Arrow's Impossibility Theorem: Two Simple Single-Profile Versions

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Abstract

In this short paper we provide two versions of Arrow's impossibility theorem, in a world with only one preference profile. Both versions are extremely simple and allow a transparent understanding of Arrow's theorem. The first version assumes a two-agent society; the second version, which is similar to a theorem of Pollak, assumes two or more agents. Both of our theorems rely on diversity of preferences axioms; our first theorem also uses a neutrality-independence assumption, commonly used in the literature; our second theorem uses a neutrality-independence-monotonicity (NIM) assumption, which is stronger and less commonly used. Using the NIM assumption results in substantial gains in terms of simplicity. We provide examples to show the logical independence of the axioms, and to illustrate our points.

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1. Introduction.

In 1950 Kenneth Arrow (1950, 1963) provided a striking answer to a basic abstract problem of democracy: how can the preferences of many individuals be aggregated into societal preferences? Arrow's answer, which has come to be known as his impossibility theorem, was that every conceivable aggregation method has some flaw. That is, a handful of reasonable-looking axioms, which one hopes an aggregation procedure would satisfy, lead to impossibility: the axioms are mutually inconsistent. In his Introduction to the *Handbook of Social Choice and Welfare*, Suzumura (2002) points out that Arrow's impossibility theorem had a revolutionary impact on the whole of social choice theory; in the same volume Campbell and Kelly (2002) indicate that social choice theory was in fact spawned by Arrow's theorem. The theorem has also had a major influence on the larger fields of economics and political science, as well as on distant fields like mathematical biology. (See, e.g., Bay and McMorris (2003).)

In this paper we shall develop two versions of Arrow's impossibility theorem. Our models are so-called single-profile models. This means impossibility is established in the context of one fixed profile of preferences, rather than in the (standard) Arrow context of many varying preference profiles. In our first version of impossibility, there are only 2 individuals. In the second version, there are 2 or more individuals. Single-profile Arrow theorems were first proved in the late 1970's by Parks (1976), Hammond (1976), Kemp and Ng (1976), Pollak (1979), and Roberts (1980). Our second version of Arrow's theorem is close to Pollak's. Rubinstein (1984) used mathematical logic to see whether or not there are single-profile versions of every multi-profile theorem of social choice.

Other related literature includes Geanakoplos (1996), who has three very short proofs of Arrow's theorem in the standard multi-profile context, and Ubeda (2004) who has another short multi-profile proof. Ubeda makes an observation that some of his arguments establish "Arrow's theorem for societies with two individuals with strict preferences." This observation, for a multi-profile model, is somewhat similar to our first version of Arrow's theorem. Reny (2001) has an interesting side-by-side pair of (multi-profile) proofs, of Arrow's theorem and the related theorem of Gibbard and Satterthwaite.

Single-profile Arrow impossibility theorems were devised in response to an argument of Samuelson (1967) against Arrow. Samuelson claimed that Arrow's model, with varying preference profiles, is irrelevant to the problem of maximizing a Bergson-Samuelson-type social welfare function (Bergson (1938)), which depends on a given set of ordinal utility functions, that is, a fixed preference profile. The single-profile Arrow theorems established that bad results (dictatorship, or illogic of social preferences, or, more generally, impossibility of aggregation) could be proved with one fixed preference profile (or set of ordinal utility functions), provided the profile is "diverse" enough. Our main purpose in this paper is to give two short and easy-to-understand single-profile Arrow theorems. Our theorems do not require the existence of large numbers of alternatives, unlike previous single-profile Arrow results. Our secondary purpose is to explore various possible assumptions regarding preference profile "diversity."

2. The Models.

We assume a society with $n \geq 2$ individuals, and 3 or more alternatives.

