ADJUSTING PRICES IN THE MANY-TO-MANY ASSIGNMENT GAME

By

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ABSTRACT

Starting with an initial price vector, prices are adjusted in order to eliminate the demand excess and at the same time to keep the transfers to the sellers as low as possible. In each step of the auction, to which sellers should those transfers be made (minimal overdemanded sets) is the key definition in the description of the algorithm. Such approach was previously used by several authors. We introduce a novel distinction by considering multiple sellers owing multiple identical objects and multiple buyers with a quota greater than one consuming at most one unit of each seller's good. This distinction induces a necessarily more complicated construction of the overdemanded sets than the constructions existing in the literature, even in the simplest case of additive utilities considered here. As the previous papers, our mechanism yields the minimum competitive equilibrium price vector. A procedure to find the maximum competitive equilibrium price vector is also provided.

Key words: matching, stable payoff, competitive equilibrium payoff, optimal stable payoff, lattice

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INTRODUCTION

The many-to-many Assignment game of our title is one of the two many-to-many matching models with additively separable utilities introduced in Sotomayor (1992). We can interprete this model as a market of buyers and sellers. Then, there are two finite and disjoint sets of players: B is a set of buyers and Q is a set of sellers. Buyers are interested in sets of objects owned by different sellers and each seller owns a bundle of identical objects. The quota of a buyer is the number of objects she is allowed to acquire; the quota of a seller is the number of identical objects he owns. For each pair of agents $(b,q)\in BxQ$ there is a number v_{bq} representing the maximum amount of money buyer b considers to pay for an object of seller q. Given the prices of the objects, each buyer demands the sets of items of size less than or equal to her quota, which maximize her additive utility payoff. That is, an element of the demand set of a buyer is one of her most desirable bundles of objects at the current prices.[2] Then a buyer is indifferent among any two elements of her demand set, if the demand set is not a singleton. The natural solution concept is that of *competitive equilibrium payoff*, introduced in Sotomayor (2007). Roughly speaking, the payoff (u,p) is a *competitive equilibrium payoff* if there is a feasible allocation, μ, under which each active buyer receives one of her demanded sets of items at the prices p, every inactive buyer has a zero payoff and every unsold object is priced at zero.

Sotomayor (2007) shows that the set of competitive equilibrium payoffs is non-empty and it is endowed with a complete lattice structure under two convenient partial order relations. Although these partial orders are not defined by the preferences of the players, the extreme points of these lattices have important meaning for the market. They reflect a coincidence of interest among agents on the same side of the market, and a corresponding conflict of interest among agents on opposite sides. All buyers, as well as all sellers, agree on the best competitive equilibrium payoff for them. These outcomes are called B-optimal competitive equilibrium payoff and Q-optimal competitive equilibrium payoff, respectively. In addition, the B-optimal competitive equilibrium payoff (respectively, Q-optimal competitive equilibrium payoff) is the worst competitive equilibrium payoff from the point of view of the sellers (respectively, buyers). That is, the

[2] This definition is slightly changed in the text due to the inclusion of dummy agents in the model.

corresponding payoff vector of the sellers is the minimum (respectively, maximum) competitive equilibrium price vector in the sense that it is smaller (respectively, greater) in each component than any other competitive equilibrium price vector.

The contributions of the present paper are: (i) to providing a simple dynamic procedure of adjusting prices for making the allocation of the objects to the buyers according to a competitive price vector; (ii) to proving that the competitive price vector yielded by this mechanism is the minimum competitive price vector; (iii) to showing that this price vector can be supported by a competitive allocation so that the *B*-optimal competitive equilibrium payoff is obtained, and (iv) to proving that, by reversing the roles between buyers and sellers, the final outcome can be adjusted to yield the *Q*-optimal competitive equilibrium payoff.

The dynamic mechanism roughly works as follows: Given the reservation prices to the sellers announced by the auctioneer, buyers indicate their demand set at the current price vector. If it is possible to satisfy the demand of every buyer, respecting sellers' quotas, then the auction stops. Otherwise, the auctioneer raises the prices of some objects. The technical contribution of this paper is the definition of the set of these objects, which must be selected by the auctioneer at each price vector. In order to choose these sets the auctioneer divides each buyer into *separate agents*, one for each slot in her quota. The auctioneer induces new demands for these separate agents. Each separate agent of a buyer is not an exact copy of the others. The first agent of a buyer is the one who demands one of the most favorite objects (there may exist more than one) of the original buyer; the second agent is the one who demands the second favorite object of the original buyer among the objects that the original buyer demands in the elements of her demand set, and so on. The last agent is the one who demands all the remaining object(s) that the original buyer demands in the elements of her demand set. (A precise and formal definition will be given in the text). The last agent is the most crucial agent in this division process. For each buyer there can be several objects in the demand of the last agent of the buyer.

In view of potential indifferences among the objects, there may be many ways of obtaining the demands of these separate agents . Each one of them induces different demand structures for the separate agents. **The auctioneer chooses a particular one, namely, the one which induces the minimum number of *minimal overdemanded sets*** (definition of a minimal overdemanded set almost follows Demange, Gale and Sotomayor (1986), with exception that the definition needs to be corrected for multiple objects that a seller can supply). The auctioneer increases the prices of the objects in one of the minimal overdemanded sets by one unit (every valuation and price are assumed to be integral values, so 1 is the minimum possible increment). This price adjustment procedure is repeated until there exists a demand structure (obtained from the division of buyers into several agents) such that there are no minimal overdemanded sets. At this point, with the help of Hall's (1935) Theorem[3], each buyer can feasibly be assigned one of the sets of objects in her demand, ending the mechanism at a competitive price.

One of the interesting features of this mechanism is that, in spite of the fact that the auctioneer may face more than one alternative whenever he needs to make a choice, the final price vector is always the same, namely the minimum competitive price vector. The intuition is that by keeping the transfers to the sellers as low as possible, the auction stops as soon as the prices increase sufficiently to become competitive. However, the proof of this result is not so straightforward and requires two technical lemmas.

An important property of the minimum competitive price vector shown here is that there exists a competitive allocation which supports it, so it is the minimum competitive equilibrium price vector. (This property is not true for an arbitrary competitive price vector. To see this, consider one buyer b and two sellers 1 and 2. Every agent has a quota of one and $v_b=(4, 5)$. Price vector $p=(1, 3)$ is competitive. Buyer b demands only the object of seller 1, which is allocated to her. The object of seller 2 is unsold. Price p is not a competitive equilibrium price because the price of the unsold object is not zero. We can also observe that the only competitive allocation assigns the buyer to seller 2 and this allocation does not supports p).

Buyers and sellers are not treated symmetrically in this model when one focus on the competitive equilibria, so if we revert the roles between buyers and sellers in the

[3] Let B and C be two finite disjoint sets. For each b in B, let D_b be a subset of C. A simple assignment is an assignment of C to B, such that each b is assigned exactly one element j of C, such that j is in D_b, and each j in C is assigned to at most one element of B. Then,

THEOREM OF HALL . *A simple assignment exists, if and only if, for every subset B' of B, the number of objects in $D(B')$ is at least as great as the number of buyers in B'.*

mechanism the resulting outcome may not even correspond to a competitive price vector.[4] We show that, in spite of this, this outcome can be used to derive the maximal competitive equilibrium price vector. It can be obtained from the outcome produced by the new mechanism by reducing the price of each of the items of every seller to his minimal individual payoff.

The present article is structured as follows. Section 2 gives the cooperative framework. In section 3 the concepts and terminology needed for the description of the mechanism and for the proofs of the main results are introduced and an illustrative example is provided. The main results are stated in Section 4. Section 5 gives some final remarks and discusses related works. All proofs are presented in the Appendices.

2.THE FRAMEWORK

The sets of buyers and sellers are denoted by B and Q, respectively. The set B has m elements and the set Q has n elements. Generically we will denote buyers by b, b', and sellers and objects by q, q'. Each $q \in Q$ has a quota $s(q)$ and each $b \in B$ has a quota $r(b)$, representing the maximum number of partnerships they can form. Quota $s(q)$ of seller q means that q owns $s(q)$ identical and indivisible objects, and quota $r(b)$ of buyer b represents the maximum number of objects buyer b is allowed to buy. Without loss of generality we can consider $r(b) \leq n$ and $s(q) \leq m$. No buyer is interested in acquiring more than one item of a given seller.

Every object has a reservation price of 0 *(which can be obtained after normalization)*. For each pair (b,q) there is a non-negative number $v_{bq} \geq 0$ which does not depend on which other partnerships are formed by buyer b and seller q. (We call this condition *separability across the pairs*). This number can be interpreted as the value of any object of seller q to buyer b. That is, v_{bq} is the gain of trade when any of the objects of seller q is sold to buyer b. If buyer b acquires some object of q at price π then b receives the individual payoff $u_{bq} \equiv v_{bq} - \pi$. Dummy players, denoted by 0, will be available in both sides of the market to fill the quotas of the non-dummy (real) players. We have that

[4] This is not the case if the focus is stability. Under this setting the resulting outcome is the optimal stable payoff for the buyers, which coincide with the *B*-optimal competitive equilibrium payoff. By reversing the roles between buyers and sellers in the mechanism we get the optimal stable payoff for the sellers, which may be different from the *Q*-optimal competitive equilibrium payoff.

$v_{b0}=v_{0q}= 0$ for all $b \in B$ and $q \in Q$. We will also include an artificial "null-object", 0, owned by the artificial seller, whose value is zero to all buyers and whose price is always zero. The many-to-many Assignment game is given by (B,Q,v,r,s), where v is the matrix of the numbers v_{bq}'s and r and s denote the array of quotas for the buyers and sellers, respectively.

