Cores of Combined Games*

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Abstract

This paper studies the core of combined games, obtained by summing two coalitional games. It is shown that the set of balanced transferable utility games can be partitioned into equivalence classes of component games to determine whether the core of the combined game coincides with the sum of the cores of its components. On the other hand, for non-balanced games, the binary relation associating two component games whose combination has an empty core is not transitive. However, we identify a class of non-balanced games which, combined with any other non-balanced game, has an empty core.

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1 Introduction

The broad subject of the paper is the study of bargaining and cooperation when multiple issues are at stake. We have two complementary objectives in mind:

1. Identify conditions under which negotiating over different issues separately is equivalent to negotiating over these issues simultaneously;

2. Identify situations in which combining issues reduces conflict in bargaining.

We use games in coalitional form, a classical model to study cooperation, to tackle these two questions. The coalitional function specifies for each coalition the surplus to be shared should its members cooperate. The simplicity of this reduced-form approach, making no direct reference to the underlying social or economic alternatives, comes at a cost. Indeed, relating the cooperative opportunities associated to different issues to the cooperative opportunities of the combined issues is possible in this framework only if the different issues are independent. In such cases, the coalitional function associated to the combined issues is simply the sum of the coalitional functions associated to each issue taken separately. Spillovers are certainly an important feature of multi-issue bargaining, and further analysis of non-welfarist models is needed to understand their implication. The present paper illustrates that *bargaining over multiple issues may have relevant implications even in the absence of such spillovers.*

Multi-issue bargaining was of central importance to Professor Shapley when studying values for games in coalitional form, as illustrated by his motivation for the additivity axiom: "*The third axiom ("law of aggregation") states that when two independent games are combined, their values must be added player by player*" (Shapley, 1953, page 309). Put differently, additivity implies that the outcome of multi-issue negotiations does not depend on the agenda chosen by the negotiators. Whether issues are discussed separately or "packaged" in different ways does not affect the result of the negotiation. In Professor Shapley's view, this agenda independence is a natural requirement to impose on a solution concept.

However, the Shapley value is the only solution concept for which additivity is posited as an axiom. Other solution concepts, whether they are based on alternative axiomatizations, like the Nash bargaining solution, or more positive considerations, like the core, do not satisfy this property of agenda independence. In this paper, we focus attention on the core primarily because of its importance in economic theory. Other solution concepts are briefly discussed in Section 5.

It is well known that the core is superadditive (see for example, Peleg's (1986) axiomatization of the core), so that the core of the combination of two games is always larger than the sum of the core of the two components. Intuitively, by combining two negotiation processes, and forcing players to make coalitional objections on the issues simultaneously, it is easier to sustain an imputation than when players can make separate objections on the two issues.

Hence, the specific question we tackle in this paper is the following: For which pairs of games is the core of the combination of the two games exactly equal to the sum of the core of the component games? This offers a formal statement to the first objective listed at the beginning of the paper.

Our main result (Proposition 1) precisely characterizes when the additivity property holds. For expositional purposes, we will restrict attention in this introduction to the simpler case where the two component games are generic in the sense that all extreme points of the cores of both games are characterized by exactly n binding coalitional constraints (where n is the number of players).[2] In such cases, the core of the sum of two games v and w is equal to the sum of the cores of v and w if and only if the extreme points of the cores of v and w are defined by the same sets of binding coalitional constraints. Because the latter property defines an equivalence relation among games, we conclude that *the set of all (generic) balanced transferable utility games can be partitioned into equivalence classes such that the core of the combination of two games is equal to the sum of the cores of the components if and only if the two games belong to the same class.* One of these equivalence classes (where the extreme points are determined by any increasing sequence of coalitions) is the set of convex games introduced by Shapley (1971). Hence, the combination of two convex games does not result in an expansion of the set of core allocations. By contrast, whenever two (generic) games v and w are taken from two different equivalence classes, the core of the combined game is strictly greater than the sum of the cores of its components. The difference can actually be extremely large, as the dimension of the core of $v + w$ may exceed the dimension of the sum of the cores (for example, even when the cores of v and w are singletons, the core of $v + w$ may be a set of full dimension in the set of imputations).

The core of two games with an empty core may be non-empty (Example 1). In such cases, bargaining over each component would lead to an impasse or to partial cooperation, but efficiency can be recovered (on both components) by combining the issues. This illustrates the relevance of the second objective introduced in the first paragraph. Formally, we would like to characterize pairs of games with an empty core whose sum has a non-empty core. Unfortunately, our characterization of the set of games for which the core is additive does not carry over to games with empty cores. The binary relation associating two games v and w whose combination has an empty core is not transitive. This is easily understood: for two games v and w to be such that the combined game $v+w$ has an empty core, it is sufficient that one of the balanced[3] collections of coalitions has a worth exceeding the worth of the grand coalition in both games v and w. Now consider a triple of games v, w, z. The worth of the balanced collection $\mathcal{C}$ may exceed the worth of the grand coalition in both v and w and the

[2] Of course, our general result also explains what happens when the component games are not necessarily generic, but it feels more natural to postpone the discussion to the main text (cf. Section 3).

[3] As in Bondareva (1963) and Shapley (1967) - the reader is reminded of the formal definition in Section 2.

worth of the balanced collection $\mathcal{D}$ may exceed the worth of the grand coalition in both w and z. However, v and z may very well not share any balanced collection whose worth exceeds the grand coalition, and be such that the core of $v + z$ is nonempty. Put differently, for a game to be unbalanced, one only requires one of the balanced collection to have a greater worth than the grand coalition, so that the set of games with empty cores is not defined by a set of linear inequalities, and is in fact typically *not convex*. In spite of this, we can identify a convex subset of the class of unbalanced games which has the following property: for any game in that class, the combination of this game with any other game with empty core also has an empty core (Proposition 2). Intuitively, this subset contains those games which are hardest to "balance" with other games, and may create the more difficulties in negotiations.

To the best of our knowledge, the only previous studies of the additivity of the core in the cooperative game theoretic literature are due to Tijs and Branzei (2002). They identify three subclasses of games on which the core is additive (including the class of convex games). Our results complement and extend their analysis by showing that in fact *the entire set of balanced games can be partitioned into subclasses on which the core correspondence is additive.* The literature on noncooperative games has paid more attention to simultaneous, multi-issue bargaining. In a two-player setting, Fershtman (1990) and Busch and Hortsmann (1997) extend Rubinstein (1982)'s alternating offers game to a multi-issue setting, where players bargain over each issue in a predefined sequence. They show that the equilibria of this multi-issue bargaining differ considerably from the single-issue model. In later contributions to this literature, Bac and Raff (1996), Inderst (2000) and In and Serrano (2004) allow players to endogenously choose on which issue to bargain, and show that players have an incentive to manipulate strategically the agenda. Issue linkage has also been studied in noncooperative games representing international negotiations across countries. It has long been argued that combining negotiations over different dimensions (trade, protection of the environment) may have beneficial effects (see for example Carraro and Siniscalco, 1994). Conconi and Perroni (2002) propose a model of issue linkage and evaluate this argument using a parameterized model of international trade and environmental negotiations. Issue linkage also appears implicitly in the literature on mergers in Industrial Organization (e.g. Perry and Porter (1985) and Farrell and Shapiro (1990)). In order to be profitable, a merger must involve two dimensions – both a cost and a market dimensions – and result in cost synergies as well as market concentration.