A specification of the preferences of all individuals is called a preference profile. There is only one preference profile. The preference profile is transformed into a social preference relation. Both the individual and the social preference relations allow indifference. The individual and social preference relations are all assumed to be complete and transitive. The following notation is used: Generic alternatives are x, y, z , w, etc. Particular alternatives are a, b, c , d, etc. A generic person is labeled i, j, k and so on; a particular person is 1, 2, 3 and so on. Person i's preference relation is R_i. xR_iy means person i prefers x to y or is indifferent between them; xP_iy means i prefers x to y; xI_iy means i is indifferent between them. Society's preference relation is R. xRy means society prefers x to y or is indifferent between them; xPy means society prefers x to y; xIy means society is indifferent between them. We will start with the following assumptions:

(1.a) **Pareto principle**. For all x and y, if xP_iy for all i, then xPy.

(1.b) **Strong Pareto principle**. For all x and y, if xR_iy for all i, and xP_iy for some i, then xPy.

(2.a) **Neutrality/independence** (**NI**). Suppose individual preferences for w vs. z are identical to individual preferences for x vs. y. Then the social preference for w vs. z must be identical to the social preference for x vs. y. More formally: For all x, y, z, and w, assume that, for all i, xP_iy if and only if wP_iz, and zP_iw if and only if yP_ix. Then wRz if and only if xRy, and zRw if and only if yRx.

(3) **No dictator**. There is no dictator. Individual i is a *dictator* if, for all x and y, xP_iy implies xPy.

(4.a) **Diverse-1 preferences**. There exists a triple of alternatives x, y, z, such that xP_iy for all i, but opinions are split on x vs. z, and on y vs. z. That is, some people prefer x to z and some people prefer z to x, and, similarly, some people prefer y to z and some people prefer z to y.

Note that we have two alternative versions of the Pareto principle here. The first is more common in the Arrow's theorem literature (e.g., see Campbell and Kelly (2002), p. 42), and when we need to distinguish this version from the second, we will call it the "standard" version. We will use the strong version of the Pareto principle in our $n = 2$ impossibility theorem below. Obviously the strong Pareto principle implies the standard Pareto principle. Neutrality-independence, assumption 2.a, and diverse-1 preferences, assumption 4.a, are so numbered because we will introduce alternatives later on.

3. Some Examples in a 2-Person Model.

We will illustrate with a few simple examples. For these examples there are 2 people and 3 alternatives, and we assume no individual is indifferent between any pair of alternatives, although society might be. Given that we aren't allowing individual indifference, the two Pareto principles collapse into one. Preferences of the 2 people are shown by listing the alternatives from top (most preferred) to bottom (least preferred). In our examples, the last columns of the tables will show what is being assumed about society's preferences.

Example 1:

Person 1	Person 2	Society (Majority Rule)
a	*c*	
b	*a*	*aPb*, *aIc* & *bIc*
c	*b*	

The social preference relation is based on majority rule. Therefore, *aPb* because *a* beats *b* in a vote, while *aIc* and *bIc,* because votes between those pairs result in ties. The Pareto principle is obviously satisfied. NI is satisfied: individual preferences for *a* vs. *c* are identical to individual preferences for *b* vs. *c*, and the social preference for *a* vs. *c* is identical to the social preference for *b* vs. *c*. There is no dictator. The preferences of the two people satisfy the diverse-1 preferences assumption, since both people prefer *a* to *b*, but opinions are split when it comes to *a* vs. *c* and *b* vs. *c*. In short, our 4 assumptions are satisfied. We should have an Arrow impossibility result here, and we do, because the social preferences are not transitive. With *aPb* and *bIc* transitivity implies *aPc,* but this contradicts *aIc*. Example 1 is therefore a 2-person voting paradox, slightly different from the classical 3-person Condorcet voting paradox, which we will visit later in this paper.

We now modify example 1 very slightly, by changing individual 1's preferences.

Example 2:

Person 1	Person 2	Society (Majority Rule)
a	*c*	
c	*a*	*aIc*
b	*b*	*aPb* & *cPb*

The Pareto principle requires that *aPb* and *cPb*, which majority voting delivers. NI still holds. There is no dictator. Opinions are split on *a* vs. *c*, but not on *b* vs. *c*; both people view *b* as the worst alternative. Since we don't have opinions split over the two pairs of alternatives, the diverse-1 preferences assumption no longer holds. Three of our 4 assumptions hold. Now examine the social preference relation. It is obviously complete. Is it transitive? The answer is yes; for example *aIc* and *cPb* should imply *aPb*, which majority rule delivers.

