then control the third equation and preserve the top two, by scalar multiples of γ. So transversality obtains if our momentary supposition on $\gamma' dv$ holds:

Write $k' \equiv \gamma'(\lambda)'\underline{\overline{z}}$. Differentiating $\gamma' dv$ with respect to the parameter of a path of risk aversion,

$$\alpha' =_{def} \frac{d}{d\xi}\gamma'(\lambda)'\left(\overline{y}_{\mathbf{1}} - \underline{\overline{z}}dP\right) = -\gamma'(\lambda)'\underline{\overline{z}}\frac{d}{d\xi}(dP) = -k'd\dot{P}$$

since λ, z (hence $\overline{y}_{\mathbf{1}}$) are locally constant by fact 2. We want to make α arbitrary, and we can if dP is k-sensitive, which holds by assumption if k has a nonzero commodity coordinate. It has: Full Externality of Price Adjustment on Welfare, $C > 1, \gamma \neq 0$ imply $\gamma'(\lambda)'\underline{\overline{z}}$ is nonzero in the coordinate $m = s1$ for some $s \leq H - 1$.

That the natural projection is proper we omit. (The numeraire asset structure is fixed.) ■

Insofar as generically supporting a Pareto improvement, a financial policy need only imply a sensitive price adjustment, and its particulars are irrelevant.

4.1 Expression for Price Adjustment

Before we can check whether a particular policy meets the Sensitivity of dP to Risk Aversion, we need an expression for dP. We express dP in terms of the Reaction of Demand to Prices and the Reaction of Demand to Policy.

Let an underbar connote the omission of the numeraire in each state, define

$$d : B' \to R_{++}^{(C-1)(S+1)} \times R^J \qquad d = \Sigma \underline{d}^h$$

and the **aggregate demand** of $(a, e, d) \in \Omega$

$$d_{a,e}(p, q) \equiv d(p, q, a, (w_s^h = e_s^{h\prime}\overline{p}_s)_s^h)$$

with domain $\mathbf{P}_{a,e} \equiv \{(p, q) \in \mathbf{P} \mid (p, q, a, (w_s^h = e_s^{h\prime}\overline{p}_s)_s^h) \in B'\}$.[6]

Now define

$$\begin{array}{rl} \nabla \equiv D_{p,q}d_{a,e} & \text{the } \textbf{Reaction of Demand to Prices} \\ \Delta \equiv D_a d_{a,e} & \text{the } \textbf{Reaction of Demand to Policy} \end{array} \tag{2}$$

Suppose a path of GEI $(p(\xi), q(\xi), a + t(\xi), (w_s^h = e_s^{h\prime}\overline{p}_s(\xi))_s^h)$ through a GEI. Then

$$d_{a,e}(P) = \begin{bmatrix} \sum \underline{e}^h \\ 0 \end{bmatrix}$$

is an identity in the path's parameter ξ. Differentiating with respect to it,

$$\nabla \dot{P} + \Delta \dot{t} = 0$$

A GEI is **regular** if ∇ is invertible. By the implicit function theorem, at a regular GEI equilibrium prices P are locally a smooth function of the financial innovation $t(\xi)$.

[6] $\mathbf{P}_{a,e}$ is open, as the preimage by a continuous function of the open B'. Recall the notation $P' = (p', q')$.

Proposition 3 (Price Adjustment) *At a regular GEI the Price Adjustment to infinitesimal financial innovation exists,*

$$dP = -\nabla^{-1}\Delta \qquad (dP)$$

where the Reactions ∇, Δ *are defined in (2).*

4.2 Primitives for the Sensitivity of Price Adjustment to Risk Aversion

Given the Logic of Pareto improvement, we want to check whether a policy meets the Generic Sensitivity of dP. We provide primitives for the Sensitivity of dP, thanks to expression (dP)[7]:

$$d\dot{P} = -\nabla^{-1}\dot{\Delta} + \nabla^{-1}\dot{\nabla}\nabla^{-1}\Delta$$

Recall equation $k'd\dot{P} = \alpha'$ from definition 1. If $\dot{\Delta} = 0$ and $\tilde{k}' \equiv_{def} k'\nabla^{-1}$ then the equation reads $\tilde{k}'\dot{\nabla}\nabla^{-1}\Delta = \alpha'$. If Δ has rank $gen\dim$ then there is a solution β to $\beta'\nabla^{-1}\Delta = \alpha'$ so it suffices to solve $\tilde{k}'\dot{\nabla} = \beta'$. Thus dP is k-Sensitive if (1) Δ has rank $gen\dim$, (2) $\tilde{k}$ is nonzero everywhere, (3) whenever $\tilde{K}$ is nonzero everywhere and $\beta \in R^{(S+1)(C-1)+J}$, there is a path of risk aversion that solves $\dot{\Delta} = 0, \tilde{K}'\dot{\nabla} = \beta'$. (Take $\tilde{k} = \tilde{K}$.) Thus Generic Sensitivity of dP obtains (independently of the $\tilde{k}$ defined) if:

Lemma 1 (Activity) *If* $H > 1$, *generically in endowments every GEI is regular.*[8]

Assumption 3 (Full Reaction of Demand to Policy) *If* $C > 1$, *generically in utilities and endowments, at every GEI* Δ *has rank* $gen\dim$.