We will say that a subset $S \subseteq Q$ is an **allowable set of partners for** $\boldsymbol{b \in B}$**,** if $|S| = r(b)$. For technical convenience, we will extend this terminology to include the sets S with k non-dummy sellers and $r(b)-k$ repetitions of the dummy seller for $0 \leq k \leq r(b)$. Analogously we define an **allowable set of partners for** $\boldsymbol{q \in Q}$**.** For an abuse of notation we will also write $S \subseteq B$ or $S \subseteq Q$ for any allowable set S of B-players or Q-players, respectively. **An allowable set of objects for buyer** $\boldsymbol{b}$ **contains** $\boldsymbol{r(b)}$ **objects**, some of which may be repetitions of the null-object. Furthermore, it does not contain more than one object belonging to the same seller (an exception is made to the fictitious seller).

A. *Utility functions*

We are considering the simplest many-to-many Assignment game, namely the model in which the agents have *additively separable utilities.* This condition means that for every coalition $T=R \cup S$, $R \subseteq B$ and $S \subseteq Q$, the payoff $v(R \cup S)$ of the coalition $R \cup S$ is given by

(1) $\boldsymbol{v(R \cup S) = max\{\sum_{b \in R, \mu(b) \in S} v_{b\mu(b)}\}}$**, over all feasible matchings** $\boldsymbol{\mu}$ **.**

Consequently, for all $S \subseteq Q$ with $|S| \leq r(b)$ and for all $R \subseteq B$ with $|R| \leq s(q)$,

(2) $\boldsymbol{v(b \cup S) = \sum_{q \in S} v_{bq}}$ **and** $\boldsymbol{v(q \cup R) = \sum_{b \in R} v_{bq}}$**.**

The number $v(b \cup S)$ also defines the value of the allowable set S of objects to buyer b.

One of the characteristics of the additively separable utility function is that if a buyer demands a set A of objects at prices p and some of these objects have their prices raised, then the buyer will continue to want to buy the objects in A whose prices were not changed. That is, the function $v(\{b\} \cup A)$ over all allowable sets A of partners for b satisfies the *gross substitute condition.*

Under *additively separable utilities* buyers can be interpreted as brokers, rather than as final consumers. Each buyer b has in hand an offer from a client who will purchase any bundle of goods of size less than or equal to $r(b)$ from the buyer and will pay v_{bq} for each unit q in the bundle, should she obtain them in the market. Since each buyer b knows that she can earn v_{bq} (but no more) by reselling the object q to her client, the buyer will not buy at a higher price. So for each object q that is sold to buyer b at a price p_q buyer b earns $v_{bq} - p_q$, and so, by reselling an allowable set of objects S she earns $\sum_{q \in S} (v_{bq} - p_q)$.

B. *Matching and allocation*

A ***feasible matching*** μ is a function that maps every agent to an allowable set of partners for him/her such that b is in $\mu(q)$ if and only if q is in $\mu(b)$, for every $(b,q) \in BxQ$. It is described by a set of partnerships between buyers and sellers of the kind (b,q), $(b,0)$ or $(0,q)$, for $(b,q) \in B$xQ, such that each b forms $r(b)$ partnerships and each q forms $s(q)$ partnerships. If b and q are matched under μ, we write $b \in \mu(q)$ or $q \in \mu(b)$. A dummy player may be matched to more than one player of the opposite side and more than once to the same player. If an agent forms all his/her partnerships with a dummy agent we say that he/she is unmatched.

The value of μ is $\sum_{q \in Q, b \in \mu(q)} v_{bq}$. The matching μ is **optimal** if it attains the maximum value among all feasible matchings.

A **feasible allocation** allocates each non-null object to one buyer (who might be the dummy buyer) so that each non-dummy buyer is assigned an allowable set of objects for her. If an object is allocated to the dummy buyer we say that it is **left unsold.** Of course, the dummy buyer may be assigned to any number of objects and the null object may be allocated to any number of buyers.

If an object is allocated to a buyer then the seller who owns this object is matched to that buyer. Thus, if μ^* is a feasible allocation, we can define a **corresponding matching**

μ such that seller $q \in \mu(b)$ if and only if one of his objects is allocated to b under μ^*[5]. We say that μ and μ^* **correspond** to each other. Clearly, μ and μ^* have the same value, so $\boldsymbol{\mu}$ **is an optimal matching if and only if** $\boldsymbol{\mu^*}$ **is an optimal allocation.**

C. *Stable payoffs and competitive equilibrium payoffs.*

For this model we have two natural solution concepts: the concept of stability and the concept of competitive equilibrium payoff. When the focus of attention is stability, buyers and sellers are treated symmetrically. The feasible outcomes are defined as follows:

DEFINITION 1: *A feasible outcome, denoted by* $(u,w;\mu)$, *is a pair* (u,w) *of the agents' individual payoffs, called feasible payoff, and a feasible matching* μ. *The individual payoffs of each* $b \in B$ *and* $q \in Q$ *are given by the arrays of numbers* u_{bq}, *with* $q \in \mu(b)$, *and* w_{bq}, *with* $b \in \mu(q)$, *respectively, with* $u_{bq} + w_{bq} = v_{bq}$, $u_{bq} \geq 0$ *and* $w_{bq} \geq 0$. *Consequently,* $u_{b0} = u_{0q} = w_{b0} = w_{0q} = 0$ *in case these payoffs are defined. The matching* μ *is said to be compatible with the feasible payoff* (u,w) *and vice-versa.*

If $(u,w;\mu)$ is a feasible outcome and μ^* is a feasible allocation corresponding to μ then we also refer to $(u,p;\mu^*)$ as a feasible outcome.

It was proved in Sotomayor (1992,2007) that the concept of stability is equivalent to pairwise-stability for this model. For this definition we will use the following notation: Given a feasible payoff (u,w), $\boldsymbol{u_b(min)}$ **is the smallest individual payoff of buyer** $\boldsymbol{b}$**;** $\boldsymbol{w_q(min)}$ **is the smallest individual payoff of seller** $\boldsymbol{q}$**.** Then,

DEFINITION 2: *The feasible outcome* $(u,w;\mu)$, *is* ***stable*** *if* $u_b(min) + w_q(min) \geq v_{bq}$ *for all pairs* (b,q) *with* $q \notin \mu(b)$.

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[5] If μ^* is a feasible allocation and a buyer buys an object from a seller, then make a link between the two. The resulting graph is the corresponding feasible matching μ. Conversely, if μ is a feasible matching and a buyer buys an object from a seller, then make a link between the buyer and the object. The resulting graph is the corresponding feasible allocation μ^*.

If this condition is not satisfied for some pair (b,q), we say that the pair *destabilizes* the outcome $(u,w;\mu.)$

The concept of competitive equilibrium payoff is closely related to the traditional concept of equilibrium from standard microeconomic theory. It is natural under the asymmetric approach where buyers demand the objects taking the prices as given. In this context, a vector of prices $p \in R^{N}{}_{+}$, with $N \equiv \sum_{q \in Q} s(q)$, is called a **feasible price vector** or **price vector**, for short**.** That is, a feasible price vector specifies a non-negative price for each object. We denote the price of object q under the price vector p by p_q.

Given a price vector p, a buyer has preferences over allowable sets of objects that are completely described by the numbers v_{bq}*'s* : For any two allowable sets of objects S and S', buyer b prefers S to S' at prices p if $\sum_{q \in S} (v_{bq}-p_q) > \sum_{q \in S'} (v_{bq}-p_q)$. She is indifferent between these two sets if $\sum_{q \in S} (v_{bq}-p_q) = \sum_{q \in S'} (v_{bq}-p_q)$. Object q is **acceptable** to buyer b at prices p if, $v_{bq}-p_q \geq 0$.

Under the structure of preferences we are assuming, each buyer b is able to determine which allowable sets of objects she would most prefer to buy at a given price vector p. We denote the set of all such allowable sets by $\boldsymbol{D_b(p)}$ and call it the **demand set of *b* at prices *p*.** (Note that $D_b(p)$ is never empty, because there is always the option of buying the allowable set with $r(b)$ repetitions of the null object. Note also that, if $S \in D_b(p)$, then every element of S is acceptable to b).

DEFINITION 3: *The outcome $(u,p;\mu^*)$ is a **competitive equilibrium outcome** if (i) it is feasible, (ii) μ^* is a feasible allocation such that, if $\mu^*(b)=S$ then $S \in D_b(p)$ for all $b \in B$ and (iii) $p_q=0$ if $\mu^*(q)=0$.*

If $(u,p;\mu^*)$ is a competitive equilibrium outcome we say that (u,p) is a **competitive equilibrium payoff,** (p,μ^*) is a **competitive equilibrium** and p is a **competitive equilibrium price** or an **equilibrium price** for short**.**

If there is an allocation μ^* satisfying condition (ii) of Definition 3, we say that p is a **competitive price vector.** The allocation μ^* is said to be **compatible** with the competitive price p. The allocation μ^* is called **competitive** if it is compatible with a competitive equilibrium price. It is proved in Sotomayor (2007) that μ^* is an optimal allocation if and only if it is competitive.

The two solution concepts capture some of the fundamental dissimilarities between the symmetric and the asymmetric approaches: Under a stable payoff the sellers can discriminate the buyers. However, **every seller sells all of his items for the same price under a competitive equilibrium payoff. (**If a seller has two identical objects, q and q', and $p_q > p_{q'}$ for some price vector p, then no buyer b will demand, at prices p, a set S of objects that contains object q. This is because, by replacing q with q' in S, b gets a more preferable allowable set of objects. But then, q will remain unsold with a positive price, which violates condition *(iii)* of Definition 3).