In short, example 2 shows that if we drop the diverse-1 preferences assumption, the remaining 3 assumptions can be mutually consistent.

In example 3 we return to the same individual preferences as example 1, but we now assume society is indifferent among all 3 alternatives. This is of course a complete and transitive preference relation for society. Obviously we are dropping the Pareto principle.

Example 3:

Person 1	Person 2	Society
a	*c*	
b	*a*	*aIbIc*
c	*b*	

Does this example violate any of the 4 assumptions, other than Pareto? NI must hold since the social preference for *w* vs. *z* is clearly always going to be the same as that for *x* vs. *y*. There is obviously no dictator. And lastly, the preferences of the 2 individuals clearly satisfy the diverse-1 preferences assumption.

In short, example 3 shows that if we drop the Pareto principle, the remaining 3 assumptions can be mutually consistent.

We turn to our fourth simple example. Here we start with the individual preferences of examples 1 and 3 above, and simply make person 1 a dictator:

Example 4:

Person 1	Person 2	Society (1 is Dictator)
a	*c*	*a*
b	*a*	*b*
c	*b*	*c*

The moral of example 4 should be clear: if we drop the no dictator assumption, the remaining 3 assumptions can be mutually consistent.

And now to our last simple example, where NI doesn't hold. We start with the individual preferences of examples 1,3 and 4, and we assume social preferences as shown:

Example 5:

Person 1	Person 2	Society
a	*c*	*a*
b	*a*	*c*
c	*b*	*b*

In example 5, the assumptions of Pareto and no dictator are obviously met, and the assumption of diverse-1 preferences is also met. However, the NI assumption doesn't hold. (Compare the social treatment of *a* vs. *c*, where the two people are split and person 1 gets his way, to the social treatment of *b* vs. *c*, where the two people are split and person 2 gets his way.) Example 5 shows that if we drop NI, the remaining assumptions can be mutually consistent.

4. Arrow Impossibility Theorem 1, $n = 2$.

We are ready to turn to a very simple and transparent version of Arrow's impossibility theorem, in our single-profile model. This version of the impossibility theorem relies on there being only 2 people in society. Therefore, throughout this section, we assume $n = 2$. We will show that our 4 assumptions, the strong Pareto principle, NI, no dictator, and diverse-1 preferences, are mutually inconsistent.

First we establish proposition 1, which is by itself a strong result. Then we prove our first simple version of Arrow's theorem[1].

Proposition 1: Assume $n = 2$. Assume the strong Pareto principle, and NI. Suppose for some pair of alternatives x and y, xP_iy and yP_jx. Suppose that xPy. Then person i is a dictator.

Proof: Let w and z be any pair of alternatives. Assume wP_iz. We need to show that wPz must hold. If wR_jz, then wPz by strong Pareto. If not wR_jz, then zP_jw by completeness for j's preference relation, and then wPz by NI. QED.

Arrow Impossibility Theorem 1: Assume $n = 2$. The assumptions of the strong Pareto principle, NI, no dictator, and diverse-1 preferences are mutually inconsistent.

[1] In our theorem we are using strong Pareto, NI, no dictator, and diverse-1 preferences to get impossibility. With an almost identical proof we could use standard Pareto, NIM, no dictator, and diverse-1 preferences to get impossibility, where NIM is a strengthened version of NI, to be discussed below.

Proof: By diverse-1 preferences there exist x, y and z such that xP_iy for $i = 1, 2$, but such that opinions are split on x vs. z, and on y vs. z.

Now xPy by the Pareto principle, standard or strong. Since opinions are split on x vs. z, one person prefers x to z, while the other prefers z to x. If xPz, then the person who prefers x to z is a dictator, by proposition 1. If zPx, then the person who prefers z to x is a dictator, by proposition 1.