The rest of the paper is organized as follows. In the next Section, we recall the standard definitions of coalitional games and the core. In Section 3, we analyze the combination of games with nonempty cores. We first provide intuition by analyzing symmetric games with three players, then provide our general result on the partitioning of the set of balanced games. We illustrate this result by computing exactly this partition in four-player symmetric games. Section 4 contains our results for games with empty cores. We provide an example to show that the set cannot be partitioned, and discuss how four-player games with empty cores can or cannot be combined to obtain a nonempty core. We finally identify the class of noncompensable games, which, combined with any other game with an empty core, still retain an empty core. The additivity of other cooperative solution concepts is briefly discussed in Section 5. Section 6 contains the proof, and in particular a key lemma on the addition of convex polyhedra defined by systems of linear inequalities.

2 Preliminaries

Let N be a set of players. A *cooperative game* is described by a *coalitional function* v which assigns to every nonempty subset S of N a real number, $v(S)$, called the *worth of the coalition*. Games will be assumed to be *superadditive*: $v(S \cup T) \geq v(S) + v(T)$, for any two disjoint coalitions S and T. We denote the set of all such n-player games by $\Gamma(n)$. A game is *convex* if the players' marginal contributions are non decreasing: $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$, for each pair (S, T) of coalitions such that $S \subseteq T$.

An *imputation* is a vector $x \in \Re^N$ that is feasible, efficient, and individually rational: $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$, for each $i \in N$. The *core* of a cooperative game v is the set of payoff vectors $x \in \Re^N$ that are feasible when all the players cooperate, and which cannot be improved upon by any coalition: $\sum_{i \in N} x_i \leq v(N)$ and $\sum_{i \in S} x_i \geq v(S)$ for each coalition S. Let A be the $(2^n - 1) \times n$ matrix encoding coalitional membership: $A_{S,i} = 1$ if $i \in S$ and $A_{S,i} = 0$ if $i \notin S$, for each coalition S and each player i. Then,

$$C(v) = \{x \in \Re^N | \sum_{i \in N} x_i = v(N), Ax \geq v\}.$$

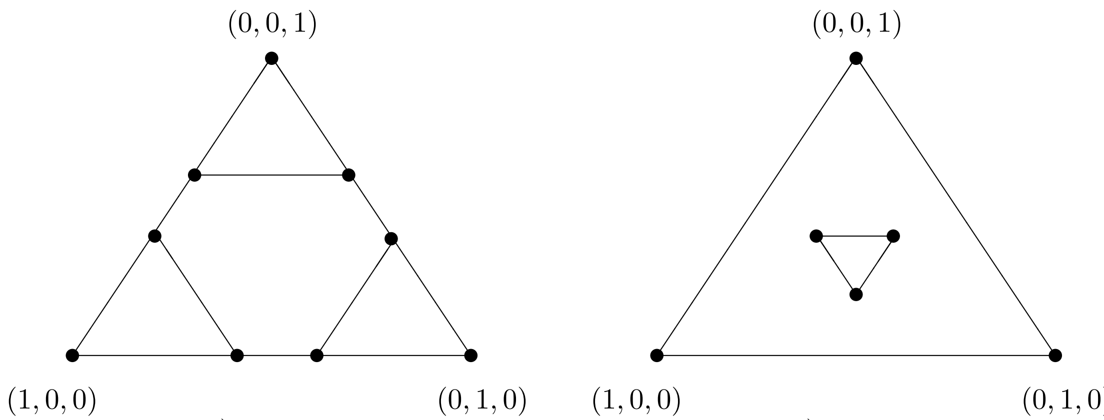
This rewriting highlights the fact that *the core is a bounded convex polyhedron* defined by a system of linear inequalities. As any such set, the core is characterized by its set of extreme points – points which cannot be obtained as convex combinations of other points in the set. Equivalently, a payoff vector x is an extreme point of the core of v if there exists a collection $(S_k)_{k=1}^n$ of coalitions such that $\sum_{i \in S_k} x_i = v(S_k)$, for each k, and these n equations are linearly independent.

The system of linear inequalities defining the core may be inconsistent, in which case the core is empty. Bondareva (1963) and Shapley (1967) proposed a characterization of games with nonempty core based on balanced collections of coalitions. A collection $(S_k)_{k=1}^K$ of coalitions is *balanced* if there exists a collection $(\delta_k)_{k=1}^K$ of real numbers between 0 and 1 (called *balancing weights*) such that $\sum_{S_k | i \in S_k} \delta_k = 1$, for each $i \in N$. A game v is *balanced* if and only if $\sum_k \delta_k v(S_k) \leq v(N)$, for each balanced collection $(S_k)_{k=1}^K$ of coalitions and each collection $(\delta_k)_{k=1}^K$ of balancing weights. The core of a game v is nonempty if and only if the game v is balanced. The set of all balanced superadditive n-player games is denoted $\beta(n)$.

3 Combining games with a nonempty core

3.1 Three-player symmetric games

In this Section, we suppose that the two component games v and w are balanced. In order to gain intuition, we first consider normalized three-player symmetric games. For these games $v(\{i\}) = 0$, $v(N) = 1$, and $v(S) = v_2 \in (0, 1)$ for any coalition S with two players. It is easy to see that the core of game v is empty if $v_2 > \frac{2}{3}$, and the game is convex if $v_2 \leq \frac{1}{2}$. Figures 1a and 1b illustrate the shape of the core of three-player symmetric games, when $v_2 \leq \frac{1}{2}$ and $\frac{1}{2} \leq v_2 \leq \frac{2}{3}$.