Lemma 2 (Mean Externality of Price Adjustment on Welfare is Regular) *Generically in utilities, at every regular GEI, whenever* k *is nonzero in some commodity coordinate,* $\tilde{k}' \equiv k'\nabla^{-1}$ *is nonzero everywhere.*

Assumption 4 (Sufficient Independence of Reactions) *If* $H > 1$, *then generically in endowments and utilities, whenever* $\tilde{k} \in R^{(S+1)(C-1)+J}$ *is nonzero everywhere and* $\beta \in R^{(S+1)(C-1)+J}$, *at every GEI there is a path of risk aversion that solves* $\dot{\Delta} = 0, \tilde{k}'\dot{\nabla} = \beta'$.

These primitives for the Generic Sensitivity of dP and the Logic of Pareto Improvement yield

Theorem 4 (Test for Pareto Improvement) *Fix a financial policy and the desired welfare impact* $\dot{v} \in R^H$. *Say the policy passes the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions under* $gen\dim, S - J \geq H > 1, C > 1$. *Then generically in utilities and endowments, at every GEI* $\dot{v}$ *is the welfare impact of some* $\dot{t} \in T$. *Hence financial innovation supports a nearby Pareto superior GEI.*

Next we illustrate how to check whether a financial policy passes this test via demand theory in incomplete markets, as developed by Turner (2003a). We show that substitution free financial innovation passes this test, and so generically supports Pareto improvement, owing to the unifying logic of a sensitive price adjustment. In contrast, financial innovation in a new unwanted asset never passes this test. At a GEI ∇ will turn out to be independent of the policy, so we will verify the lemma on the Mean for one and all policies.

[7] Applying the chain rule to $JJ^{-1} = I$ gives $\frac{d}{d\xi}J^{-1} = -J^{-1}(\frac{d}{d\xi}J)J^{-1}$.

[8] We do not argue this standard result. For these endowments, both Δ and dP are defined.

5 Summary of demand theory in incomplete markets

We must check whether each policy meets the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions. For this we report the theory of demand in incomplete markets as developed by Turner (2003a). The basic idea is to use decompositions of Δ, ∇ in terms of Slutsky matrices, and then to perturb these Slutsky matrices by perturbing risk aversion, while preserving neoclassical demand at the budget variables under consideration. We stress that this theory is applied to, but independent of, equilibrium.

5.1 Slutsky perturbations

Define $H : R^{C^* \times C^*} \to R^{C^*+J+(S+1) \times C^*+J+(S+1)}$ as

$$H(D) = \begin{bmatrix} D & 0 & -[\overline{p}] \\ 0 & 0 & W \\ -[\overline{p}]' & W' & 0 \end{bmatrix}$$

where $p, W = [-q : a] \in R^{J \times S+1}$ of rank J are given, and $C^* = C(S+1)$. In other notation,

$$H(D) = \begin{bmatrix} M(D) & -\rho \\ -\rho' & 0 \end{bmatrix} \qquad \text{where } M(D) = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \rho = \begin{bmatrix} [\overline{p}] \\ -W \end{bmatrix}$$

In showing the differentiability of demand, the key step is the invertibility of $H(D^2u)$. **Slutsky matrices** are $H(D^2u)^{-1}$. If D is symmetric, so are $H(D), H(D)^{-1}$ when defined. Thus we write

$$H(D)^{-1} = \begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix}$$

where S, c are symmetric of dimensions $C^* + J, S+1$ and $m = (m_x, m_y)$ is $C^* + J \times S + 1$.[9] A **Slutsky perturbation** is of the form $\nabla = H(D)^{-1} - H(D^2u)^{-1}$, for some symmetric $D \approx D^2u$ that is close enough for the inverse to exist. A Slutsky perturbation is a perturbation of Slutsky matrices rationalizable by some perturbation of the Hessian of utility. Being symmetric, we write

$$\nabla = \begin{bmatrix} \dot{S} & -\dot{m} \\ -\dot{m}' & -\dot{c} \end{bmatrix}$$

and view a Slutsky perturbation as a triple $\dot{S}, \dot{m}, \dot{c}$. We identify Slutsky perturbations, without reference to the inversion defining them, in terms of independent linear constraints on ∇ :

$$\boxed{\begin{array}{ll} \text{on } \dot{S} & \rho'\dot{S} = 0 \text{ and } \dot{S} \text{ is symmetric} \\ \text{on } \dot{m} & \rho'\dot{m} = 0 \text{ and } \dot{m}_x W' = 0 \\ \text{on } \dot{c} & \dot{c}W' = 0 \text{ and } \dot{c} \text{ is symmetric} \end{array}} \qquad \text{(constraints)}$$

[9] It turns out that $m = D_w d$.