DEFINITION 4: *A stable (respectively, competitive equilibrium) payoff is called the B-optimal stable (respectively, competitive equilibrium) payoff if it gives to each player in B the highest total payoff among all stable (respectively, competitive equilibrium) payoffs. Similarly we define the Q-optimal stable (respectively, competitive equilibrium) payoff.*

REMARK 1: When every seller sells his identical objects for the same feasible price, we can identify a seller with any of his objects. Thus, we do not cause any confusion by using the same notation for a seller and for any of his objects. Then, if S is an allowable set of objects for buyer b we can use the same notation S to mean the set of owners of these objects, and so this set can also be referred as an allowable set of partners for buyer b. Under this identification, if b demands the allowable set S of objects at prices p we will say that b also demands the allowable set S of sellers at prices p. The notation $D_b(p)$ stands for the demand set of buyer b *for objects* at prices p as well as for the demand set of buyer b *for sellers* at prices p.

Under this observation, if μ^* is a feasible allocation and μ is its corresponding matching, $q \in \mu^*(b)$ means that object q is allocated to buyer b (there is only one object q belonging to seller q allocated to buyer b), and $q \in \mu(b)$ means that buyer b and seller q are partners at μ.. On the other hand, since the array of payoffs for any seller q is given by the array of prices of his objects, then, in order to represent the array of the $s(q)$ identical individual payoffs for any seller q, we need not make any reference to the buyers who are matched to q. For example, $(p_q, p_q, \ldots, p_q)$ denotes the array of payoffs of seller q and p_q denotes the price of any of his objects.■

3. THE MECHANISM

In order to describe our price adjusting mechanism for the many-to-many Assignment game (B,Q,v,r,s) we need a few more notations and definitions.

A. Demand structure for a buyer.

In every step of the mechanism all the objects of a seller have the same price. Then, by Remark 1, we can identify the seller with any of his objects. For every buyer b, each pair (b,i), $i=1,\ldots,r(b)$, is called ***b-agent***. We will provide a demand structure for b at prices p which specifies a demand set for every b-agent.

Given a price vector p, the demand set of buyer b for sellers may have several allowable sets C_i's of **partners** for b. If $|D_b(p)|=k$, set

(3) $D_b(p) \equiv \{C_1,\ldots,C_k\}$ and $B_b(p) \equiv \cup C_i$, $i=1,\ldots,k$.

Therefore, $\boldsymbol{q \in B_b(p)}$ **if and only if the number of sellers strictly preferred to** $\boldsymbol{q}$ **by** $\boldsymbol{b}$ **at prices** $\boldsymbol{p}$ **is less than** $\boldsymbol{r(b)}$**.** Of course, the number of elements listed in $B_b(p)$ (this set must include all repetitions of the dummy seller that appear in the allowable sets C_i's,, if any) is greater than or equal to $r(b)$. Also, if $q \in B_b(p)$ and b weakly prefers q' to q at prices p, then $q' \in B_b(p)$.

Now, each b-agent (b,i) demands the set of sellers $A_{b,i}(p)$ defined as follows:

(4) $A_{b,1}(p)=\{q\}$, for some q such that $v_{bq}-p_q \geq v_{bq'}-p_{q'}$, $\forall q' \in B_b(p)$;

$A_{b,i}(p)=\{q\}$, for some q such that $v_{bq}-p_q \geq v_{bq'}-p_{q'}$, $\forall q \in B_b(p)-[A_{b,1}(p) \cup \ldots \cup A_{b,i-1}(p)]$, $\forall i=2,\ldots,r(b)-1$;

$A_{b,r(b)}(p)= B_b(p) - [A_{b,1}(p) \cup \ldots \cup A_{b,r(b)-1}(p)]$.

Therefore, $A_b(p)=\{A_{b,1}(p),.., A_{b,r(b)}(p)\}$ is a partition of $B_b(p)$. It will be called a **demand structure for b at prices p.** Due to the potential indifferences among the sellers, there may be more than one way of obtaining the demands of the *b*-agents. Consequently we may have different demand structures for *b* at prices *p.* The set of all $A_b(p)$'s will be called a **demand structure at p** and will be denoted by $A(p)$.

In order to illustrate these definitions suppose that $D_b(p)=\{\{q_1,q_2,0\},\{q_1,0,0\},\{q_2,0,0\},\{0,0,0\}\}$. Then $B_b(p)=\{q_1,q_2,0,0,0\}$. Observe that in this case *b* is indifferent between any two elements of $B_b(p)$. A demand structure for *b* at prices *p* is: $A_{b,1}(p)=\{0\}$, $A_{b,2}(p)=\{0\}$ and $A_{b,3}(p)=\{q_1,q_2,0\}$; another demand structure for *b* at prices *p* is: $A'_{b,1}(p)=\{q_1\}$, $A'_{b,2}(p)=\{0\}$ and $A'_{b,3}(p)=\{q_2,0,0\}$, and so on.

From (4) it follows that, at prices *p*, *b* likes $A_{b,i}(p)$ as well as $A_{b,i+1}(p)$ for all i=1,…,r(b)-1. It is worth to point out that, at prices *p*, ***b*** **is indifferent between any two sellers from** $\boldsymbol{A_{b,r(b)}(p)}$ when this set is not a singleton. In fact, if there are two sellers *q* and *q'* in $A_{b,r(b)}(p)$ such that *b* prefers *q* to *q'* at prices *p,* then *b* strictly prefers the single seller in $A_{b,i}(p)$ to q', for all $i=1,…,r(b)-1,$ so the number of sellers strictly preferred to *q'* by *b* at prices *p* is greater than or equal to $r(b)$, and so *q'* is not in $B_b(p)$ as remarked before, which is absurd.

B. *Overdemanded set.*

We now introduce the main concept that will be used in the description of our mechanism, which is that of *overdemanded set for a given demand structure.* Given a demand structure $A(p)$,

(5) *b-agent (b,i) is* ***a loyal demander*** *of S if* $A_{b,i}(p)\subseteq S$.

DEFINITION 5: *Given the feasible price vector* $p\in R^n_+$ *we will say that the set* $S\subseteq Q$ *is* ***overdemanded*** *for the demand structure* $A(p)$, *if there is a set T of loyal demanders of S, such that* $|T|>\sum_{q\in S}\sigma(q)$, *where* $\sigma(q)=\min\{s(q), \text{number of } (b,i)\in T \text{ with } q\in A_{b,i}(p)\}$.

The overdemanded set S is said to be **minimal,** if no proper subset of S is overdemanded. Thus, suppose for example, that b has quota $r(b)=1$ and $A_{b,1}(p)=\{q_3,q_4\}$; b' has quota $r(b')=2$ and $A_{b',1}(p)=\{q_3\}$ and $A_{b',2}(p)=\{q_4\}$; and b'' has quota $r(b'')=2$ and $A_{b'',1}(p)=\{q_1\}$, $A_{b'',2}(p)=\{q_3\}$. Then, $T=\{(b,1),(b',1),(b',2),(b'',2)\}$ is a set of loyal demanders of $S=\{q_3,q_4\}$. If $s(q_3)=1$ and $s(q_4)=3$ then $\sigma(q_3)=min\{1,3\}=1$ and $\sigma(q_4)=min\{3,2\}=2$. Then $4=|T|>3=\sigma(q_3)+ \sigma(q_4)$. Set S is overdemanded, but it is not minimal. In fact, the set $S'=\{q_3\}$ is overdemanded by $T'=\{(b',1),(b'',2)\}$. Indeed, S' is a minimal overdemanded set.

It is not hard to see that a set of sellers S is minimal overdemanded if and only if the set of all objects of the sellers in S is minimal overdemanded. This is because for all $q \in S$, the number of (b,i)'s, loyal demanders of S with $q \in A_{b,i}(p)$, is strictly greater than $s(q)$, so $\sigma(q)=s(q)$. Therefore, our mechanism is able to operate if we change the vector of prices in R^n by their extension in R^N_+.

REMARK 2. It follows from the definition of competitive equilibrium that, if $p \in R^n_+$ is a competitive equilibrium price then each buyer b can be matched to her most preferred allowable set of sellers at prices p (this may include repetitions of the dummy-seller). Therefore, there is some demand structure $A(p)$ for which each pair (b, i) can be matched to exactly one seller q, with $q \in A_{b,i}(p)$ (q might be the dummy-seller). In addition, every non-dummy seller q is matched $s(q)$ times at most. Hence, **by Hall's theorem, there is no overdemanded set for $A(p)$.**

If $p \in R^n_+$ is not a competitive equilibrium price, then there is no way to match each buyer to her $r(b)$ most preferred sellers under a feasible matching. Again, **by Hall's theorem, every demand structure $A(p)$ has an overdemanded set.** ■

C. Description of the price adjusting mechanism for (B,Q,v,r,s).

We now can describe how the prices will be adjusted along the steps of the mechanism. We will take all prices and valuations to be integers. Then,

Step (1): The auctioneer announces an initial price vector, $p(1)= (0,...,0) \in R^n_+$. Each buyer b "bids" by announcing $D_b(1) \equiv D_b(p(1))$.