a) $v_2 = 0.4$ b) $v_2 = 0.6$

Figure 1 Core of symmetric three-player games

When the game is balanced but not convex, the core has the shape of a triangle in the simplex, and has three extreme points given by $(2v_2 - 1, 1 - v_2, 1 - v_2)$ (and the permutations). When the game is convex, the core has the shape of an hexagon, characterized by six extreme points given by $(0, v_2, 1 - v_2)$ (and the permutations). Clearly, by summing up two balanced, nonconvex games v and w, one obtains a new balanced nonconvex game $v + w$. This combined game again has a core shaped like a triangle, and all the extreme points of $C(v + w)$ are be decomposed as sums of extreme points of $C(v)$ and $C(w)$. Similarly, by summing up two convex games v and w, one obtains a new convex game $v + w$. The core of $v + w$ is shaped like an hexagon and all the extreme points of $v + w$ can be decomposed as sums of extreme points of $C(v)$ and $C(w)$. However, if one combines a convex game with a nonconvex balanced game, the core of the combined game *cannot be equal to the sum of the cores of the games.* To see this, let us combine a nonconvex game – $v_2 \in (\frac{1}{2}, \frac{2}{3}]$ – with a strictly convex game – $w_2 \in [0, \frac{1}{2})$. Observe that, for any point in $C(v)$, $x_i > 2v_2 - 1$. Hence, for any points in $C(v) + C(w)$, $x_i > 2v_2 - 1$. However, the core of $v + w$ is either shaped as a triangle or as an hexagon. In the latter case, it contains extreme points on the boundary, so that $C(v + w)$ is a strict superset of $C(v) + C(w)$. In the former case, the core of $v + w$ contains an extreme point for which a player's payoff equals $x_i = 2(v_2 + w_2) - 2 = (2v_2 - 1) + (2w_2 - 1) < 2v_2 - 1$, where the last inequality is obtained because $w_2 < 1/2$. Yet that player's minimal payoff is $2v_2 - 1$ in $C(v)$ and 0 in $C(w)$. Hence, again $C(v + w)$ is a strict superset of $C(v) + C(w)$.

3.2 A general result

The study of three-player symmetric games thus shows that the set of balanced games can essentially[4] be partitioned into two subclasses on the basis of the extreme points of the core. The core of the combined game is equal to the sum of the cores of the component games if and only if the two component games belong to the same subclass. Our main result shows that this intuition can be generalized to any n-player transferable utility game. We will prove this statement as a corollary to a general result on convex polyhedra.

In the Appendix, we define an equivalence relation between two bounded convex polyhedra[5] $P(A, b) = \{x \in \Re^N | Ax \geq b\}$ and $P(A, b') = \{x \in \Re^N | Ax \geq b'\}$ if the extreme points of the two polyhedra are defined by the same constraints. To gain some intuition, we consider the simpler and generic[6] case where all extreme points of both $P(A, b)$ and $P(A, b')$ are characterized by exactly N equalities. If $P(A, b) + P(A, b') = P(A, b + b')$, then any extreme point of $P(A, b + b')$ can be decomposed as the sum of two elements of $P(A, b)$ and $P(A, b')$. These vectors have to be extreme points of the polyhedra $P(A, b)$ and $P(A, b')$, and furthermore neither $P(A, b)$ nor $P(A, b')$ can possess additional extreme points. This shows that, whenever $P(A, b) + P(A, b') = P(A, b + b')$, the extreme points of $P(A, b)$ and $P(A, b')$ must be defined by the same constraints. To prove the converse statement, we need to show that, when extreme points are defined by the same constraints, $P(A, b + b') \subset P(A, b) + P(A, b')$ (the other inclusion being always trivially true). This is proven by induction on the dimension N. For $N = 1$, the polyhedra are subsets of the line, and the inclusion is verified. For higher values of n, we pick an extreme point of $P(A, b + b')$ and show that it can be decomposed as the sum of two vectors in $P(A, b)$ and $P(A, b')$. This is done by isolating one of the players, i, redefining an $N - 1$ dimensional polyhedron by using one of the binding constraints to define x_i as a function of x_{-i} and applying the induction hypothesis to the lower dimensional polyhedron.

The equivalence relation described in the previous paragraph captures most of the cases where the additivity property holds, but not all (see e.g. footnote 4). The general result states that $P(A, b) + P(A, b') = P(A, b + b')$ if and only if one can construct sequences of b^k and b'^k

[4] The statement does not include the special case $v_2 = 1/2$. Each extreme point of the core satisfies four coalitional constaints with equality in this non-generic situation, and the additivity property holds when combining the game with any other symmetric game with a nonempty core.

[5] In this paragraph, as in the Appendix, N is an arbitrary strictly positive integer that does not need to be related to the set of players, or its cardinality.

[6] If $P(A, b)$ has an extreme point with more than N binding inequalities, then at least one of these equations can be written as a linear combination of the other equations, which implies that b satisfies at least one affine equation and is thus contained in a hyperplane, a non-generic feature.

converging to b and b' such that $P(A, b^k)$ and $P(A, b'^k)$ are equivalent for all k. Applying this lemma to the core of cooperative games, we obtain the following result.

Proposition 1 *Consider the equivalence relation $\mathcal{R}$ on $\beta(n)$, where $v\mathcal{R}w$ if and only if the extreme points of $C(v)$ and $C(w)$ are defined by the same constraints. Then $C(v) + C(w) = C(v + w)$ if and only if there exist two sequences of games v^k and w^k in $\beta(n)$ that converge to v and w respectively, and such that $v^k\mathcal{R}w^k$, for all k. In the generic case where exactly n coalitional constraints are binding at each extreme point of the core of both v and w, we have that $C(v) + C(w) = C(v + w)$ if and only if $v\mathcal{R}w$.*

We conclude this subsection with a remark on the shape of the equivalence classes defined by R. For each set $\mathbb{S}$ of coalitions, the set of games in $\beta(n)$ that have an extreme point of the core for which the set of binding constraints is exactly $\mathbb{S}$ forms a convex cone in $\mathbb{R}^{2^n-1}$. The equivalence classes defined by $\mathcal{R}$ are thus the intersection of convex cones, and thus form cones as well.

3.3 Four-player symmetric games

We illustrate the partition of the set of balanced games into equivalence classes by considering normalized four-player symmetric games – $N = \{1, 2, 3, 4\}$, $v(N) = 1$ and $v(\{i\}) = 0$, for each $i \in N$. Let v_2 denote the value of two-player coalitions and v_3 the value of three-player coalitions. Superadditivity requires that $v_2 \in [0, 1/2]$ and $v_3 \in [v_2, 1]$. Figure 2 depicts the subsets of games where the extreme points of the cores are defined by the same constraints. The computations underlying Figure 2 are given in Section 6.2.

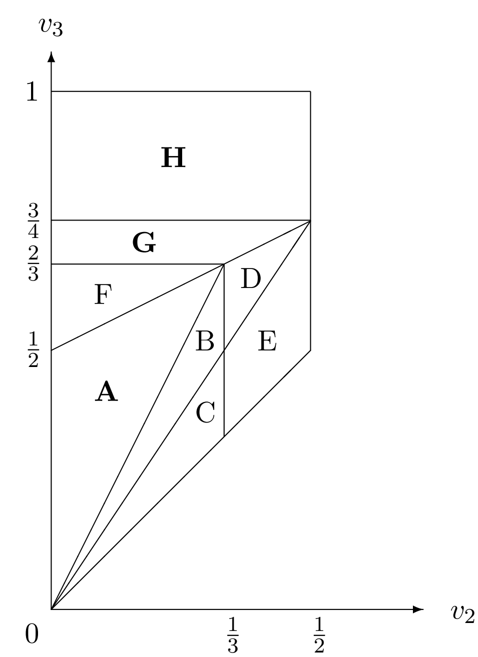


Figure 2 Equivalence classes of four-player symmetric games

The equivalence relation described in the previous subsection leads to a partition of the set of normalized four-player symmetric games. The seven regions labeled from A to G correspond to the partition induced on the class of generic games for which exactly n constraint are binding at each extreme point of the core. The equivalence relation leads to the lines (e.g. the line between regions A and B) and intersecting points (e.g. the point that falls next to all seven regions) separating these regions when considering non-generic games.