Theorem 5 (Identification of Slutsky perturbations, Turner 2003a) *Given* u *smooth in Debreu's sense and* b *in* B' *with* $t = 0$*, consider the Slutsky matrices* $H(D^2u)^{-1}$*. Every small enough Slutsky perturbation* ∇ *satisfies (constraints). Conversely, every small enough perturbation* ∇ *that satisfies (constraints) is Slutsky:* $H(D^2u)^{-1} + \nabla$ *is the inverse of* $H(D)$ *for some* D *that is negative definite and symmetric.*

We use only Slutsky perturbations with $\dot{m}, \dot{c} = 0$ by choosing $\dot{S}$ as follows. A matrix $\underline{\dot{S}} \in R^{(C-1)(S+1)+J \times (C-1)(S+1)+J}$ is extendable in a unique way to a matrix $\dot{S} \in R^{C^*+J \times C^*+J}$ satisfying $\rho' \dot{S} = 0$; we call $\dot{S}$ the **extension** of $\underline{\dot{S}}$. It is easy to verify that if $\underline{\dot{S}}$ is symmetric, so is its extension. In sum, any symmetric $\underline{\dot{S}}$ defines a unique Slutsky perturbation with $\dot{m}, \dot{c} = 0$.

Now we turn to decompositions of Δ, ∇ in terms of Slutsky matrices, which in turn make up the inverse the Hessian H matrix.

5.2 Slutsky decomposition of the Reaction to prices

The relevance of Slutsky perturbations is that they allow us to perturb demand functions directly, while preserving their neoclassical nature, without having to think about utility. This is because Slutsky matrices appear in the **decomposition** of demand $D_{p,q}\underline{d}$ at b :[10]

$$D_{p,q}\underline{d}^h = \underline{S}^h L^h_+ - \underline{m}^h \cdot ([\underline{x}^h]' : \overline{y}^h_0) \tag{dec}$$

Here L^h_+ a diagonal matrix displaying the marginal utility of contingent income

$$L^h_+ \equiv \begin{bmatrix} L^h & 0 \\ 0 & \lambda^h_0 I_J \end{bmatrix} \qquad L^h \equiv \begin{bmatrix} \cdot & & 0 \\ & \lambda^h_s I_{C-1} & \\ 0 & & \cdot \end{bmatrix}$$

$m^h = D_{w^h} d^h$, and $([\underline{x}^h]' : \overline{y}^h_0)$ is the transpose of $\underline{d}^h$:[11]

$$[\underline{x}^h]' = \begin{bmatrix} \cdot & 0 & 0 \\ 0 & \underline{x}^{h\prime}_s & 0 \\ 0 & 0 & \cdot \end{bmatrix}_{(S+1)\times(C-1)(S+1)} \qquad \overline{y}^h_0 = \begin{bmatrix} y^{h\prime} \\ 0 \end{bmatrix}_{S+1\times J}$$

Writing $(e^{h\prime}_s \overline{p}_s)_s$ as $[e^h]'\overline{p}$, we have $D_{p,q}[e^h]'\overline{p} = ([\underline{e}^h]' : 0)$, so from (2) we have

$$\nabla = \Sigma D_{p,q}\underline{d}^h + D_{w^h}\underline{d}^h \cdot ([\underline{e}^h]' : 0)$$

Inserting decomposition (dec),

$$\nabla = \Sigma \underline{S}^h L^h_+ - D_{w^h}\underline{d}^h \cdot ([\underline{x}^h - \underline{e}^h]' : \overline{y}^h_0)$$

Writing $\underline{z}^{h\prime} \equiv ([\underline{x}^h - \underline{e}^h]' : \overline{y}^h_0)$ this reads

$$\boxed{\nabla = \Sigma \underline{S}^h L^h_+ - D_{w^h}\underline{d}^h \cdot \underline{z}^{h\prime}} \tag{∇}$$

[10] Gottardi and Hens (1999) have this in the case $C = 1$. They do not address or define Slutsky perturbations.

[11] We view p as one long vector, state by state, and p, q as an even longer one; $(* : \#)$ denotes concatenation of $*, \#$.

This **decomposition** of the aggregate demand of $(a, e, t, t_*) \in \Omega$ generalizes Balasko 3.5.1 (1988) to incomplete markets.

One implication of the decomposition is that ∇ is independent of the policy.

Proof that Mean Externality of Price Adjustment on Welfare is Regular. See Turner (2003b).

■

5.3 Slutsky decomposition of the Reaction to insurance

There is another **decomposition** of demand $D_a\underline{d}$ at b with $t = 0$:

$$D_a\underline{d}^h = \underline{S}^h \begin{bmatrix} 0 \\ \Lambda_{\mathbf{1}}^h \end{bmatrix} - \underline{m}^h \cdot \overline{y}_{\mathbf{1}}^h$$

Here $\Lambda_{\mathbf{1}}^h$ is a matrix displaying the marginal utility of contingent income

$$\Lambda_{\mathbf{1}}^h \equiv [\lambda_1^h I_J : ... : \lambda_s I_J]_{J\times JS}$$

and $\overline{y}_{\mathbf{1}}^h$ is a repeated display of y^h : [12]

$$\overline{y}_{\mathbf{1}}^h = \begin{bmatrix} 0 & \cdot & 0 \\ y' & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & y' \end{bmatrix}_{S+1\times JS}$$

Specializing to a single asset's payoff, this reads

$$\boxed{D_{a^j}d = \Sigma \underline{S}_j^h \lambda_{\mathbf{1}}^{h\prime} - \underline{m}_{\mathbf{1}}^h \cdot y_j^h} \qquad (D_{a^j}d)$$

where $\underline{S}_j^h$ is column $(C-1)(S+1)+ j$ of $\underline{S}^h$.