Step (t+1): After bids $D_b(t)$ are announced, the auctioneer determines all the demand structures at $p(t)$. If there is some demand structure $A(t)\equiv A(p(t))$, for which it is possible to match each b-agent (b, i) to a seller $q\in A_{(b,i)}(t)$, so that no real seller is matched more times than his quota, the algorithm stops. **If no such demand structure exists, Hall's Theorem implies that there is some overdemanded set for every demand structure.** Then, the auctioneer chooses some demand structure that has the **minimum number of minimal overdemanded sets**, among all demand structures. Next, he selects a minimal overdemanded set for the demand structure chosen and raises the price of all objects belonging to each seller in the set by one unit. All other prices remain at level $p(t)$. This defines $p(t+1)$.

It is clear that the algorithm stops at some step, because, as soon as the price of the objects of a given seller becomes higher than any buyer's valuation for them, the seller will not be in the bid of any buyer. It follows from the construction of the algorithm that the final price is a competitive price vector. What is less clear is that this algorithm yields the same price, independent of the demand structures selected by the auctioneer. We will prove this fact in section 4, by showing that the price obtained in the algorithm is the minimum equilibrium price vector. Before, we will illustrate the mechanism with an example.

D. Example

The following example illustrates the price adjusting mechanism. There are four non-dummy-buyers, 1, 2, 3 and 4, and six non-dummy sellers, q_1, q_2, ..., q_6. Seller q_1 has two identical objects and the other sellers have each only one object. The maximum number of objects that each buyer can purchase is given by 3, 2, 1 and 1, respectively. These numbers define the quotas of the buyers. The values of the buyers to the non-null objects are given by the following vectors: v_1=(4,3,3,3,1,1), v_2=(2,2,1,0,1,1), v_3=(2,0,0,0,0,2) and v_4=(1,0,1,1,1,2), where the j-th coordinate of v_i is the value of any object of seller q_j to buyer i.

Step 1. $p(1)$=(0,0,...,0). The matrix of surpluses $(v_{bq}-p_q(1))$ is given in the table below. For each row the entries corresponding to the sellers belonging to the demand set of the corresponding buyer are in boldface .

	q_1	q_2	q_3	q_4	q_5	q_6	0
1	**4**	**3**	**3**	**3**	1	1	0
2	**2**	**2**	1	0	1	1	0
3	**2**	0	0	0	0	**2**	0
4	1	0	1	1	1	**2**	0

Then, $B_1(1)=\{q_1,q_2,q_3,q_4\}$; $B_2(1)=\{q_1,q_2\}$; $B_3(1)=\{q_1,q_6\}$; $B_4(1)= \{q_6\}$. There are three demand structures. The first one is: $A_{1,1}(1)=\{q_1\}$, $A_{1,2}(1)=\{q_2\}$, $A_{1,3}(1)=\{q_3,q_4\}$; $A_{2,1}(1)=\{q_1\}$, $A_{2,2}(1)=\{q_2\}$, $A_{3,1}(1)=\{q_1,q_6\}$ and $A_{4,1}(1)=\{q_6\}$. It is not possible to find a competitive matching. There are two minimal overdemanded sets: $\{q_2\}$ and $\{q_1,q_6\}$. The second demand structure is: $A'_{1,1}(1)=\{q_1\}$, $A'_{1,2}(1)=\{q_3\}$, $A'_{1,3}(1)=\{q_2,q_4\}$, $A'_{2,1}(1)=\{q_1\}$, $A'_{2,2}(1)=\{q_2\}$, $A'_{3,1}(1)=\{q_1,q_6\}$ and $A'_{4,1}(1)=\{q_6\}$. The only minimal overdemanded set is $\{q_1,q_6\}$. The third demand structure is given by: $A''_{1,1}(1)=\{q_1\}$, $A''_{1,2}(1)=\{q_4\}$, $A''_{1,3}(1)=\{q_2,q_3\}$, $A''_{2,1}(1)=\{q_1\}$, $A''_{2,2}(1)=\{q_2\}$, $A''_{3,1}(1)=\{q_1,q_6\}$ and $A''_{4,1}(1)=\{q_6\}$. As before, it is not possible to find a competitive matching. The only minimal overdemanded set is $\{q_1,q_6\}$. The auctioneer must choose a demand structure with the **minimum number of minimal overdemanded sets.** Suppose the auctioneer chooses A'. As a result, he raises the price of all objects of q_1 and q_6 by one unit.

Step 2. $p(2)=(1,0,0,0,0,1,0)$. The matrix of surpluses $(v_{bq}-p_q(2))$ is given in the table below. For each row the entries corresponding to the sellers belonging to the demand set of the corresponding buyer are in boldface .

	q_1	q_2	q_3	q_4	q_5	q_6	0
1	**3**	**3**	**3**	**3**	1	0	0
2	**1**	**2**	**1**	0	**1**	0	0
3	**1**	0	**0**	**0**	**0**	**1**	**0**
4	**0**	0	**1**	**1**	**1**	**1**	0

Then, $B_1(2)=\{q_1,q_2,q_3,q_4\}$; $B_2(2)=\{q_2,,q_1,q_3,q_5\}$; $B_3(2)=\{q_1,q_6\}$ and $B_4(2)=\{q_3,q_4,q_5,q_6\}$. There are several demand structures. Under $A_{1,1}(2)=\{q_1\}$, $A_{1,2}(2)=\{q_2\}$, $A_{1,3}(2)=\{q_3,q_4\}$; $A_{2,1}(2)=\{q_2\}$, $A_{2,2}(2)=\{q_1,q_3,q_5\}$; $A_{3,1}(2)=\{q_1,q_6\}$ and $A_{4,1}(2)=\{q_3,q_4,q_5,q_6\}$, for example, it is not possible to find a competitive matching and the minimal overdemanded set is $\{q_2\}$. However, under $A'_{1,1}(2)=\{q_1\}$, $A'_{1,2}(2)=\{q_3\}$, $A'_{1,3}(2)=\{q_2,q_4\}$ and $A'_{2,1}(2)=\{q_2\}$, $A'_{2,2}(2)=\{q_1,q_3,q_5\}$; $A'_{3,1}(2)=\{q_1,q_6\}$ and $A'_{4,1}(2)=\{q_3,q_4,q_5,q_6\}$, there is a competitive matching that matches buyer 1 to $\{q_1,q_3,q_4\}$, buyer 2 to $\{q_1,q_2\}$, buyer 3 to q_6 and buyer 4 to q_5. Therefore, the final price is (1,0,0,0,1,0).■

4. MAIN RESULTS

The first theorem states that the price vector p produced by the price adjusting mechanism is the minimum competitive price.

Theorem 1. *Let p be the price vector produced by the price adjusting mechanism. Let p' be any competitive price vector. Then, $p_q \leq p'_q \;\; \forall q \in Q$.*

The idea of the proof of this theorem is the following. If p is not the minimum competitive price vector then there is some competitive price vector y such that $p\neq y$ and p is not smaller than y. In such a case there exists at least one seller who gets a final price for his objects higher than the price given by y. On the other hand, we have that $p(1)=(0,...,0)$, so $p(1)\leq y$. Then, since we are working with all integers, there is at least one-step t of the auction in which $p_q(t)=y_q$ for some q with $p_q > y_q$. By choosing t such that t is the first such step we still have $p(t) \leq y$, because if there would be some q^* such that $p_{q^*}(t)>y_{q^*}$ this would contradict the choice of t. At this point Lemma 2 implies that **the auctioneer will never raise the price of the objects owned by q at any step further, which is a contradiction.**

Lemma 2 is a key result which uses in its proof a very technical lemma, Lemma 1, which in its turn has a very long proof.

Theorem 2 states that the final matching μ can be chosen so that (p,μ) is the minimum competitive equilibrium.

Theorem 2. *If p is the minimum competitive price vector then it is the minimum competitive equilibrium price vector.*

For the proof of this theorem we argue that, since p is competitive then there exists some matching μ compatible with p. If p is not an equilibrium then there must exist some seller who owns an object, with a positive price, which is unsold at μ The fact that p is the smallest price vector among all competitive price vectors allows us to construct a procedure for altering the matching so as to allocate this object to a non-dummy buyer. We can repeat this procedure until obtaining a matching compatible with p in which every unsold object is zero priced.

According to Theorem 2, the outcome produced by the mechanism allocates the sellers to the buyers according to the B-optimal competitive equilibrium payoff. By Proposition 1 below, whose proof can be seen in Sotomayor (2007), under the B-optimal stable payoff no seller discriminates the buyers. Then, the B-optimal stable payoff is competitive. It then easily follows that it coincides with the B-optimal competitive equilibrium payoff.

Proposition 1. *Let (u,w) be the B-optimal (respectively, Q-optimal) stable payoff for M. Let μ be an optimal matching. Then, $w_{bq}=w_{b'q}$ for all $q\in Q$ and all b and b' in $\mu(q)$ (respectively $u_{bq}=u_{bq'}$ for all $b\in B$ and all q and q' in $\mu(b)$).*

As remarked in foot-note (4), buyers and sellers are treated symmetrically in this model when we focus on the stable payoffs. Then, if we revert the roles between buyers and sellers in the mechanism **we get the *Q*-optimal stable payoff. This outcome need not be competitive,** as it is illustrated by an example in Sotomayor (2007). However we can obtain the Q-optimal competitive equilibrium payoff by using Proposition 2 below, from Sotomayor (2007):

Proposition 2. *Let* $(u,w;\mu)$ *be a Q-optimal stable outcome. Construct the payoff* (u',w') *such that the payoffs of a seller* q *are given by a vector with* $s(q)$ *repetitions of the number* $w'_q=w_q(min)$*, and the* u'_{bq}*'s are given by* $u'_{bq}=v_{bq}-w'_q$ *if* $q\in\mu(b)$*. Then,* (u',w') *is the Q-optimal competitive equilibrium payoff.*

5. FINAL REMARKS AND RELATED LITERATURE

A. *Final remarks*

We considered the many-to-many Assignment game introduced in Sotomayor (1992). In Sotomayor (2007), the relationship between the optimal stable payoffs and the corresponding optimal competitive equilibrium payoffs for this model is established: The *B*-optimal stable payoff equals the *B*-optimal competitive equilibrium payoff, but the *Q*-. optimal competitive equilibrium payoff may be different from the *Q*-optimal stable payoff. The *Q*-optimal competitive equilibrium payoff can be obtained from the *Q*-optimal stable payoff by reducing the price of each of the items of every seller to his minimal individual payoff. (Propositions 1 and 2).