As explained in the previous subsection, the additivity property holds if one chooses two component games that fall in the same equivalence class, but not only in those cases. It would also hold for instance if we combine a game that falls on the line between A and F with a game that falls on the line between A and B, since both games can be approximated by games that belong to A. This extended property with limits, on the other hand, characterizes all the cases where the additivity property holds. The core of the sum of a game that belongs to A with a game that belongs to E is strictly larger than the sum of the cores, or the core of the sum of a game that falls on the line between A and F with a game that falls on the line between C and E is strictly larger than the sum of the cores. The difference between the core of the combined game and the sum of the cores of the component games can be extremely large. In fact, it is possible to combine two component games where the core collapses to a single point, and obtain a full dimensional core. For example, pick two games v and w such that $v_2 = \frac{1}{2}$ and $w_3 = \frac{3}{4}$. For each of these games, the core is a single point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. However, the sum of the two games can belong to any of the regions A,B,D,E,F or G, where the core is a full-dimensional set.

Three classes of games stand out. Region A and its closure corresponds to the class of *convex games*. The work of Shapley (1971) and Ichiishi (1981) imply that a game is convex if and only if the extreme points of the core coincide with the vectors of marginal contribution. Proposition 1 confirms the known-result that the core of the sum of any two convex games is equal to the sum of the cores (see also Tijs and Branzei (2002) on that point). Region G and its closure corresponds to games where the extreme points of the core are characterized by constraints involving only three-player coalitions, or the dual imputation set. This is the class of games K_d introduced by Driessen and Tijs (1983) – and for which Tijs and Branzei (2002) also note that the core is additive. Finally, region H (for which $v_3 > \frac{3}{4}$) corresponds to games with empty cores.

4 Combining games with an empty core

4.1 Examples

We now consider the combination of unbalanced games. First, as the next example shows, the combination of two games with empty cores may very well possess a nonempty core. In this sense, it is worthwhile to combine games, or to link negotiations which would otherwise result in an impasse.

Example 1 *Let $n = 4$. Let $v(S) = 2/3$ if $|S| = 2$ and $4 \in S$, or $|S| = 3$, $v(N) = 1$, and $v(S) = 0$ for all other coalition S. Let $w(S) = 0$ if $|S| < 3$, $w(S) = 4/5$ if $|S| = 3$, and $w(N) = 1$.*

In Example 1, game v has an empty core, because the worth of the grand coalition is smaller than the weighted sum of worths associated to the balanced collection $\mathcal{C}_v = \{\{123\}, \{14\}, \{24\}, \{34\}\}$, and the balancing weights $\delta_v(\{1,4\}) = \delta_v(\{2,4\}) = \delta_v(\{3,4\}) = 1/3$ and $\delta_v(\{1,2,3\}) = 2/3$. Game w is unbalanced with respect to the collection $\mathcal{C}_w = \{\{123\}, \{124\}, \{134\}, \{234\}\}$ and the balancing weights $\delta_w(\{1,2,3\}) = \delta_w(\{1,2,4\}) = \delta_w(\{1,3,4\}) = \delta_w(\{2,3,4\}) = 1/3$. However, it is easy to check that the imputation $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ belongs to the core of $v + w$.

Example 1 illustrates a crucial difference between the conditions characterizing a nonempty core (the equations defining the extreme points of the core), and the conditions guaranteeing emptiness of the core (the inequalities for balanced collections). Whereas a nonempty core is characterized by *all the equations determining extreme points*, emptiness of the core is guaranteed *as long as one of the balanced collections has a worth greater than the grand coalition*. As the next example shows, this difference implies that the binary relation linking two unbalanced games whose combination is also unbalanced may not be transitive.

Example 2 *Let $n = 5$. Let $v(S) = 0$ if $|S| < 4$, $v(S) = 5/6$ if $|S| = 4$, and $v(N) = 1$. Let $w(S) = 0$ if $|S| < 3$, $w(S) = 3/4$ if $|S| = 3, 4$, and $w(N) = 1$. Let $z(S) = 0$ if $|S| < 3$, $z(S) = 3/4$ if $S = 3$, $z(S) = 5/6$ if $S = 4$, and $z(N) = 1$.*

In Example 2, v, w and z are unbalanced because of the collection of coalitions of size 4 for v, because of the collection of coalitions of size 3 for w, and because of both collections of coalitions of size 3 and 4 for z. Hence, the combined game $v + z$ (respectively $w + z$) is unbalanced, because the worth of the grand coalition is smaller than the weighted sum of worths of the balanced collection of coalitions of size 4 (respectively 3). However, the core of the combined game $v + w$ is nonempty, and contains for example the symmetric allocation $(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$.

Put differently, the difficulty is that the set of games with empty cores (for which only one of the balanced collections may pose problems) *need not be convex*, whereas classes of games with nonempty cores whose extreme points satisfy the same constraints are always convex. This is illustrated in Figure 3, which graphs unbalanced symmetric games with 5 players, as a function of the values v_3 and v_4 of the three- and four-player coalitions.

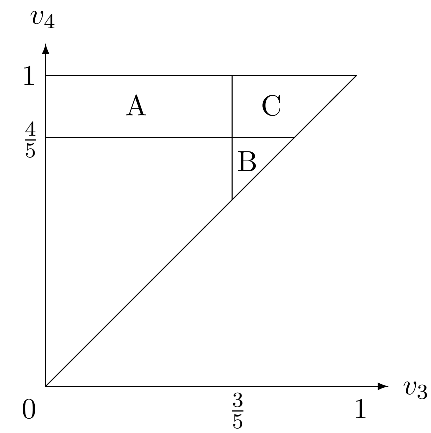


Figure 3 Five-player symmetric games with empty cores

In region A, the core is empty because any imputation is blocked by a four-player coalition; in region B the blocking coalitions are of size 3. In region C, both coalitions of sizes 3 and 4 are blocking. From Figure 3, it is clear that combinations of games of regions A and B may have nonempty cores. On the other hand, any combination of symmetric games involving one component in region C has an empty core.

4.2 Noncompensable games

Figure 3 suggests that there exist unbalanced games which, combined with any other unbalanced game, produce an empty core. We will term these games noncompensable. Formally, a game v with an empty core is *noncompensable* if $C(v+w)=\emptyset$, for any game w such that $C(w)=\emptyset$.