5.4 Preparation for genericity

We investigate for each policy the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions. In computing

$$\Delta \equiv D_a d_{a,e}$$

we use the following notation for $\underline{S}^h$, where A^h, B^h are symmetric of dimensions $(C-1)(S+1), J$:

$$\underline{S}^h = [\underline{S}_p^h : \underline{S}_q^h] = \begin{bmatrix} A^h & P^h \\ P^{h\prime} & B^h \end{bmatrix} \qquad (\underline{S}^h)$$

We can perturb P^h arbitrarily and get a Slutsky perturbation.

Remark 1 *In checking the Sufficient Independence of Reactions, all marginal utilities* λ^i *and excess demands* z *are automatically fixed by the* $\dot{\underline{S}}^h$ *Slutsky perturbations. Their only effect is on the Jacobian* $\dot{\nabla} = \Sigma \dot{\underline{S}}^h L_+^h$ *in* (∇)*. Also, we solve* $\tilde{k}'\dot{\nabla} = \beta'$ *piecemeal, solving* $\tilde{k}'\dot{\nabla}_p = \beta_p', \tilde{k}'\dot{\nabla} = \beta_q'$ *by splitting* $\beta' = (\beta_p', \beta_q')$, $\dot{\nabla} = [\dot{\nabla}_p : \dot{\nabla}_q]$.

[12] We view a as one long vector, state by state.

6 Pareto improving innovation in an existing asset

Here we prove the generic existence of Pareto improving innovation in an existing asset, targeting the income effect, by showing this policy passes the sensitivity test in theorem 4. Key is the generic position of

6.1 The insurance deficit

In equilibrium, every household's marginal utility of future income projects to a common point in the asset span[13]

$$\frac{\lambda_{\mathbf{1}}^h}{\lambda_0^h} = u^h + c \in a^{\perp} \oplus span(a)$$

We summarize the **insurance deficit** by

$$U \equiv \begin{bmatrix} \dots & u^h & \dots \end{bmatrix}_{S \times H}$$

Lemma 3 (Insurance deficit in general position) *If $S - J \geq H$ then generically in endowments, at every GEI every H rows of the insurance deficit U are linearly independent.*[14]

Proof. Fix $K \subset \{1, ..., S\}$ with cardinality H, and apply transversality to

$$\begin{gathered} \text{nonnumeraire excess demand equations} \\ \pi' U = 0 \\ \pi_K' \pi_K - 1 = 0 \end{gathered}$$

where $\pi_{S \backslash K} = 0$. Endowments make this transverse to zero. The burden of the argument is to control the second equations independently of the others. Given $t \in R^H$ we want $\pi' \dot{u}^h = t^h$, where $\dot{u}^h \equiv \frac{d}{d\xi} \Pr_{a^{\perp}} \left(\frac{\lambda_{\mathbf{1}}^h}{\lambda_0^h} \right)$, via appropriate $\dot{\lambda}^h$, i.e. $\dot{\lambda}^h$ must preserve first order conditions $\dot{\lambda}_0^h q = a \dot{\lambda}_{\mathbf{1}}^h$. Any $\dot{\lambda}^h$ is implementable by an endowment perturbation $\dot{e}^h = \dot{x}^h$ as we show last. If $\dot{\lambda}_0^h = 0$ and $0 = a \dot{\lambda}_{\mathbf{1}}^h$ then first order conditions remain and

$$\frac{\partial}{\partial \cdot} \left(\frac{\lambda_{\mathbf{1}}^h}{\lambda_0^h} \right) = \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h} - \frac{\lambda_{\mathbf{1}}^h}{\lambda_0^{h2}} \dot{\lambda}_0^h = \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h} \quad \text{so} \quad \dot{u}^h \equiv \frac{\partial}{\partial \cdot} \Pr_{a^{\perp}} \left(\frac{\lambda_{\mathbf{1}}^h}{\lambda_0^h} \right) = \Pr_{a^{\perp}} \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h} = \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h}$$

So set $\dot{\lambda}_0^h = 0$ and seek $\dot{\lambda}_{\mathbf{1}}^h$ with $0 = a \dot{\lambda}_{\mathbf{1}}^h, \pi' \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h} = t^h$. To find $\dot{\lambda}_{\mathbf{1}}^h$, say $\pi_s \neq 0, s \in K$ and set $\dot{\lambda}_K^h$ to $\dot{\lambda}_s^h = \frac{\lambda_0^h t^h}{\pi_s}, \dot{\lambda}_{t \neq s}^h = 0$ for $t \in K$ so that, thanks to $\pi_{S \backslash K} = 0$, $\pi' \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h} = t^h$ regardless of $\dot{\lambda}_{S \backslash K}^h$. Having set $\dot{\lambda}_K^h$, define $\dot{\lambda}_{S \backslash K}^h$ as a solution to $0 = a \dot{\lambda}_{\mathbf{1}}^h = a_K \dot{\lambda}_K^h + a_{S \backslash K} \dot{\lambda}_{S \backslash K}^h$, which exists since these are J equations in $|S \backslash K| = S - H \geq J$ variables and every J columns of a are linearly independent.