In the present paper we provided a dynamic mechanism to finding such optimal competitive equilibrium payoffs. We proved that the mechanism operates in a finite number of steps, and converges to the extreme point of the lattice of stable payoffs favored by players on the offer-making side of the market. The two extreme points of the lattice of competitive equilibrium payoffs can then be obtained via the application of the results of Sotomayor (2007) mentioned above.

The intuition behind the mechanism is quite simple. At any step, transfers are made to some set of sellers. Since a transfer is made to a seller by raising the prices of all his objects equally by the same increment, then, at the end of the mechanism, the identical objects of any seller are sold for the same price. In order to yield the minimum price vector that clears the market the transfers to the sellers were kept as low as possible. Since all prices and values are integers, the increment in each step was conveniently taken as one unit.

This idea has been previously explored in several papers on multi-unit auctions existing in the literature, where the key concept in designing these auction mechanisms has been that of minimal overdemanded set. (We can cite Demange, Gale and Sotomayor (1986) and Gul and Stacchetti (2000), among others). In despite of the similarities found with several of these procedures, our approach reveals a novel distinction: The fact that there are multiple sellers with multiple identical objects, that each buyer can consume at most one unit of each seller's good and may have a quota greater than one. Even using additive utility functions this distinction induces a necessarily more complicated construction of the overdemanded sets than the constructions existing in the literature. This complexity reveals why and in which ways our model is distinct. For example, in Demange *et al* each buyer can consume at most one object and each seller owns at most one object. For this model, as well as for the case in which the sellers own multiple identical objects and the buyers have a quota of one, it is very simple to construct an over-demanded set: There is only one demand structure in each step of the algorithm given by the demand set of every buyer. Then *a set* S *of objects is over-demanded if there is a set* T *of buyers, loyal demanders of* S*, such that* $|T|>\sum_{q\in S}\sigma(q)$*, where* $\sigma(q)$ *is the minimum between* q*'s quota and the number of buyers in* T *who have* q *in their demand set.* In the generalization of the ascending bid auction of Demange *et al* presented by Gul *et al* the buyers have gross substitutes quasi-linear utility functions. Buyers consume a bundle of objects, all objects are distinct and they are all owned by the same seller. Thus, if the buyers in our model could consume as many of a seller's good as they want up to their consumption capacity, then one could apply Gul and Stacchetti directly, as if one had only one seller. Also if the agents could be divided into smaller pieces, by creating repetitions of a buyer that can consume at most one good and repetitions of a seller that own one object at most, (which might be incorrectly suggested by our additive utility assumption), one could apply Demange, Gale and Sotomayor (1986) auction. Nevertheless, both approaches would not work in our case, since it would create possibilities for a buyer to consume more than one unit of a seller's good. Instead we found a different way of defining the overdemanded set.

This paper was not intended to provide a general auction mechanism, instead it shows that even the simplest case of additive utilities can rise a very complex mechanism.

The understanding of its similarities and dissimilarities with the existing mechanisms can help the understanding of the limitations that should be taken into account in the design of new mechanisms under more general utility functions.

B. *Some related literature*

Competitive equilibria have been used by several authors, besides those cited along this paper, to produce allocations with desirable properties of fairness and efficiency. There is by now a vast theoretic literature about two-sided matching markets, providing mechanisms to produce such allocations. For the buyer-seller market game proposed by Shapley and Shubik, Sotomayor (2002a) proposes a new descending bid method for auctioning multiple objects, which generalizes the Dutch auction and produces the maximum competitive equilibrium price. Demange, Gale and Sotomayor (1986) also consider a second auction mechanism which approximates the minimum competitive equilibrium price to any desired degree of accuracy. For the same model, Perez-Castrillo and Sotomayor (2002b) analyze a two-stage mechanism that produces the maximum equilibrium price vector when buyers and sellers play subgame perfect equilibrium strategies. For the case in which the utility functions are piecewise linear, Alkan (1988) presents a dynamic mechanism that finds an equilibrium price in finitely many steps and approximates an equilibrium price for general continuous utilities. Sotomayor (1992) presents a procedure to obtain the optimal stable payoffs of the many-to-many case which consists in solving three linear programming problems.

APPENDIX I.

In this section we will demonstrate the results stated in section 4. In the proof of Theorem 1 we argue by contradiction that p is not the minimum competitive price vector. Then, there is some competitive price vector y such that $p\neq y$ and p is not smaller than y. We have that $p(1)=(0,...,0)$, so $p(1)\leq y$. For each step t of the auction we define

$$\boldsymbol{U(t)\equiv\{q\in Q;\ p_q(t)=y_q\}}.$$

The proof of Theorem 1 uses lemmas 1 and 2 below. For the statement of these lemmas we need some more notation. Since we are working with all integers, there is at least one-step t of the auction such that $\boldsymbol{U(t)\neq\phi}$ **and** $\boldsymbol{p(t)\leq y}$**.** From the competitivity of y it follows that there is some demand structure for y with no overdemanded set. Choose one of such demand structures and call it $A^*(y)$. Let $A(t)$ be any demand structure at step t. Now set

(P1) $\boldsymbol{B_b(t)\equiv F(t)\cup G(t)}$**,** where $G(t)\neq\phi$, $F(t)\cap G(t)=\phi$, $(\forall q,q'\in G(t))\ v_{bq}-p_q(t)=v_{bq'}-p_{q'}(t)$ and $(\forall q\in F(t)$ and $\forall\ q'\in G(t))\ v_{bq}-p_q(t)>v_{bq'}-p_{q'}(t)$.

Since $G(t)\neq\phi$ and $F(t)$ is the set of objects which are strictly preferred by b to any object of $G(t)$ then $|F(t)|<r(b)$. From (4) it follows that $\boldsymbol{|A_{b,j}(t)|=1}$ for all $j\neq r(b)$**.** It is then clear that, for all $j\neq r(b)$, **either** $\boldsymbol{A_{b,j}(t)\subseteq F(t)}$ **or** $\boldsymbol{A_{b,j}(t)\subseteq G(t)}$**. Since** $\boldsymbol{b}$ **is indifferent between any two sellers from** $\boldsymbol{A_{b,r(b)}(t)}$ when this set is not a singleton, we also have that $\boldsymbol{A_{b,r(b)}(t)\subseteq G(t)}$**.**

Also set

(P2) $\boldsymbol{B_b(y)\equiv F(y)\cup G(y)}$**,** $G(y)\neq\phi$, $F(y)\cap G(y)=\phi$, $(\forall q,q'\in G(y))\ v_{bq}-y_q=v_{bq'}-y_{q'}$ and $(\forall q\in F(y)$ and $\forall\ q'\in G(y))\ v_{bq}-y_q>v_{bq'}-y_{q'}$.

By using the same argument as before, for all $j\neq r(b)$, **either** $\boldsymbol{A^*_{b,j}(y)\subseteq F(y)}$ **or** $\boldsymbol{A^*_{b,j}(y)\subseteq G(y)}$**.** We also have that $\boldsymbol{A^*_{b,r(b)}(y)\subseteq G(y)}$**.**

Now, suppose there is some $C\subseteq Q$ such that

(a) $\boldsymbol{p_q(t)=y_q\ \ \forall q\in C}$ **and**

(b) $\boldsymbol{C\subseteq G(t)}$ **and** $\boldsymbol{C\subseteq G(y)}$**.**

From (b) it follows that:

(c) $q''\in F(t)\Leftrightarrow(\forall q\in C)\ \ v_{bq''}-p_{q''}(t)>v_{bq}-p_q(t)$;

(d) $q''\in F(y)\ \Leftrightarrow(\forall q\in C)\ v_{bq''}-y_{q''}>v_{bq}-y_q$;

(e) $q''\in F(t)\cup G(t)\Leftrightarrow(\forall q\in C)\ \ v_{bq''}-p_{q''}(t)\geq v_{bq}-p_q(t)$ and

(f) $q''\in F(y)\cup G(y)\ \Leftrightarrow(\forall q\in C)\ \ v_{bq''}-y_{q''}\geq v_{bq}-y_q$.

It is, therefore, a matter of verification that, if there is some set $C \subseteq Q$ satisfying (a) and (b) then:

$\boldsymbol{F(y) \cup G(y) \subseteq F(t) \cup G(t)}$ **and** $\boldsymbol{F(y) \subseteq F(t)}$ **(A1)**

$\boldsymbol{G(y) \cap U(t) = G(t) \cap U(t)}$ **and** $\boldsymbol{F(y) \cap U(t) = F(t) \cap U(t)}$**.** **(A2)**

If $\boldsymbol{A_{b,i}(t) \subseteq F(t) \cap U(t)}$**, for some** $\boldsymbol{(b,i)}$**, then** $\boldsymbol{A_{b,i}(t) = A^*_{b,j}(y)}$**, for some** $\boldsymbol{(b,j)}$**.** **(A3)**

The original proof of Lemma 1 is very technical and long. The details of this proof are in Appendix 2. For a first reading we suggest the sketch of the proof presented below.