Suppose xIz. Then zIx. By transitivity, zIx and xPy implies zPy. But opinions are split on y vs. z. One person, say i, is getting his way on z vs. y, in face of the opposition of the other person, say j. By proposition 1, person i would then be a dictator. As a dictator he would have to get his way on x vs. z also. (Remember, opinions are split on x vs. z, so he is not indifferent). We conclude that xIz is impossible. QED.

5. Trying to Generalize to an *n*-Person Model.

In what follows we will seek to generalize our version of Arrow's theorem to societies with arbitrary numbers of people. From this point on in the paper we will assume that $n \geq 2$. In order to get an impossibility theorem when $n \geq 2$, we will need to strengthen some of our basic assumptions. We start with the neutrality/independence assumption. We will strengthen it to a single-profile version of what is called neutrality/independence/monotonicity, or NIM. (See Blau & Deb (1977), who call the multi-profile analog "full neutrality and monotonicity"; Sen (1977), who calls it NIM; and Pollak (1979), who calls it "nonnegative responsiveness.")

(2.b) **Neutrality/independence/monotonicity** (**NIM**). Suppose the support for *w* over *z* is as strong or stronger than the support for *x* over *y*, and suppose the opposite support, for *z* over *w*, is as weak or weaker than the support for *y* over *x*. Then, if the social preference is for *x* over *y*, the social preference must also be for *w* over *z* . More formally: For all *x*, *y*, *z*, and *w*, assume that for all *i*, xP_iy implies wP_iz, and that for all *i*, zP_iw implies yP_ix. Then xPy implies wPz.

Does this strengthening of the neutrality assumption, by itself, give us an Arrow impossibility theorem when $n \geq 2$? The answer is No. In example 6 there are 3 people and 4 alternatives, *a*, *b*, *c* and *d*. The preferences of individuals 1, 2 and 3 are shown in the first 3 columns of the table. The fourth column shows social preferences under majority rule, which is used here, as in examples 1 and 2, to generate the social preference relation.

Example 6:

Person 1	Person 2	Person 3	Society (Majority Rule)
a	*c*	*a*	*a*
b	*a*	*c*	*c*
c	*b*	*d*	*b*
d	*d*	*b*	*d*

Let us comment on these preferences. We start with exactly the people and preferences of example 1 above, although we throw in a fourth alternative *d*, at the bottoms of their lists. Then we add the third person, with preferences somewhat like person 1's (although he switches *b* and *c*). Person 3 also puts *d* next to last on his list, rather than last.

The three people use simple majority rule to define social preferences. The resulting social preferences are as follows: aPb, aPc, aPd, cPb, bPd, and cPd. This (strict) social ordering is shown in the last column of the table. Note that the social preference relation is logically fine.

Let's go back and check the 4 conditions that we thought might produce an impossibility result. Given the use of simple majority rule, and given the particular preference profile we have assumed, do the conditions hold? For Pareto (either standard or strong), the answer is obviously yes. NIM is not instantly obvious, but, since majority rule simply counts instances of xP_iy and yP_ix and compares the counts, it has to hold. How about dictatorship? If there were a dictator, the social preference relation would have to be identical to one of the individual preference relations, which it isn't. Finally, what about diverse-1 preferences? Take (a,b,c) as the (x,y,z) triple in the definition of diverse-1 preferences. Note that aP_ib for all i, and that opinions are split on *a* vs. *c*, as required by the assumption, and that opinions are similarly split on *b* vs. *c*. So the diverse-1 preferences assumption holds.

In short, example 6 shows that the assumptions of the Pareto principle, neutrality/independence/monotonicity, no dictator, and diverse-1 preferences are not mutually inconsistent. When $n \geq 2$ there is no Arrow's impossibility theorem, with these assumptions.

6. Diversity.

In this section we will modify the diverse preferences assumption.