To implement this $\dot{\lambda}^h$, solve $D^2 u^h \cdot \dot{x}^h = (\overline{p}_s \dot{\lambda}_s^h)_s$ for $\dot{x}^h$, possible by the negative definiteness of $D^2 u^h$ and the inverse function theorem. Implement this $\dot{x}^h$ by setting $\dot{e}^h = \dot{x}^h$, while preserving the other equations.

[13] This is the same as the decomposition $\lambda_{\mathbf{1}}^h \in a_+^{\perp} \oplus span(a_+)$ by definition of new unwanted asset.

[14] This requires that every J columns of a are linearly independent.

By the transversality theorem, generically in endowments, the system is transverse to zero in the remaining variables. These are $\dim p + \dim q + \dim \pi$ variables and $\dim p + \dim q + H + 1$ equations, with $\dim \pi = H$, so the associated zero set is a submanifold of dimension -1, hence empty. For these endowments E_K, the K rows of U are linearly independent. The intersection of the generic E_K over the finitely many such K is generic still. ■

6.2 Applying the sensitivity test

Substitution free innovation in an existing asset satisfies $\lambda_{\mathbf{1}}^{h\prime} \dot{a}^j = 0$. We parameterize financial innovation by $T(b) = span(a, U)^{\perp}$. Note, $\dot{a}^j \in T(b) \Rightarrow \lambda_{\mathbf{1}}^{h\prime} \dot{a}^j = 0$.

Substitution free innovation provokes only the income effect on demand; formula $(D_{a^j} d)$ implies

$$\Delta \cdot \dot{a}^j = D_{a^j} d \cdot \dot{a}^j = -\Sigma \underline{m}_{\mathbf{1}}^h \cdot y_j^h \dot{a}^j$$

That is,

$$\Delta = -\Sigma \underline{m}_{\mathbf{1}}^h \cdot y_j^h \quad \text{on} \quad T(b)$$

Corollary 6 *Fix the desired welfare impact* $\dot{v} \in R^H$. *Assume* $S - J \geq 2H; H, C > 1$. *Then generically in utilities and endowments, at every GEI* $\dot{v}$ *is the welfare impact of some* $\dot{t} \in T$. *Hence there is a nearby Pareto superior GEI with substitution free innovation in an existing asset.*

Proof. The next lemmas with $gen \dim = S - J - H$ and the hypothesis $S - J - H \geq H$ enable theorem 4. ■

Lemma 4 (Generic Dimension of Innovation) *If* $S - J \geq H$, *then* $gen \dim = S - J - H$. *That is, generically in endowments, at every GEI the vector subspace* $\dot{t} \in T = T(b)$ *parameterizing financial innovation has dimension* $S - J - H$.

Proof. Lemma 3 says that generically in endowments U has rank H, and then $span(a, U)'s$ dimension is $J + H$. ■

Lemma 5 (Full Reaction of Demand to Policy) *If* $C > 1, S - J \geq H > 1$, *generically in utilities and endowments, at every GEI* Δ *has rank* $gen \dim$.

Proof. We recall $\Delta = -\Sigma \underline{m}_{\mathbf{1}}^h \cdot y_j^h$ has domain $k \in T(b) = span(a, U)^{\perp}$, and take for granted the very standard result that with $H > 1$ generically in numeraire endowments, at every GEI asset j is traded. Taking generic endowments from this result and the previous lemma's, we apply transversality to

$$\begin{gathered} \text{nonnumeraire excess demand equations} \\ \left(\Sigma \nabla^h \cdot y_j^h \right) k = 0 \\ k'k - 1 = 0 \end{gathered}$$

where $\nabla_{S \times S}^h$ selects from $\underline{m}_{\mathbf{1}}^h \in R^{S(C-1) \times S}$ only the rows of commodities $(s1)_{s \geq 1}$. Utilities make this transverse to zero. The burden of the argument is to control the middle equations independently of the top and bottom ones. Say $k_{s\geq 1} \neq 0$; we want to perturb arbitrarily column s of the parenthetical sum, as $\frac{d}{d\xi}\left(\Sigma\nabla^h \cdot y_j^h\right) = \frac{a}{k_s}$, and no other. There is h^* with $y_j^{h^*} \neq 0$. From the identification of Slutsky perturbations 5, we may perturb arbitrarily any row of $m_x^{h^*}$, hence any row of ∇^{h^*}, subject only to $\dot{m}_x^{h^*} W' = 0$, where $W = [-q : a]$. So perturb it as $\dot{\nabla}_s^{h^*} = [0 : \frac{a_s}{y_j^h} k']$ so that $\frac{d}{d\xi}\left(\Sigma\nabla_s^h \cdot y_j^h\right) k = \frac{d}{d\xi}\left(\dot{\nabla}_s^{h^*} y_j^{h^*}\right) k = a_s(k'k) = a_s$ is arbitrary. Indeed, $\dot{\nabla}_s^{h^*} W' = 0$ since $k \in T(b) \equiv span(a, U)^{\perp} \subset a^{\perp}$.