LEMMA 1: *Let* t *be some step of the auction at which* $U(t) \neq \phi$ *and* $p(t) \leq y$. Let $A^*(y)$ *be some demand structure for* y *with no overdemanded set. Let* $A(t)$ *be any demand structure at step* t *under price* $p(t)$. *Let* $T' = \{(b,i);\ A_{b,i}(t) \cap U(t) \neq \phi\}$. *Suppose that* $T' \neq \phi$. *Then, there is some demand structure* $A'(t)$, *such that for each* $(b,i) \in T'$, *there exists some* (b,j), *with* $A^*_{b,j}(y) \subseteq U(t)$, *and such that* $A^*_{b,j}(y) = A'_{b,i}(t)$, *if* $i \neq r(b)$ *and* $A^*_{b,j}(y) \subseteq A'_{b,i}(t)$, *otherwise. Furthermore,* $A'_{b,i}(t) = A_{b,i}(t)$ *for all* $(b,i) \notin T'$.

SKETCH OF THE PROOF: Define $A'(t)$ as follows. If $(b,i) \notin T'$, set $A'_{b,i}(t) \equiv A_{b,i}(t)$. If for all $(b,i) \in T'$ there is some (b,j), such that $A^*_{b,j}(y) \subseteq A_{b,i}(t) \cap U(t)$, define $A'_{b,i}(t) \equiv A_{b,i}(t)$ for all $(b,i) \in T'$ and we are done. Otherwise, there is some $(b,i) \in T'$, with $A_{b,i}(t) \equiv C \cup E$, where $C = A_{b,I}(t) \cap U(t)$, such that,

for all $\boldsymbol{(b,j)}$**,** $\boldsymbol{A^*_{b,j}(y)}$ **is not contained in** $\boldsymbol{C}$ **.**

We want to show that it is possible to define $A'(t)$ so that, for all (b,j)) $\in$T', with $j \neq r(b)$, there exists some (b,k) such that $A'_{b,j}(t) = A^*_{b,k}(y) \subseteq U(t)$; if $(b,r(b)) \in T'$, there exists some (b,k) such that $A^*_{b,k}(y) \subseteq A'_{b,r(b)}(t) \cap U(t)$.

The plan of the proof is the following: By defining $F(t)$, $G(t)$, $F(y)$ and $G(y)$ as in (P1) and (P2) we first show that $C \subseteq G(t)$ and $C \subseteq G(y)$, so (a) is satisfied. Since $C \subseteq U(t)$ then we have that (b) is satisfied. Then, (A1) and (A2) hold. From $C \subseteq B_b(y)$ it follows that all of C must be demanded by b at prices y, so every element of C must be in some $A^*_{b,j}(y)$ for some b-agent (b,j). Then, by (L1), we conclude that such a copy of b is

$(b,r(b))$. Then, $A^*_{b,r(b)}(y)=C\cup D$, with $D\cap C=\phi$ and $D\neq\phi$ by (L1). It is clear that $A^*_{b,r(b)}(y)\subseteq G(y)$, because $(b,r(b))$ is the last copy of b. Now set:

$\Gamma\equiv\{(b,j);\ A_{b,j}(t)\subseteq G(t)$ and $A_{b,j}(t)\cap U(t)\neq\phi\}$

$\Gamma'\equiv\{(b,j);\ A^*_{b,j}(y)\subseteq G(y)$ and $A^*_{b,j}(y)\cap U(t)\neq\phi\}$

$\mathfrak{I}\equiv\{(b,j);\ A_{b,j}(t)\subseteq G(t)$ and $A_{b,j}(t)\cap U(t)=\phi\}$

$\mathfrak{I}'\equiv\{(b,j);\ A^*_{b,j}(y)\subseteq G(y)$ and $A^*_{b,j}(y)\cap U(t)=\phi\}$

We have that $\Gamma\neq\phi$, since $(b,i)\in\Gamma$. Also, $\Gamma'\neq\phi$, since $(b,r(b))\in\Gamma'$.

The next step is to define a one-to-one map f from $\Gamma-\{(b,r(b)\}$ into $\Gamma'-\{(b,r(b)\}$. This can be done by establishing that $|\Gamma|\leq|\Gamma'|$ and $(b,r(b))\in\Gamma$. Then, define

$\boldsymbol{A'_{b,j}(t)\equiv A_{b,j}(t)}$ **if** $\boldsymbol{A_{b,j}(t)\subseteq F(t)}$ **or** $\boldsymbol{(b,j)\in\mathfrak{I}}$**.**

$\boldsymbol{A'_{b,j}(t)\equiv A^*_{f(b,j)}(y)}$ **if** $\boldsymbol{(b,j)\in\Gamma-\{(b,r(b))\}}$**.**

$\boldsymbol{A'_{b,r(b)}(t)\equiv G(t)-\bigcup_{j\neq r(b)}A'_{b,j}(t)}$**.**

To see that $A'(t)$ is well defined and is the desired demand structure, use (A1) and (A2).■

LEMMA 2: *Let t be some step of the auction at which $U(t)\neq\phi$ and $p(t)\leq y$.* Let *$A^*(y)$ be some demand structure for y with no overdemanded set. Let $A(t)$ be any demand structure at step t under price $p(t)$. Let T' and $A'(t)$ be defined as in Lemma 1. Then, a) $A'(t)$ has no minimal overdemanded set containing elements of $U(t)$; b) every minimal overdemanded set for $A'(t)$, if any, is a minimal overdemanded set for $A(t)$.*

PROOF: For part a), suppose by way of contradiction that S is a minimal overdemanded set for $A'(t)$ and $S_1\equiv S\cap U(t)\neq\phi$. Let T be the set of loyal demanders of S. The fact that S is overdemanded means exactly that

$$|\boldsymbol{T}| > \sum_{q\in S} S(q)| \qquad \textbf{(L*1)}$$

We will show that $S-S_1$ is non-empty and overdemanded for $A'(t)$, so S is not a minimal overdemanded set for $A'(t)$, which is a contradiction. To see this, define $T_1=\{(b,i)\in T;\ A'_{b,i}(t)\cap S_1\neq\phi\}$. Let T' be as defined in Lemma 1. Now, observe that $T_1\subseteq T'$. In fact, if $(b,i)\notin T'$ then $A_{b,i}(t)=A'_{b,i}(t)$, so $A'_{b,i}(t)\cap U(t)=\phi$, and so $(b,i)\notin T_1$. By Lemma 1, for each $(b,i)\in T_1$ there is some (b,j), such that $A^*_{b,j}(y)\subseteq A'_{b,i}(t)\cap U(t)$. On the other hand, the fact that $(b,i)\in T$ implies that $A'_{b,i}(t)\subseteq S$, so $A'_{b,i}(t)\cap U(t) = A'_{b,i}(t)\cap S_1$. Then, $A^*_{b,j}(y)\subseteq A'_{b,i}(t)\cap S_1$, so $A^*_{b,j}(y)\subseteq S_1$. Thus, since $A'_{b,i}(t)\cap A'_{b,k}(t)=\phi$ if $i\neq k$,

$$|\boldsymbol{T_1}|\leq|\{(\boldsymbol{b,j});\ \boldsymbol{A^*_{b,j}(y)\subseteq S_1}\}|\leq \sum_{q\in S_1} S_1(q)\,, \qquad \textbf{(L*2)}$$

where the last inequality is due to the competitivity of y. But then, (L*1) and (L*2) imply that $|T-T_1|=|T|-|T_1|>\sum_{q\in S} S(q)-\sum_{q\in S_1} S_1(q)=\sum_{q\in S-S_1} S(q)\geq 0$, from which follows that $T-T_1\neq\phi$. However, $T-T_1=\{(b,i)\in T;\ A'_{b,i}(t)\subseteq S-S_1\}$, so $S-S_1$ is non-empty and overdemanded for $A'(t)$, as we wanted to show.