Before doing so, let's revisit the assumption in the n = 2 world. In that world, diverse-1 preferences says there must exist a triple of alternatives x, y, z, such that xP_iy for i = 1,2, but such that opinions are split on x vs. z and on y vs. z. That is, one person prefers x to z, while the other prefers z to x, and one person prefers y to z, while the other prefers z to y. Given our assumption that individual preferences are transitive, it must be the case that the two people's preferences over the triple can be represented as follows:

Diverse-1 preferences array, *n* = 2.

Person i	Person j
x	z
y	x
z	y

Note that this is exactly the preference profile pattern of examples 1, 3, 4 and 5. The reader familiar with social choice theory may recognize the preferences in this table as being two thirds of the Condorcet voting paradox preferences, as shown below:

Condorcet voting paradox array.

Person i	Person j	Person k
x	z	y
y	x	z
z	y	x

A similar array of preferences is used by Arrow in the proof of his impossibility theorem (e.g. Arrow (1963), p. 58), and by many others since, including us (Feldman & Serrano (2006), p. 294). For the moment, assume V is any non-empty set of people in society, that V^C is the complement of V, and that V is partitioned into two non-empty subsets V_1 and V_2. (Note that V^C may be empty.) The standard preference profile used in many versions of Arrow's theorem looks like this:

Standard Arrow array.

People in V_1	People in V_2	People in V^C
x	z	y
y	x	z
z	y	x

Now, let's return to the question of how to modify the diverse preferences assumption. Example 6 shows that we cannot stick with the diverse-1 preferences array. We might start with the Condorcet voting paradox array, but if $n \geq 4$, we would have to worry about the preferences of people other than i, j and k. That suggests using something like the standard Arrow array. However, assuming the existence of a triple x, y, and z, and preferences as per that array, for *any subset of people V* and any partition of V, is an unnecessarily strong diversity assumption.

An even stronger diversity assumption was in fact used by Parks (1976), Pollak and other originators of single-profile Arrow theorems. Pollak (1979) is clearest in his definition. His condition of "unrestricted domain over triples" requires the following: Imagine "any logically possible sub-profile" of individual preferences over 3 "hypothetical" alternatives x, y and z. Then there exist 3 actual alternatives a, b and c for which the sub-profile of preferences exactly matches that "logically possible sub-profile" over x, y and z. We will call this *Pollak diversity*. Let us consider what this assumption requires in the simple world of strict preferences, 2 people, and 3 alternatives. Pollak diversity would require that every one of the following arrays be represented, somewhere in the actual preference profile of the two people over the actual alternatives:

Pollak diversity arrays, *n* = 2.

1	2	1	2	1	2	1	2	1	2	1	2
x	*x*	*x*	*x*	*x*	*y*	*x*	*y*	*x*	*z*	*x*	*z*
y	*y*	*y*	*z*	*y*	*x*	*y*	*z*	*y*	*x*	*y*	*y*
z	*z*	*z*	*y*	*z*	*z*	*z*	*x*	*z*	*y*	*z*	*x*

The first pair of columns says 1 and 2 agree on what's best, what's in the middle, and what's worst. The second pair says they agree on what's best only. The third pair says they agree on what's worst only. The fourth and fifth pairs are diverse-1 preferences; the existence of either pattern would satisfy our diverse-1 assumption, but Pollak diversity would require both, since Pollak diversity does not allow free permuting of individuals 1 and 2 . The sixth pair says they agree only on what's in the middle.

Note that the number of arrays in the table above is 3! = 6. If *n* were equal to 3 we would have triples of columns instead of pairs, and there would have to be $(3!)^2 = 36$ such triples. With *n* people, the number of required *n*-tuples would be $(3!)^{n-1}$. In short, the number of arrays required for Pollak diversity rises exponentially with *n*. The number of alternatives rises with the number of required arrays, although not as fast because of array overlaps. Parks (1976) uses an assumption ("diversity in society") that is very similar to Pollak's, although not so clear, and he indicates that it "requires at least 3^n alternatives..."