By the transversality theorem, generically in endowments and utilities, this system is transverse to zero in the remaining endogenous variables. These number $\dim p + \dim q + gen \dim$ and there are $\dim p + \dim q + S$ equations, and $gen \dim = S - J - H$, so by the preimage theorem, for these endowments and utilities the associated solution set is empty–every GEI has $\Sigma\nabla^h \cdot y_j^h$ (a fortiori Δ) with linearly independent columns. ■

Lemma 6 (Sufficient Independence of Reactions) *Generically in endowments and utilities, whenever* $\tilde{k} \in R^{(S+1)(C-1)+J}$ *is nonzero everywhere and* $\beta \in R^{(S+1)(C-1)+J}$*, at every GEI there is a path of risk aversion that solves* $\dot{\Delta} = 0, \tilde{k}'\dot{\nabla} = \beta'$.

Proof. Fix such a $\tilde{k}$, and follow remark 1. Since $\Delta = -\Sigma\underline{m}_{\mathbf{1}}^h \cdot y_j^h$ is independent of the substitution matrices $\underline{S}^h$, which is all we perturb, automatically $\dot{\Delta} = 0$ and $\dot{\nabla} = \Sigma\dot{\underline{S}}^h L_+^h$. Left to solve $\tilde{k}'\dot{\nabla} = \Sigma\dot{\underline{S}}^h L_+^h = \beta'$, we set $\dot{\underline{S}}^{h\neq H} = 0$ and so seek to solve $\tilde{k}'\dot{\underline{S}}^H = \beta'\left(L_+^H\right)^{-1} \equiv \tilde{\beta}'$. This is made trivial by a diagonal hence symmetric $\dot{\underline{S}}^H$, with $\dot{\underline{S}}_{mm}^H = \frac{\tilde{\beta}_m}{\tilde{k}_m}$. ■

7 The insurance deficit bound on the rate of improvement

We bound the rate of Pareto improvement by the equilibrium's *insurance deficit*, which vanishes exactly at Pareto optimality. The bound turns out to be the covariance of the insurance deficit with the marginal purchasing power.

Recall that the welfare impact is $\dot{v}^h = \lambda^{h\prime} dm^h$ where dm^h is marginal purchasing power, for some matrices $\Sigma dm^h = 0$. ($dm^h = \overline{y}_{\mathbf{1}} - \overline{z} dP$.) Converting marginal welfare from utils to the numeraire at time 0, marginal utility becomes $\frac{\lambda^h}{\lambda_0^h}$, which we rewrite as λ^h with $\lambda_0^h = 1$. In this common unit,

$$dW = \frac{1}{H}\Sigma\lambda^{h\prime} dm^h \qquad \text{the } \textbf{mean welfare impact}$$

Every household's marginal utility of future income projects to a common point in the asset span,

$$\lambda_{\mathbf{1}}^h = \delta^h + c \in a^{\perp} \oplus a$$

by the first order condition, being unique only in its **insurance deficit** δ^h. If the **mean insurance deficit** is $\overline{\delta} = H^{-1}\Sigma\delta^h$, then the GEI's **insurance deficit** is

$$\Delta = [\delta^1 - \overline{\delta} : ... : \delta^H - \overline{\delta}]_{S\times H}$$

Note that the GEI is Pareto optimal exactly when $\Delta = 0$[15]. Computing the mean welfare impact,

$$\begin{aligned}
H \cdot dW &= \Sigma\lambda_0^h dm_0^h + \Sigma\lambda_{\mathbf{1}}^{h\prime} dm_{\mathbf{1}}^h \\
&= \Sigma dm_0^h + \Sigma(\delta^h + c)' dm_{\mathbf{1}}^h \\
&= 0 + \Sigma\delta^{h\prime} dm_{\mathbf{1}}^h + c'\Sigma dm_{\mathbf{1}}^h \\
&= \Sigma\delta^{h\prime} dm_{\mathbf{1}}^h \\
&= \Sigma(\delta^h - \overline{\delta})' dm_{\mathbf{1}}^h \\
&= H \cdot cov(\Delta, dm_{\mathbf{1}})
\end{aligned}$$

since $\Sigma dm^h = 0$. The **rate of Pareto improvement** is the norm of the functional $dW \mid_{dv \geq 0}$.

Remark 2 *At a regular GEI, the mean welfare impact equals the covariance across households of the insurance deficit and the marginal purchasing power, $dW = cov(\Delta, dm_{\mathbf{1}})$. So the rate of Pareto improvement is bounded above by the norm of this covariance.*

If the tax policy targets only current income, i.e. $\tau_{\mathbf{1}}^h, \tau_{\mathbf{1}} = 0$, then $dm_{\mathbf{1}}^h = -\underline{z}_{\mathbf{1}}^h dP_{\mathbf{1}}$ and

$$dW = -cov(\Delta, \underline{z}_{\mathbf{1}}) dP_{\mathbf{1}}$$

The sole control is the future price adjustment, since the GEI sets the insurance deficit and net trade. In a nutshell, the mean welfare impact of the sole control is minus the covariance of insurance deficit and net trade.