For part b), suppose that $A'(t)$ has overdemanded sets. Let S be some minimal overdemanded set for $A'(t)$. Let T be the set of loyal demanders of S. Let T' be as defined in Lemma 1. By part a), $S\cap U(t)=\phi$. Then, if $(b,i)\in T$, $A'_{b,i}(t)\subseteq S$, so $A'_{b,i}(t)\cap U(t)=\phi$, so $(b,i)\notin T'$. By Lemma 1, $A'_{b,i}(t)=A_{b,i}(t)$. Then, for all $(b,i)\in T$, $A_{b,i}(t)\subseteq S$. Then, T is a set of loyal demanders of S under $A(t)$ and $min\{s(q),$ number of $(b,i)\in T$ with $q\in A_{b,i}(t)\} = min\{s(q),$ number of $(b,i)\in T$ with $q\in A'_{b,i}(t)\}$. Hence, S is also minimal overdemanded for $A(t)$, and the proof is complete. ■

PROOF OF THEOREM 1: Suppose by way of contradiction that p is not the minimum competitive price. Let y and $A^*(y)$ be as defined in Lemma 1. Let t be the last step of the auction at which $p(t)\leq y$ and let

$$S_1=\{q\in Q;\ p_q(t+1)>y_q\}. \qquad \text{(T1)}$$

Then, $S_1\neq\phi$. Since we are working with all integers, $S_1\subseteq U(t)$. **Let $A(t)$ be the demand structure chosen by the auctioneer at prices p(t) which has the minimum number of minimal overdemanded sets.** Let S be the minimal overdemanded set for $A(t)$ whose prices are raised at stage $t+1$. Thus,

$$S=\{q\in Q;\ p_q(t+1)>p_q(t))\}, \qquad \text{(T2)}$$

so $S_1= S\cap U(t)$, and so

$$S\cap U(t)\neq\phi. \qquad \text{(T3)}$$

By Lemma 1 and Lemma 2-a, there is some demand structure $A'(t)$, defined from $A(t)$ and $A^*(y)$, **that has no minimal overdemanded set containing some element of $U(t)$.** Then, by (T3), S is not a minimal overdemanded set for $A'(t)$. On the other hand, Lemma 2-b asserts that **every minimal overdemanded set for $A'(t)$, if any, is a minimal overdemanded set for $A(t)$.** Therefore, since $S \subseteq A(t)-A'(t)$, $A'(t)$ has less minimal overdemanded sets than $A(t)$, contradiction. Hence, p is the minimum competitive price. ■

PROOF OF THEOREM 2: Let μ be a matching compatible with p. Call the objects of seller q *overpriced* if q does not complete his quota under μ but $p_q>0$. Suppose (p, μ) is not a competitive equilibrium, so there is at least one seller k whose objects are overpriced. We will give a procedure for altering μ so as to eliminate the overpriced objects of seller k. For this purpose we construct a directed graph whose vertices are $B \cup Q$. There are two types of arcs. If $q \in \mu(b)$ and b likes q' at least as well as q for all $q' \in \mu(b)$, at prices p, there is an arc from b to q. If q is in $B_b(p)-\mu(b)$ there is an arc from q to b. (Observe that, since every buyer is matched under μ to her favorite set of allowable sellers, it follows that if there is an arc from q to b and an arc from b to q' then b is indifferent between q and q' at prices p). We have that k is in $B_b(p)$ for some $b \notin \mu(k)$, for if not we could decrease p_k a little bit and still have competitive prices, which contradicts the minimality of p. Let $B^* \cup Q^*$ be all vertices that can be reached by a directed path starting from k, followed by $b_1 \notin \mu(k)$.

Case 1: B^* contains a buyer b such that $\mu(b)$ contains a dummy-seller. Then, there is an arc from b to 0. Let $(k=q_1, b_1,q_2,b_2,q_3,...,q_t,b,0=q_{t+1})$ be a path from k to 0. Then, we may change μ by replacing q_2 by k in $\mu(b_1)$; q_3 by q_2 in $\mu(b_2)$; ..., the dummy-seller q_{t+1} by q_t in $\mu(b)$. Since each b_j is indifferent between q_j and q_{j+1}, for all $j=1,...,t,$ the matching is still competitive and k has less one unsold object, and hence he has less one overpriced object.

Case 2: No dummy-seller is in $\mu(b)$ for every $b \in B^*$. Then, we claim that there must be some q in Q^* such that $p_q=0$, for suppose not. By definition of $B^* \cup Q^*$ we know that if $b \notin B^*$ then $Q^* \cap [B_b(p)- \mu(b)] = \phi$. On the other hand, if $b \in B^*$, $q \notin Q^*$, $q' \in Q^*$ and q and q' are in $\mu(b)$, then b prefers q to q'. Therefore we can decrease the price of the objects of each seller in Q^* by some positive ε and still have competitiveness, contradicting the minimality of p. So choose q in Q^* such that $p_q=0$ and let $(k=q_1,$

$b_1,q_2,b_2,q_3,...,q_t,b_t,q$) be a path from k to q where $b_1 \notin \mu(k)$. Again change μ by replacing q_2 by k in $\mu(b_1)$, q_3 by q_2 in $\mu(b_2)$, ..., q by q_t in $\mu(b)$ and leaving one object of q unsold. The resulting matching is still competitive. Again the number of unsold objects of k has been reduced and so does the number of overpriced objects. ■

APPENDIX II.

Detailed proof of Lemma 1

PROOF OF LEMMA 1: Define $A'(t)$ as follows. If $(b,i) \notin T'$, set $A'_{b,i}(t) \equiv A_{b,i}(t)$. If for all $(b,i) \in T'$ there is some (b,j), such that $A^*_{b,j}(y) \subseteq A_{b,i}(t) \cap U(t)$, define $A'_{b,i}(t) \equiv A_{b,i}(t)$ for all $(b,i) \in T'$ and we are done. Otherwise, there is some $(b,i) \in T'$ such that,

for all (b,j), $A^*_{b,j}(y)$ is not contained in $A_{b,i}(t) \cap U(t)$. (L1)

We want to show that it is possible to define $A'(t)$ so that, for all (b,j)) $\in T'$, with $j \neq r(b)$, there exists some (b,k) such that $A'_{b,j}(t) = A^*_{b,k}(y) \subseteq U(t)$; if $(b,r(b)) \in T'$, there exists some (b,k) such that $A^*_{b,k}(y) \subseteq A'_{b,r(b)}(t) \cap U(t)$.

Set $A_{b,i}(t) \equiv C \cup E$, where $C = A_{b,i}(t) \cap U(t)$ and $C \cap E = \phi$. We have that $C \neq \phi$, due to the fact that $(b,i) \in T'$. Define $F(t)$, $G(t)$, $F(y)$ and $G(y)$ as in (P1) and (P2). Then, C is contained in the set of elements listed in $B_b(t)$. We claim that

$$A_{b,i}(t) \subseteq G(t). \qquad \text{(L2)}$$

In fact, if $A_{b,i}(t) \subseteq F(t)$, then $|A_{b,i}(t)|=1$, and so $A_{b,i}(t)=\{q\}$, for some $q \in U(t)$. We do not have that $q \in F(y)$, for if not there would be some (b,j) such that $A^*_{b,j}(y)=\{q\}= A_{b,i}(t)$, which contradicts (L1). Then suppose $q \notin F(y)$. We are going to show that $F(y) \cup G(y) \subseteq F(t)$, so $|F(y) \cup G(y)| \leq |F(t)| < r(b)$. But this is absurd since $|B_b(y)| \geq r(b)$. Then, take any $q' \in G(y)$. If b prefers q to q' at $p(t)$ then $v_{bq}-y_q = v_{bq}-p_q(t) > v_{bq'} - p_{q'}(t) \geq v_{bq'} - y_{q'}$, so $v_{bq}-y_q > v_{bq'} - y_{q'}$, from which follows that $q \in F(y)$, contradiction. Then, b prefers q' to q or is indifferent between q and q' at prices $p(t)$. Then $q' \in F(t)$, because $q \in F(t)$, so $G(y) \subseteq F(t)$. To see that $F(y) \subseteq F(t)$ take any $q'' \in F(y)$. Then b strictly prefers q'' to q at prices y (recall that $q \notin F(y)$), so $v_{bq''} - p_{q''}(t) \geq v_{bq''} - y_{q''} > v_{bq} - y_q = v_{bq} - p_q(t)$, so b strictly prefers q'' to q at prices $p(t)$. (In the last inequality we used that $q \in U(t)$).

Since $q \in F(t)$, it follows that $q'' \in F(t)$. Therefore, $F(y) \cup G(y) \subseteq F(t)$ and we have obtained the desired contradiction. Hence, $A_{b,i}(t) \subseteq G(t)$ and we have proved (L2). Then,

$$\boldsymbol{C \subseteq G(t)}, \qquad \textbf{(L3)}$$

We also have that

C is contained in the set of elements of list $B_b(y)$, **(L4)**

because otherwise all elements of C are out of $F(y) \cup G(y)$, so $v_{bq''} - y_{q''} > v_{bq} - y_q$, $\forall q'' \in F(y) \cup G(y)$ and $\forall q \in C$. In this case we would have that $\forall q'' \in F(y) \cup G(y)$ and $q \in C$ we can use that $p_q(t) = y_q$ and $y_{q''} \geq p_{q''}(t)$ to get that $v_{bq''} - p_{q''} \geq v_{bq''} - y_{q''} > v_{bq} - y_q = v_{bq} - p_q(t)$, so $q'' \in F(t)$ (we used here that $q \in C \subseteq G(t)$). This implies that $F(y) \cup G(y) \subseteq F(t)$, so $|F(y) \cup G(y)| \leq |F(t)| < r(b)$, absurd.

Now observe that if there is some $q \in C$ such that $q \in F(y)$, then there is some (b,j) such that $A^*_{b,j}(y) = \{q\} \subseteq C = A_{b,i}(t) \cap U(t)$, which contradicts (L1). Hence, it follows by (L4) that

$$\boldsymbol{C \subseteq G(y)}. \qquad \textbf{(L5)}$$

It also follows from (L4) (or (L5)) that all of C are demanded by b at prices y, so every element of C must be in some $A^*_{b,j}(y)$ for some (b,j). Let (b,j) be such that $A^*_{b,j}(y) \cap C \neq \phi$. Since $A^*_{b,j}(y)$ is not contained in C, by (L1), we must have that $|A^*_{b,j}(y)| > 1$, so $j = r(b)$. Then $C \subseteq A^*_{b,r(b)}(y)$ and we can write $A^*_{b,r(b)}(y) = C \cup D$, where $D \neq \phi$ and $D \cap C = \phi$. It is clear that $A^*_{b,r(b)}(y) \subseteq G(y)$, because $(b,r(b))$ is the last copy of b.