Recall that a balanced collection is *minimal* if any subcollection is unbalanced. It is *proper* if no two sets in the collection are disjoint. It is easy to verify that there exists a unique collection $\delta(\mathcal{C})$ of balancing weights associated to each minimal balanced collection $\mathcal{C}$ of coalitions. In addition, the core is non-empty if and only if $\sum_{S\in\mathcal{C}}\delta_S(\mathcal{C})v(S)\leq v(N)$, for each balanced collection $\mathcal{C}$ that is both proper and minimal (see for instance Owen, 1982, Chapter 8). Clearly the class of noncompensable games contains those games for which imputations are blocked for all proper minimal balanced collections. For example, if $n=3$, the only proper minimal balanced collection is the collection $\mathcal{C}=\{\{1,2\},\{1,3\},\{2,3\}\}$. Hence any game such that: $v(12)+v(13)+v(23)>2v(123)$ is unbalanced and noncompensable. If $n=4$, the proper minimal balanced collections are (up to a permutation) $\mathcal{C}=\{\{1,2,3\},\{1,4\},\{2,4\},\{3,4\}\},\mathcal{C}'=\{\{1,2,3\},\{1,2,4\},\{1,3\},\{1,4\}\},\mathcal{C}''=\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$. Hence, a symmetric four-player game is noncompensable if and only if $2v_3+3v_2>1$ and $v_3>\frac{3}{4}$. Notice in particular that this set is smaller than region H in Figure 2. In fact, some four-player symmetric games (for which $v_3>\frac{3}{4}$ but $2v_3+3v_2\leq 1$) cannot be compensated by another unbalanced symmetric game, but can be compensated by nonsymmetric games (see Example 1).

We now prove that the set of noncompensable games is *exactly equal* to the set of games for which imputations are blocked by all minimal proper coalitions.

Proposition 2 *A game v with an empty core is noncompensable if and only if $\sum_{S \in \mathcal{C}} \delta_S(\mathcal{C})v(S) \geq v(N)$, for all proper minimal balanced collections $\mathcal{C}$ of coalitions.*

5 Additivity of other cooperative solution concepts

In this paper, we characterize the classes of cooperative games on which the core is additive. In this concluding section, we briefly comment on the generalization of our results to other cooperative solution concepts, and discuss the existing literature on additivity axioms in cooperative game theory.

We first note that, whenever a solution is defined by a system of linear inequalities, a direct application of the Lemma from Section 6.1 shows that the set of cooperative games can be partitioned into equivalence classes where the solution is additive. For example, Laussel and Le Breton (2001) analyze the Pareto frontier of sets $U(v) = \{(u_1, ..., u_n) | u_i \geq 0, \sum_{i \in S} u_i \leq v(N) - v(N \setminus S)\}$ for a given cooperative game v. From our analysis, it is clear that the convex polyhedron corresponding to the sum of two games v and w is equal to the sum of the convex polyhedra, $U(v + w) = U(v) + U(w)$ if and only if the extreme points of $U(v)$ and $U(w)$ are defined by the same coalitions.[7] On the other hand, the Lemma does not apply if the solution concept is not a unique polyhedron but a finite union of polyhedra, like the $\mathcal{M}_1^i$ bargaining set (Davis and Maschler (1963) and Maschler (1966)), or the kernel (Davis and Maschler (1965) and Maschler and Peleg (1966)). Suppose for illustration that a solution can be written as the union of two polyhedra: $\mathcal{S}(v) = \mathcal{A}(v) \cup \mathcal{B}(v)$. Even if we consider two games v and w with the same binding coalitions in the two polyhedra $\mathcal{A}$ and $\mathcal{B}$, so that $\mathcal{A}(v + w) = \mathcal{A}(v) + \mathcal{A}(w)$ and $\mathcal{B}(v + w) = \mathcal{B}(v) + \mathcal{B}(w)$, there is no guarantee that $\mathcal{S}(v + w) = \mathcal{S}(v) + \mathcal{S}(w)$. In fact, it is easy to check that $(\mathcal{A}(v + w) \cup \mathcal{B}(v + w)) \subseteq (\mathcal{A}(v) \cup \mathcal{B}(v)) + (\mathcal{A}(w) \cup \mathcal{B}(w))$, with strict inclusion for generic games.

We next consider solutions defined as unique points rather than convex polyhedra. Of course, the Shapley value satisfies additivity. Peters (1985) and (1986) provides an axiomatic characterization of solutions to Nash's bargaining problem which satisfy additivity and variants of superadditivity. Charnes and Kortanek (1969) and Kohlberg (1971) prove that the nucleolus is piecewise linear in the following sense. For any imputation x, and any coalition S, compute the excess function $e(x, S) = v(S) - x(S)$, and order the coalitions, by decreasing values of the excess, to obtain an array of coalitions $\mathbf{b}(x, v) = (b_1(x, v), ..., b_{2^n-1}(x, v)$. Partition

[7] We are grateful to Hideo Konishi for pointing this reference to us.

then the set of coalitional games in such a way that v and w belong to the same equivalence class if and only if, at the nucleolus of the two games, $\nu(v)$ and $\nu(w)$, the array of coalitions satisfy $\mathbf{b}(\nu(v), v) = \mathbf{b}(\nu(w), w)$. Then, for any two v and w in the same equivalence class, $\nu(v + w) = \nu(v) + \nu(w)$.

Finally, we would like to emphasize that, in our opinion, the study of the additivity of the core is only a first step in a research program on multi-issue cooperation. In the future, we hope to extend the analysis by studying alternative models of multi-issue bargaining in non-welfarist environments.

6 Appendix

6.1 A useful lemma

For each positive integers M and N, let $\mathcal{A}_{M,N}$ be the set of couples (A, b), where A is an $(M\text{x}N)$-matrix and b is an N-vector such that $P(A, b) = \{x \in \Re^N | Ax \geq b\}$ is non-empty and bounded. For each extreme point of $P(A, b)$, let $M_e(A, b)$ be the set of binding constraints at e, i.e. $M_e(A, b) = \{m \in \{1, \ldots, M\} | A_m e = b_m\}$. Two vectors b and b' are *equivalent* (given A), $b \sim b'$, if there exists a bijection f between the set of extreme points of $P(A, b)$ and the set of extreme points of $P(A, b')$ such that $M_e(A, b) = M_{f(e)}(A, b')$, for each extreme point e of $P(A, b)$.

Lemma 1 *Let (A, b) and (A, b') be two elements of $\mathcal{A}_{M,N}$, for some integers M and N. The two following properties are equivalent:*

1. *$P(A, b + b') = P(A, b) + P(A, b')$.*
2. *There exist two sequences $(b^k)_{k\in\mathbf{N}}$ and $(b'^k)_{k\in\mathbf{N}}$ in $\Re^N$ such that $(b^k)_{k\in\mathbf{N}}$ converges to b, $(b'^k)_{k\in\mathbf{N}}$ converges to b', (A, b^k) and (A, b'^k) belong to $\mathcal{A}_{M,N}$, and $b^k \sim b'^k$ for each $k \in \mathbf{N}$.*

The proof requires another lemma.