8 Appendix

8.1 Notation

An underbar connotes the omission of the $sC, s \geq 0$ coordinates, as in $\underline{x}^h$; an upperbar on a price $\overline{p}$ connotes the addition of sC coordinates with value $p_{sC} = 1, s \geq 0$.

When differentiating with respect to p, q, a, w, we parameterize these as long vectors:

$$p = \begin{bmatrix} \cdot \\ p_s \\ \cdot \end{bmatrix}_{(C-1)(S+1)\times 1} \qquad q = \begin{bmatrix} \cdot \\ q_j \\ \cdot \end{bmatrix}_{J\times 1} \quad a = \begin{bmatrix} \cdot \\ a_s \\ \cdot \end{bmatrix}_{SJ\times 1} \quad w = \begin{bmatrix} \cdot \\ w^h \\ \cdot \end{bmatrix}_{H(S+1)\times 1}$$

8.2 Derivation of formula for welfare impact

It is standard how Debreu's smooth preferences, linear constraints, and the implicit function theorem imply the smoothness of neoclassical demand. It is standard also that the envelope property follows from the value function's local smoothness, which is the case for v^h as the composition of smooth functions:

$$D_b v^h = D_b L(x, y, \lambda^h) \mid_{(x^h, y^h)(b)}$$

[15] Also, a household's commodity demand is as though asset markets were complete exactly when $\delta^h = 0$.

where $b = (p, q, a, w^h)$ and

$$L(x, y, \lambda^h) \equiv u^h(x) - \lambda^{h\prime}\left([\overline{p}]'x - w^h - \begin{bmatrix} -q' \\ a' \end{bmatrix} y\right)$$

Thus

$$D_b v^h = -\lambda^{h\prime}\left([\underline{x}^h]' : \overline{y}_0^h : -\overline{y}_{\mathbf{1}}^h : -I\right) \qquad \text{where } \overline{y}_0^h = \begin{bmatrix} y^{h\prime} \\ 0 \end{bmatrix}, \overline{y}_{\mathbf{1}}^h = \begin{bmatrix} 0 & \cdot & 0 \\ y^{h\prime} & & 0 \\ & \cdot & \\ 0 & & y^{h\prime} \end{bmatrix}$$

So much for demand theory. Recalling regular GEI from the subsection on the Expression for the Price Adjustment, $dP' = (dp', dq')$ exists and

$$\begin{aligned} w^h &= [\overline{p}]'e^h \Rightarrow \\ dw^h &= [\underline{e}^h]'dp \\ &= ([\underline{e}^h]' : 0)dP \end{aligned}$$

Thus the welfare impact at a regular GEI is

$$\begin{aligned} dv^h &= D_b v^h \cdot db \\ &= -\lambda^{h\prime}\left(([\underline{x}^h]' : \overline{y}_0^h) : -\overline{y}_{\mathbf{1}}^h : -I\right) \cdot \left(dP : I_{SJ} : ([\underline{e}^h]' : 0)dP\right) \\ &= -\lambda^{h\prime}\left(([\underline{x}^h]' : \overline{y}_0^h)dP - \overline{y}_{\mathbf{1}}^h - ([\underline{e}^h]' : 0)dP\right) \\ &= -\lambda^{h\prime}\left(\underline{z}^{h\prime}dP - \overline{y}_{\mathbf{1}}^h\right) \end{aligned}$$

where $\underline{z}^{h\prime} \equiv ([\underline{x}^h - \underline{e}^h]' : \overline{y}_0^h)$ by definition. In sum,

$$\boxed{dv^h = \lambda^{h\prime}\left(\overline{y}_{\mathbf{1}}^h - \underline{z}^h dP\right)}$$

8.3 Aggregate notation

We collect marginal utilities of contingent income, and denote stacking by an upperbar

$$(\lambda)' \equiv \begin{bmatrix} \cdot & & 0 \\ & \lambda^{h\prime} & \\ 0 & & \cdot \end{bmatrix}_{H\times H(S+1)} \qquad \overline{y}_{\mathbf{1}} = \begin{bmatrix} \cdot \\ \overline{y}_{\mathbf{1}}^h \\ \cdot \end{bmatrix}_{H(S+1)\times SJ} \qquad \overline{\underline{z}} \equiv \begin{bmatrix} \cdot \\ \underline{z}^{h\prime} \\ \cdot \end{bmatrix}_{H(S+1)\times(S+1)(C-1)+J}$$

Thus

$$\boxed{dv = (\lambda)'\left(\overline{y}_{\mathbf{1}} - \overline{\underline{z}}dP\right)}$$

To visualize the bracket notation $[\cdot]$ defined in footnote 7, it staggers state contingent vectors:

$$[p] \equiv \begin{bmatrix} \cdot & & & \\ & p_{s-1} & & 0 \\ & & p_s & \\ & 0 & & p_{s+1} \\ & & & & \cdot \end{bmatrix}_{C(S+1)\times S+1}$$

8.4 Transversality

A function $F : M \times \Pi \to N$ defines another one $F_\pi : M \to N$ by $F_\pi(m) = F(m, \pi)$. Given a point $0 \in N$ consider the "equilibrium set" $E = F^{-1}(0)$ and the natural projection $E \to \Pi, (m, \pi) \mapsto \pi$. A function is *proper* if it pulls back sequentially compact sets to sequentially compact sets.