Set:

$\Gamma \equiv \{ (b,j);\ A_{b,j}(t) \subseteq G(t) \text{ and } A_{b,j}(t) \cap U(t) \neq \phi \}$

$\Gamma' \equiv \{ (b,j);\ A^*_{b,j}(y) \subseteq G(y) \text{ and } A^*_{b,j}(y) \cap U(t) \neq \phi \}$

$\Im \equiv \{ (b,j);\ A_{b,j}(t) \subseteq G(t) \text{ and } A_{b,j}(t) \cap U(t) = \phi \}$

$\Im' \equiv \{ (b,j);\ A^*_{b,j}(y) \subseteq G(y) \text{ and } A^*_{b,j}(y) \cap U(t) = \phi \}$

We have that $\Gamma \neq \phi$, since $(b,i) \in \Gamma$ by (L2) and by the fact that $(b,i) \in T'$. Also, $\Gamma' \neq \phi$, since $(b,r(b)) \in \Gamma'$. We are going to show that we can define a one-to-one map from $\Gamma - \{(b,r(b)\}$ into $\Gamma' - \{(b,r(b)\}$. In fact, it is clear that

$$\boldsymbol{|\Gamma| = r(b) - |F(t)| - |\Im| \text{ and } |\Gamma'| = r(b) - |F(y)| - |\Im'|}. \qquad \textbf{(L6)}$$

We claim that

$$\boldsymbol{|F(t)| \geq |F(y)| + |\Im'| + |D - U(t)|} \qquad \textbf{(L7)}$$

To see this, first observe that, due to the fact that $(b,r(b))\in\Gamma'$, then every $A^*_{b,j}(y)$ with (b,j) in $\Im'$ is singleton, so $|\bigcup_{(b,j)\in\Im'} A^*_{b,j}(y)|=|\Im'|$. Furthermore, $[\bigcup_{(b,j)\in\Im'} A^*_{b,j}(y)]\cap(D-U(t))=\phi$. Thus, since $G(y)-U(t)=[\bigcup_{(b,j)\in\Im'} A^*_{b,j}(y)]\cup(D-U(t))$, it follows that $|G(y)-U(t)|=|\Im'|+|D-U(t)|$. Therefore, it is enough to prove that $F(t)\supseteq F(y)\cup[G(y)-U(t)]$. That $F(t)\supseteq F(y)$ follows from (A1). Then, let $q\in G(y)-U(t)$. It also follows from (A1) that $q\in F(t)\cup G(t)$. If q was in $G(t)$ then, for all q' in C, $v_{bq'}-y_{q'}=v_{bq'}-p_{q'}(t)=v_{bq}-p_q(t)>v_{bq}-y_q=v_{bq'}-y_{q'}$, contradiction, where in the second equality we used (L3), in the inequality we used that $q\notin U(t)$ and in the last equality we used that $q\in G(y)$ and (L5). Therefore, $q\notin G(t)$, so $q\in F(t)$, and so $G(y)-U(t)\subseteq F(t)$. Hence, $F(t)\supseteq F(y)\cup[G(y)-U(t)]$ and we have proved (L7).

Using (L6) and (L7), we get that $|\Gamma|=r(b)-|F(t)|-|\Im|\leq r(b)-|F(y)|-|\Im'|-|D-U(t)|-|\Im|=|\Gamma'|-|D-U(t)|-|\Im|$. That is,

$$|\Gamma|\leq|\Gamma'|-|D-U(t)|-|\Im|. \qquad \textbf{(L8)}$$

Since $A_{b,r(b)}(t)\subseteq G(t)$ (because $(b,r(b))$ is the last copy of b), there must be that, either $(b,r(b))\in\Im$, or $(b,r(b))\in\Gamma$. If $(b,r(b))\in\Im$ then $|\Im|\geq 1$ and $|\Gamma|=|G(t)\cap U(t)|$, because every $A_{b,j}(t)$ is singleton for $b\neq r(b)$. But then, $|\Gamma|\leq|\Gamma'|-|D-U(t)|-1\leq|\Gamma'|-1<|\Gamma'|$, so $|\Gamma'|>|\Gamma|$. Since $G(t)\cap U(t)=G(y)\cap U(t)$ by (A2), we must have that $|\Gamma'|>|\Gamma|=|G(t)\cap U(t)|=|G(y)\cap U(t)|$, which would be a contradiction (for each $(b,j)\in\Gamma'$, $A^*_{b,j}(y)$ has at least one element of $G(y)\cap U(t)$). Then, $(b,r(b))\in\Gamma$. In this case, $|\Gamma-\{(b,r(b))\}|\leq|\Gamma'-\{(b,r(b))\}|$. Hence, **we can define a one-to-one map f from $\Gamma-\{(b,r(b)\}$ into $\Gamma'-\{(b,r(b)\}$.** Then set:

$$A'_{b,j}(t)\equiv A_{b,j}(t) \text{ if } A_{b,j}(t)\subseteq F(t) \text{ or } (b,j)\in\Im.$$

$$A'_{b,j}(t)\equiv A^*_{f(b,j)}(y) \text{ if } (b,j)\in\Gamma-\{(b,r(b))\}.$$

$$A'_{b,r(b)}(t)\equiv G(t)-\bigcup_{j\neq r(b)} A'_{b,j}(t).$$

It is a matter of verification that $A'_b(t)$ is well defined (use that $G(t)\cap U(t)=G(y)\cap U(t)$ given by (A2)). We have to check that, if $(b,j)\in T'$ and $j\neq r(b)$, there exists some (b,h) such that $A'_{b,j}(t)=A^*_{b,h}(y)\subseteq U(t)$; if $(b,r(b))\in T'$, there exists some (b,h) such that $A^*_{b,h}(y)\subseteq A'_{b,r(b)}(t)\cap U(t)$. Then, let $(b,j)\in T'$. This means that

$A_{b,j}(t) \cap U(t) \neq \phi$. If $A_{b,j}(t) \subseteq F(t)$, then $j \neq r(b)$, so $A_{b,j}(t)=\{q\}$ for some $q \in U(t)$, and so $q \in F(t) \cap U(t) = F(y) \cap U(t)$, where the equality follows from (A2). Thus, there is h such that $A^*_{b,h}(y)=\{q\}$ and so $A'_{b,j}(t)= A^*_{b,h}(y) \subseteq U(t)$, by definition of $A'(t)$. If $(b,j) \in T'$ and $A_{b,j}(t) \subseteq G(t)$, then $(b,j) \in \Gamma$ because $A_{b,j}(t) \cap U(t) \neq \phi$. We distinguish two cases:

Case 1. $j \neq r(b)$. Then, $A'_{b,j}(t)= A^*_{f(b,j)}(y)$ and $f(b,j) \in \Gamma' - \{(b,r(b))\}$, so $A^*_{f(b,j)}(y)=\{q\}$, for some q. The definition of Γ' implies that $q \in G(y) \cap U(t)$. Hence, $A'_{b,j}(t)= A^*_{f(b,j)}(y) \subseteq U(t)$.

Case 2. $j=r(b)$. The definition of A'_b implies that

$$\boldsymbol{A'_{b,r(b)}(t) \cap U(t) = [G(t) \cap U(t)] - \bigcup_{(b,k) \in \Gamma - (b,r(b))} A^*_{f(b,k)}(y)} \qquad \textbf{(L9)}$$

Suppose that $|D-U(t)|=0$. Then, we have that $A^*_{b,r(b)}(y)=C \cup D \subseteq U(t)$. Therefore, $A^*_{b,r(b)}(y)=[G(y) \cap U(t)] - \bigcup_{k \neq r(b)} A^*_{b,k}(y)=[G(y) \cap U(t)] - \bigcup_{(b,k) \in \Gamma' - (b,r(b))} A^*_{b,k}(y) \subseteq [G(y) \cap U(t)] - \bigcup_{(b,k) \in \Gamma - (b,r(b))} A^*_{f(b,k)}(y)$. Using (L9) and the fact that $G(y) \cap U(t)=G(t) \cap U(t)$, given by (A2), we get that $A^*_{b,r(b)}(y) \subseteq A'_{b,r(b)}(t) \cap U(t)$.

Now, suppose that $|D-U(t)|>0$. Then $|\Gamma| \leq |\Gamma'| - 1$ by (L8), so $|Im(f)| \leq |\Gamma' - \{(b,r(b))\}|-1$.[6] Thus, there is at least one $(b,h) \in \Gamma'$, with $h \neq r(b)$, such that $(b,h) \notin Im(f)$. The fact that $h \neq r(b)$ implies that $A^*_{b,h}(y)=\{q\}$, for some q. The fact that $(b,h) \notin Im(f)$ implies that $q \notin \bigcup_{(b,k) \in \Gamma - (b,r(b))} A^*_{f(b,k)}(y)$. That $(b,h) \in \Gamma'$ implies that $q \in G(y) \cap U(t)$ and so $q \in G(t) \cap U(t)$ by (A2). That is, $q \in [G(t) \cap U(t)] - \bigcup_{(b,k) \in \Gamma - (b,r(b))} A^*_{f(b,k)}(y)$. Now, use (9) to conclude that $A^*_{b,h}(y)=\{q\} \subseteq A'_{b,r(b)}(t) \cap U(t)$.

Thus, if $j=r(b)$, there is some (b,h) such that $A^*_{b,h}(y) \subseteq A'_{b,r(b)}(t) \cap U(t)$.

Hence we have demonstrated the desired result and the proof is complete. ■

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[6] We are using the abbreviation $Im(f)$ to denote the image set of f: $f(\Gamma - \{(b,r(b))\})$.

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