Lemma 2 *Let α be a strictly positive real number, and let (A, b) and (A, b') be two elements of $\mathcal{A}_{M,N}$, for some integers M and N. If $P(A, b + b') = P(A, b) + P(A, b')$, then $P(A, \alpha b + b') = P(A, \alpha b) + P(A, b')$.*

<u>Proof of Lemma 2</u>: It is always true that $P(A, \alpha b) + P(A, b') \subseteq P(A, \alpha b + b')$. So we have to prove the other inclusion. We first assume that $\alpha > 1$. Let x be an element of $P(A, \alpha b + b')$. Consider the correspondence $F : P(A, b') \to 2^{P(A,b')}$ defined as follows:

$$F(y') = \{z' \in P(A, b') | (\exists z \in P(A, b)) : z + z' = \frac{x - y'}{\alpha} + y'\},$$

for each $y' \in P(A, b')$. Observe that $A(\frac{x-y'}{\alpha} + y') \geq b + b'$ (the total coefficient of y', $\frac{\alpha-1}{\alpha}$, is positive because $\alpha > 1$). Hence F is non-empty valued. It is easy to check that it is also convex-valued, and has a closed graph. Kakutani's fixed point theorem implies that there exists y' in $P(A, b')$ such that $y' \in F(y')$. Hence $\frac{x-y'}{\alpha} \in P(A, b)$, and $x = (x-y')+y' \in P(A, \alpha b)+P(A, b')$.

Suppose now that $\alpha < 1$. We have: $P(A, \alpha b) + P(A, b') = \alpha P(A, b) + \alpha P(A, \frac{b'}{\alpha}) = \alpha[P(A, b) + P(A, \frac{b'}{\alpha})] = \alpha P(A, b + \frac{b'}{\alpha}) = P(A, \alpha b + b')$. The penultimate equality follows from the previous paragraph. The other equalities are straightforward. ■

Proof of Lemma 1: $(1 \Rightarrow 2)$ For each $k \in \mathbb{N}$, let $b^k = \frac{k}{k+1}b + \frac{1}{k+1}b'$ and $b'^k = \frac{1}{k+1}b + \frac{k}{k+1}b'$. Notice that if e^k is an extreme point of $P(A, b^k)$, then there exists a unique extreme point x of $P(A, b)$ and a unique extreme point x' of $P(A, b')$ such that $e^k = \frac{k}{k+1}x + \frac{1}{k+1}x'$. In addition, $M_{e^k}(A, b^k) = M_x(A, b) \cap M_{x'}(A, b')$. Indeed, if e^k is an extreme point of $P(A, b^k)$, then there exists a set L of N independent lines such that $A_L e^k = b^k_L$. By Lemma 2, there exist $x \in P(A, b)$ and $x' \in P(A, b')$ such that $e^k = \frac{k}{k+1}x + \frac{1}{k+1}x'$. It must be that $A_L x = b_L$ and $A_L x' = b'_L$. So x and x' are the unique vectors in $P(A, b)$ and $P(A, b')$ whose weighted sum coincides with e^k. It must also be that x and x' are extreme points of $P(A, b)$ and $P(A, b')$, respectively. Finally, $A_m e^k = b^k_m$ if and only if $A_m x = b_m$ and $A_m x' = b'_m$ (the necessary condition follows from the fact that $x \in P(A, b)$ and $x' \in P(A, b')$). Conversely, observe that if there exists an extreme point x of $P(A, b)$ and an extreme point x' of $P(A, b')$ such that $M_x(A, b) \cap M_{x'}(A, b')$ contains N independent lines, then $\frac{k}{k+1}x + \frac{1}{k+1}x'$ is an extreme point of $P(A, b^k)$. A similar argument holds to show that $\frac{1}{k+1}x + \frac{k}{k+1}x'$ is an extreme point of $P(A, b'^k)$.

For each extreme point e^k of $P(A, b^k)$, let $f(e^k)$ be the vector $\frac{1}{k+1}x + \frac{k}{k+1}x'$, where x is the unique extreme point of $P(A, b)$ and x' is the unique extreme point of $P(A, b')$ such that $e^k = \frac{k}{k+1}x + \frac{1}{k+1}x'$. The previous paragraph implies that $f(e^k)$ is an extreme point of $P(A, b'^k)$. It also implies that f is a bijection, and that $M_{e^k}(A, b^k) = M_{f(e^k)}(A, b'^k)$, for each extreme point e^k of $P(A, b^k)$. We thus have established Condition 2, since $(b^k)_{k\in\mathbb{N}}$ converges to b, and $(b'^k)_{k\in\mathbb{N}}$ converges to b'.

$(2 \Rightarrow 1)$ It is always true that $P(A, b)+P(A, b') \subseteq P(A, b+b')$. So we have to prove the other inclusion. Consider the correspondence ϕ associating to any vector b the nonempty bounded convex polyhedron $P(A, b)$. Because $\lambda P(A, b)+(1-\lambda)P(A, b') \subseteq P(A, \lambda b+(1-\lambda)b')$, the graph of ϕ is convex, and by Corollary 9.2.3 in Peleg and Sudhölter (2003), the correspondence ϕ is lower hemi continuous. Because $P(A, b)$ is defined by a set of continuous, linear inequalities, the correspondence ϕ is clearly upper hemi continuous, and hence fully continuous.

Now take a point x in $P(A, b + b')$. Because ϕ is lower hemi continuous, there exist sequences b^k and b'^k converging to b and b' such that $x \in P(A, b^k + b'^k)$. Furthermore, because the polyhedron $P(A, b + b')$ is only determined by the sum $b + b'$, we are free to choose two sequences b^k and b'^k such that $b^k \sim b'^k$. Suppose that we have proven that

$P(A, b + b') = P(A, b) + P(A, b')$ for each pair (b, b') of N-vector such that $b \sim b'$. Then, $x \in P(A, b^k) + P(A, b'^k)$ for all k. Because the correspondence ϕ is upper hemi continuous, this implies that $x \in P(A, b) + P(A, b')$.

The preceding argument shows that in order to finish the proof of Lemma 1, it is sufficient to show that $P(A, b+b') = P(A, b) + P(A, b')$ for each pair (b, b') of N-vector such that $b \sim b'$. We proceed by induction on N. Suppose first that $N = 1$. Then

$$P(A, b) = [\frac{b_k}{A_k}, \frac{b_l}{A_l}],$$

where

$$k = \arg \max_{m|A_m>0} \frac{b_m}{A_m} \text{ and } l = \arg \min_{m|A_m<0} \frac{b_m}{A_m}.$$

Since $b \sim b'$, we must have:

$$P(A, b') = [\frac{b'_k}{A_k}, \frac{b'_l}{A_l}],$$

and hence

$$P(A, b) + P(A, b') = [\frac{b_k + b'_k}{A_k}, \frac{b_l + b'_l}{A_l}].$$

The desired conclusion follows from the fact that $P(A, b + b') \subseteq [\frac{b_k+b'_k}{A_k}, \frac{b_l+b'_l}{A_l}]$.