Remark 3 (Transversality) *Suppose F is a smooth function between finite dimensional smooth manifolds. If 0 is a regular value of F, then it is a regular value of F_π for almost every $\pi \in \Pi$. The set of such π is open if in addition the natural projection is proper.*

A subset of Π is **generic** if its complement is closed and has measure zero. Write $C^* = C(S+1)$. Here the set of parameters is

$$\Pi = O \times O' \times (0, \epsilon)$$

where O, O' are an open neighborhoods of zero in $R^{C^* H}, R^{\frac{C^*(C^*+1)}{2} H}$ relating to endowments and symmetric perturbations of the Hessian of utilities. We have in mind a fixed assignment of utilities, which we perturb by $O' \times (0, \epsilon)$. Specifically, given an equilibrium commodity demand $\overline{x}$ by some household and $\Box \in R^{\frac{C^*(C^*+1)}{2}}, \alpha \in (0, \epsilon)$ we define $u_{\Box,\alpha}$ as

$$u_{\Box,\alpha}(x) \equiv u(x) + \frac{\omega_\alpha(\|x - \overline{x}\|)}{2}(x - \overline{x})'\Box(x - \overline{x})$$

where $\omega_\alpha : R \to R$ is a smooth bump function, $\omega_\alpha \mid_{(-\frac{\alpha}{2}, \frac{\alpha}{2})} \equiv 1$ and $\omega_\alpha \mid_{R \setminus (-\alpha, \alpha)} \equiv 0$. In a neighborhood $x \approx \overline{x}$ we have

$$\begin{aligned} u_{\Box,\alpha}(x) &= u(x) + \frac{1}{2}(x - \overline{x})'\Box(x - \overline{x}) \\ Du_{\Box,\alpha}(x) &= Du(x) + (x - \overline{x})'\Box \Rightarrow Du_{\Box,\alpha}(\overline{x}) = Du(x) \\ D^2 u_{\Box,\alpha}(x) &= D^2 u(x) + \Box \end{aligned}$$

So in an α-neighborhood the Hessian changes, by $\Box$, but the gradient, demand do not. For small enough $\alpha, \Box$ this utility remains in Debreu's setting, so neoclassical demand is defined and smooth when active.

In the Sufficient Independence of Reactions, the path of risk aversions is identified with a linear path $(\Box^h, \alpha^h)(\xi) \equiv (\Box^h \xi, \frac{\|\overline{x}^h\|}{2})$ for each household, so that $\frac{d}{d\xi} D^2 u^h_{\Box,\alpha}(x) = \Box^h$.

References

[1] Balasko, Y.,1988, *Foundations of the Theory of General Equilibrium.* New York: Academic Press.

[2] Cass, D. and A. Citanna, 1998, "Pareto Improving Financial Innovation in Incomplete Markets," *Economic Theory* 11: 467-494.

[3] Dixit, A., 1983, "The Multi-Country Transfer Problem," *Economic Letters* 13: 49-53.

[4] Duffie, D. and W. Shafer, 1985, "Equilibrium in Incomplete Markets: I A Basic Model of Generic Existence," *Journal of Mathematical Economics* 14, 285-300.

[5] Elul, R., 1995, "Welfare Effects of Financial Innovation in Incomplete Markets Economies with Several Consumption Goods," *Journal of Economic Theory* 65: 43-78.

[6] Geanakoplos, J. and H. Polemarchakis, 1980,"On the Disaggregation of Excess Demand Functions," *Econometrica* 48(3): 315-335.

[7] Geanakoplos, J. and H. Polemarchakis, 1986, "Existence, Regularity and Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete." In Heller, W., Starr, R., Starrett, D. (eds.) *Uncertainty, Information and Communication: Essays in Honor of Kenneth J. Arrow, Vol.III.* Cambridge: Cambridge University Press, 1986.

[8] Geanakoplos, J., 2000, *Lecture Notes on Incomplete Markets.* Yale University.

[9] Gottardi, P. and T. Hens, 1999, "Disaggregation of Excess Demand and Comparative Statics with Incomplete Markets and Nominal Assets," *Economic Theory* 13: 287-308.

[10] Magill, M. and M. Quinzii, 1996, *Theory of Incomplete Markets.* Cambridge, MA: The MIT Press.

[11] Turner, S., 2003a, "Theory of Demand in Incomplete Markets," Brown University.

[12] Turner, S., 2003b, "Pareto Improving Taxation in Incomplete Markets," Brown University.