We believe Pollak diversity is much stronger than necessary, and we will proceed as follows. We will not assume the existence of a triple *x*, *y* and *z* to give every conceivable array of preferences on that triple. We will not even assume a triple *x*, *y* and

z to give every possible array for given V, V_1, V_2, and V^C, as per the description of the standard Arrow array. We will only assume the existence of the required Arrow-type triple, and we will only assume that much when the Arrow array matters. For the purposes of our proof, the Arrow array assumption only matters when V is a decisive set.

We say that a set of people V is *decisive* if it is non-empty and if, for any alternatives x and y, if xP_iy for all i in V, then xPy.

It is appropriate to make a few comments about the notion of decisiveness. First, note that if person i is a dictator, then i by himself is a decisive set, and any set containing i is also decisive. Also, note that the Pareto principle (standard or strong) implies the set of all people is decisive. Second, in a multi-preference profile world, decisiveness for V would be a stronger assumption that it is in the single-profile world, since it would require that (the same) V prevail no matter how preferences might change. We only require that V prevail under the given fixed preference profile.

Our diversity assumption is now modified as follows:

(4.b) **Diverse-2 preferences**. For any decisive set V with 2 or more members, there exists a triple of alternatives x, y, z, such that xP_iy for all i in V; such that yP_iz and zP_ix for everyone outside of V; and such that V can be partitioned into non-empty subsets V_1 and V_2, where the members of V_1 all put z last in their rankings over the triple, and the members of V_2 all put z first in their rankings over the triple.

The assumption of diverse-2 preferences means that for any decisive set V with 2 or more members, there is a triple x, y, and z, and a partition of V, which produces exactly the standard Arrow array shown above. One disadvantage of this particular definition is that one must know what sets of individuals are decisive, before one can say whether preferences are diverse, and to know what sets of individuals are decisive one has to know the social preference relation. Nonetheless, it is a logical definition.

Referring back to example 6 of the previous section, consider persons 2 and 3. Under simple majority rule, which was assumed in the example, they constitute a decisive coalition. The reader is invited to show that the diverse-2 preferences assumption *fails* in this example, because there is no way to define the triple x, y, z so as to get the standard Arrow array, when $V_1 = \{2\}$, $V_2 = \{3\}$, and $V^C = \{1\}$. Therefore the assumption of diverse-2 preferences rules out example 6.

We now make 3 simple observations about the assumptions of diverse-1 preferences, diverse-2 preferences, and Pollak diversity. (For the sake of brevity we omit the obvious proofs.)

Observation 1: Assume $n = 2$. Then diverse-1 preferences implies diverse-2 preferences.

Observation 2: Assume $n = 2$. Assume the (standard) Pareto principle. Then diverse-2 preferences implies diverse-1 preferences.

Observation 3: Assume $n \geq 2$. Pollak diversity implies diverse-1 preferences and diverse-2 preferences.

7. Arrow/Pollak Impossibility Theorem 2, $n \geq 2$.

We now proceed to a proof of our second single-profile Arrow's theorem, which, unlike our first proof, is not restricted to a 2-person society.[2] Although Pollak made a much stronger diversity assumption than we use, and although Parks (1976), Hammond (1976), and Kemp and Ng (1976), preceded Pollak with single-profile Arrow theorems, we will call this the Arrow/*Pollak* Impossibility Theorem, because of the similarity of our proof to his. But first, we need a proposition paralleling proposition 1:

Proposition 2: Assume $n \geq 2$, and NIM. Assume there is a non-empty group of people V and a pair of alternatives x and y, such that xP_iy for all i in V and yP_ix for all i not in V. Suppose that xPy. Then V is decisive.

Proof: Let w and z be any pair of alternatives. Assume wP_iz for all i in V. We need to show that wPz must hold. This follows immediately from NIM. QED.

Arrow/Pollak Impossibility Theorem 2: Assume $n \geq 2$. The assumptions of the Pareto principle, NIM, no dictator, and diverse-2 preferences are mutually inconsistent.