Let $N \geq 2$ be such that the desired inclusion holds for all $N' \leq N - 1$. We prove now that $P(A, b + b') \subseteq P(A, b) + P(A, b')$, for all (A, b) and (A', b) in $\mathcal{A}_{M,N}$ such that $b \sim b'$. Assume first that (A, b) is such that for each $m \in \{1, \ldots, M\}$, there exists an extreme point e of $P(A, b)$ such that $A_m e = b_m$. Let e be an extreme point of $P(A, b + b')$. We will be done with this part of the proof after showing that $e \in P(A, b) + P(A, b')$. Let m be such that $A_m e = b_m + b'_m$.[8] Let $i \in N$ be such that $A_{m,i} \neq 0$.[9] Observe that

$$P(A, b) \cap \{x \in \Re^N | A_m x = b_m\} = \{x \in \Re^N | x_{-i} \in P(\bar{A}, \bar{b}),\ x_i = \frac{b_m - A_{m,-i} x_{-i}}{A_{m,i}}\}$$

$$P(A, b') \cap \{x \in \Re^N | A_m x = b'_m\} = \{x \in \Re^N | x_{-i} \in P(\bar{A}, \bar{b}'),\ x_i = \frac{b'_m - A_{m,-i} x_{-i}}{A_{m,i}}\}$$

$$P(A, b+b') \cap \{x \in \Re^N | A_m x = b_m + b'_m\} = \{x \in \Re^N | x_{-i} \in P(\bar{A}, \bar{b} + \bar{b}'),\ x_i = \frac{b_m + b'_m - A_{m,-i} x_{-i}}{A_{m,i}}\}$$

[8] The vector e is characterized by n equations of this type. The difficulty in this part of the proof stems from the fact that we are not sure a priori that these equations applied to b and b' lead to an element of $P(A, b)$ and $P(A, b')$, respectively.

[9] We assume without loss of generality that A does not have a line with only zero entries. Such lines are redundant and can be deleted anyway.

where $\bar{A}$, $\bar{b}$ and $\bar{b}'$ are defined as follows:

$$\bar{b}_k = b_k - \frac{A_{k,i} b_i}{A_{m,i}}$$

$$\bar{b}'_k = b'_k - \frac{A_{k,i} b'_i}{A_{m,i}}$$

$$\bar{A}_{k,j} = A_{k,j} - \frac{A_{m,j}}{A_{m,i}}$$

for all $k \in \{1, \ldots, M\} \setminus \{m\}$ and all $j \in \{1, \ldots, N\} \setminus \{i\}$. Notice that the first two sets above must be non-empty because $m \in M_e(A, b)$ for some extreme point e of $P(A, b)$ (and hence $m \in M_{f(e)}(A, b')$, since $b \sim b'$). $P(\bar{A}, \bar{b})$ and $P(\bar{A}, \bar{b}')$ are thus non-empty, and belong to $\mathcal{A}_{M-1,N}$. Take an extreme point $\bar{e}$ in $P(\bar{A}, \bar{b})$. Then $e = (\bar{e}, e_i = \frac{b_m - A, m, -i\bar{e}_{-i}}{A_{m,i}})$ is an extreme point of $P(A, b)$ such that $m \in M_e(A, b)$. Because $b \sim b'$, there exists an extreme point e' of $P(A, b')$ such that $m \in M_e(A, b')$ and all other constraints are defined by the same equalities as $\bar{e}$. It is easy to check that the $N-1$ vector $\bar{e}'$ such that $e'_i = \frac{b'_m - A, m, -i\bar{e}'_{-i}}{A_{m,i}}$ is an extreme point of $P(\bar{A}, \bar{b}')$, so that $\bar{b} \sim \bar{b}'$. The induction hypothesis implies that $P(\bar{A}, \bar{b} + \bar{b}') = P(\bar{A}, \bar{b}) + P(\bar{A}, \bar{b}')$. Hence there exists $x_{-i} \in P(\bar{A}, \bar{b})$ and $x'_{-i} \in P(\bar{A}, \bar{b}')$ such that $e_{-i} = x_{-i} + x'_{-i}$ and $e_i = \frac{b_m + b'_m - A_{m,-i} e_{-i}}{A_{m,i}}$. Taking $x_i = \frac{b_m - A_{m,-i} x_{-i}}{A_{m,i}}$ and $x'_i = \frac{b'_m - A_{m,-i} x'_{-i}}{A_{m,i}}$, we obtain that $e = x + x'$, where $x \in P(A, b)$ and $x' \in P(A, b')$, as desired.

Let us finally drop the assumption that each inequality appearing in $Ax \geq b$ is binding at some extreme point of $P(A, b)$. Let $z \in P(A, b + b')$, and let $L \subseteq \{1, \ldots, M\}$ be the set of inequalities that are binding at some extreme point of $P(A, b)$ (or $P(A, b')$, since $b \sim b'$). In particular, we have $z \in P(A_L, b_L + b'_L)$. By our previous argument, there exists $x \in P(A_L, b_L)$ and $x' \in P(A_L, b'_L)$ such that $z = x + x'$. Let $m \in \{1, \ldots, M\} \setminus L$. Notice that $P(A_{-m}, b_{-m}) \cap \{y \in \Re^N | A_m y = b_m\} = \emptyset$ (as otherwise the extreme points of this convex polyhedron would be extreme points of $P(A, b)$, contradicting the fact that $m \notin L$). Since $P(A_{-m}, b_{-m})$ is convex and has a nonempty intersection with $\{y \in \Re^N | A_m y \geq b_m\}$, we conclude that $P(A_{-m}, b_{-m}) \subseteq \{y \in \Re^N | A_m y > b_m\}$. Iterating the argument, we conclude that $A_L x \geq b_L$ implies that $Ax \geq b$. Similarly, $A_L x' \geq b'_L$ implies that $Ax' \geq b'$. Hence $z \in P(A, b) + P(A, b')$. ■

6.2 Equivalence classes of four-player symmetric games

We characterize (up to a permutation) the different categories of extreme points, and the conditions on the games for which those extreme points belong to the core. We restrict attention to the generic case where each extreme point is characterized by a set of three coalitions (in addition to N) for which the inequalities are binding. By superadditivity, we can restrict attention to coalitions which have a nonempty intersection – if two coalitions S and

T with $S \cap T = \emptyset$ are used, this must imply that $v(S \cup T) = v(S) + v(T)$, a nongeneric condition. Furthermore, we only have to consider collections of coalitions for which the conditions are independent. This leaves us with the following possible extreme points:

E1 Coalitions $\{1\}$, $\{1,2\}$, and $\{1,2,3\}$ lead to the extreme point $(0, v_2, v_3 - v_2, 1 - v_3)$. This vector belongs to the core if and only if $v_3 \geq 2v_2$ and $1 \geq 2v_3 - v_2$.