Proof: By the Pareto principle, the set of all individuals is decisive. Therefore decisive sets exist. Let V be a decisive set of minimal size, that is, a decisive set with no proper subsets that are also decisive. We will show that there is only one person in V, which will make that person a dictator. This will establish Arrow's theorem.

Suppose to the contrary that V has 2 or more members. By the diverse-2

[2] We have a similar proof for a multi-profile Arrow's theorem in Feldman & Serrano (2006).

preferences assumption there is a triple of alternatives x, y, and z, and a partition of V into non-empty subsets V_1 and V_2, giving the standard Arrow array as shown above. Since V is decisive, it must be true that xPy. Next we consider the social preference for x vs. z.

Case 1. Suppose zRx. Then zPy by transitivity. Then V_2 becomes decisive by proposition 2 above. But this is a contradiction, since we assumed that V was a decisive set of minimal size.

Case 2. Suppose *not* zRx. Then the social preference must be xPz, by completeness. But in this case V_1 is getting its way in the face of opposition by everyone else, and by proposition 2 above V_1 is decisive, another contradiction. QED.

8. Last Examples.

The reader is invited to revisit examples 2, 3, 4, and 5, using the diverse-2 preferences assumption instead of the diverse-1 assumption. If she does, she will find that the comments surrounding all 4 examples stay the same. That is, in each example, one Arrow assumption is relaxed, and the remaining 3 can co-exist. Example 2 fails both diverse-1 preferences and diverse-2 preferences. Example 3 satisfies diverse-1 preferences, and also satisfies diverse-2 preferences vacuously, because there are no decisive sets. Examples 4 and 5 satisfy both diverse-1 preferences and diverse-2 preferences. In fact, observations 1 and 2 together imply that when n = 2 and the Pareto principle holds, the diverse-1 and diverse-2 assumptions are equivalent, and we have the

Pareto principle in examples 2, 4 and 5. In example 3 we don't have the Pareto principle, but both diversity assumptions are nonetheless satisfied.

Example 7 below shows that the assumption of diverse-2 preferences is not an impossibly strong assumption, when the Pareto principle is assumed, and when there are 3 or more people. In this example, we assume person 1 is a dictator; so the social preferences are exactly his preferences. The decisive sets are {1}, {1,2}, {1,3}, and {1,2,3}. The diverse-2 preferences assumption allows us to ignore the decisive set {1}, because it doesn't have 2 or more members. In the right hand column of the table below we indicate which decisive set is playing the role of the *V* of the assumption, for the corresponding triple of alternatives, which is playing the role of *x*, *y*, and *z*.

Example 7:

Person 1	Person 2	Person 3	Society (1 is Dictator)	*V*
a	*b*	*c*	*a*	{1,2}
b	*c*	*a*	*b*	{1,2}
c	*a*	*b*	*c*	{1,2}
d	*f*	*e*	*d*	{1,3}
e	*d*	*f*	*e*	{1,3}
f	*e*	*d*	*f*	{1,3}
g	*h*	*h*	*g*	{1,2,3}
h	*i*	*i*	*h*	{1,2,3}
i	*g*	*g*	*i*	{1,2,3}

For $V = \{1,2\}$, we let $(x,y,z) = (a,b,c)$, and we see that the array condition for diverse-2 preferences holds. Similarly, for $V = \{1,3\}$, we let $(x,y,z) = (d,e,f)$; and for $V = \{1,2,3\}$, we let $(x,y,z) = (g,h,i)$. Therefore this preference profile satisfies diverse-2 preferences.

9. Conclusions.

We have presented two single-profile Arrow impossibility theorems which are simple and transparent. The first theorem, which requires $n = 2$, relies on a very simple and modest assumption about diversity of preferences within the given preference profile, and on a relatively modest neutrality/independence assumption. The second theorem, which allows $n \geq 2$, uses a substantially more complicated assumption about diversity of preferences within the given profile, and uses a stronger neutrality/independence/monotonicity assumption. Both theorems support the claim that Arrow impossibility happens even if individual preferences about alternatives are given and fixed.

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