E2 Coalitions $\{1\}$, $\{1,2\}$, and $\{1,3\}$ lead to the extreme point $(0, v_2, v_2, 1 - 2v_2)$. This vector belongs to the core if and only if $v_2 \leq \frac{1}{3}$ and $2v_2 \geq v_3$.

E3 Coalitions $\{1\}$, $\{1,2,3\}$, and $\{1,2,4\}$ lead to the extreme point $(0, 2v_3 - 1, 1 - v_3, 1 - v_3)$. This vector belongs to the core if and only if $v_3 \leq \frac{2}{3}$ and $2v_3 \geq v_2 + 1$.

E4 Coalitions $\{1,2\}$, $\{1,3\}$, and $\{1,2,3\}$ lead to the extreme point $(2v_2 - v_3, v_3 - v_2, v_3 - v_2, 1 - v_3)$. This vector belongs to the core if and only if $2v_2 \geq v_3$ and $v_2 + 1 \geq 2v_3$.

E5 Coalitions $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$ lead to the extreme point $(\frac{3v_2-1}{2}, \frac{1-v_2}{2}, \frac{1-v_2}{2}, \frac{1-v_2}{2})$. This vector belongs to the core if and only if $v_2 \geq \frac{1}{3}$ and $v_2 + 1 \geq 2v_3$.

E6 Coalitions $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ lead to the extreme point $(\frac{v_2}{2}, \frac{v_2}{2}, \frac{v_2}{2}, 1 - 3\frac{v_2}{2})$. This vector belongs to the core if and only if $\frac{3}{2}v_2 \geq v_3$.

E7 Coalitions $\{1,2,3\}$, $\{1,2,4\}$, and $\{1,3,4\}$ lead to the extreme point $(3v_3 - 2, 1 - v_3, 1 - v_3, 1 - v_3)$. This vector belongs to the core if and only if $\frac{3}{4} \geq v_3 \geq \frac{2}{3}$ and $2v_3 \geq 1 + v_2$.

Figure 2 can be understood as follows. Games in region A (resp. F; G) have extreme points of the E1-type (resp. E3-; E7-type) only. Games in region B have extreme points of both E2- and E4-type. Games in region C have extreme points of both E2- and E6-type. Games in region D have extreme points of both E4- and E5-type. Games in region E have extreme points of both E5- and E6-type.

6.3 Proof of Proposition 2

($\Leftarrow$) This implication is obvious. If imputations are blocked for all proper minimal balanced collections, clearly the game is noncompensable.

($\Rightarrow$) Suppose that there exists one proper minimal balanced collection $\mathcal{C} = \{S_1, ..., S_j, ..., S_J\}$ for which $\sum_{j=1}^{J} \delta_j v(S_j) < v(N)$, where δ_j stands for $\delta_{S_j}(\mathcal{C})$. Construct then the game w as follows:

$$w(T) = \begin{cases} 0 & \text{if there does not exist } j \text{ such that } S_j \subseteq T, \\ \alpha t & \text{if there exists } j \text{ such that } T = S_j, \\ \alpha t(1 - \frac{1}{n}) & \text{if } T \neq N \text{ and there exists } j \text{ such that } T \supset S_j, \\ \alpha t - \gamma & \text{if } T = N, \end{cases}$$

where $t = |T|$, and γ is a strictly positive number that is lower than $v(N) - \sum_{j=1}^{J} \delta_j v(S_j)$, and α is a strictly positive number that is larger than γ.

We first check that the game w is superadditive. Because the collection $\mathcal{C}$ is proper, $S_j \cap S_k \neq \emptyset$ for all j, k. As the game w only puts positive worth on supersets of sets in $\mathcal{C}$, we conclude that if $S \cap T = \emptyset$, either $w(S) = 0$ or $w(T) = 0$. This implies that, in order to check that w is superadditive, we only need to verify that w is monotonic, that is $w(S \cup \{i\}) \geq w(S)$, for each coalition S and each player i that does not belong to S. The inequality is obvious if there does not exist j such that $S_j \subseteq S$, since w has non-negative values. Suppose now that $S = S_j$ for some j. We have: $w(S \cup \{i\}) = \alpha(s+1)(1 - \frac{1}{n}) = \alpha s + \alpha \frac{n-(s+1)}{n} \geq \alpha s = w(S)$, since $s \leq n-1$ (N is never part of a minimal balanced collection). Suppose now that $S_j \subset S$, for some j. Then $w(S \cup \{i\}) = \alpha(s+1)(1 - \frac{1}{n}) = \alpha s(1 - \frac{1}{n}) + \alpha(1 - \frac{1}{n}) \geq w(S)$. Finally, if $s = n-1$, we have: $w(S \cup \{i\}) = \alpha n - \gamma \geq \alpha n - \alpha \geq \alpha(n-1)(1 - \frac{1}{n}) = w(S)$, where the first inequality follows from the fact that $\gamma \leq \alpha$.

We now prove that w is not balanced with respect to $\mathcal{C}$, but satisfies the inequality with a slack for each other proper minimal balanced collection. We have:

$$\sum_{j=1}^{J} \delta_j w(S_j) = \sum_{j=1}^{J} \delta_j \alpha s_j = \sum_{i \in N} \sum_{j | S_j \ni i} \delta_j \alpha = \alpha n.$$

Hence $C(w) = \emptyset$, since $w(N) < \alpha n$. Consider any other proper minimal balanced collection, $\mathcal{C}' = \{T_1, ..., T_k, ..., T_K\}$. Following the same convention as before, δ'_k will stand for $\delta_{T_k}(\mathcal{C}')$. We compute:

$$\begin{aligned}
\sum_{k=1}^{K} \delta'_k w(T_k) &= \sum_{k | T_k = S_j} \delta'_k \alpha t_k + \sum_{k | T_k \supset S_j} \delta'_k \alpha t_k (1 - \frac{1}{n}) \\
&= \sum_{i \in N} (\sum_{k | T_k \ni i, T_k = S_j} \delta'_k \alpha + \sum_{k | T_k \ni i, T_k \supset S_j} \delta'_k \alpha (1 - \frac{1}{n})) \\
&= \alpha \sum_{i \in N} (\sum_{k | T_k \ni i, T_k \supseteq S_j} \delta'_k - \sum_{k | T_k \ni i, T_k \supset S_j} \frac{\delta'_k}{n}) \\
&< \alpha \sum_{i \in N} n.
\end{aligned}$$

The last inequality is obtained because $\sum_{k | T_k \ni i, T_k \supseteq S_j} \delta'_k \leq 1$, and equals 1 for all agents i only if $T_k \supseteq S_j$ for all sets T_k in the collection $\mathcal{C}'$. However, there must exist some k such that $T_k \supset S_j$ in such cases, because $\mathcal{C}' \neq \mathcal{C}$, which implies that $\sum_{k | T_k \ni i, T_k \supset S_j} \delta'_k \neq 0$ for some player i.Hence we can define, for any proper minimal balanced collection $\mathcal{C}' \neq \mathcal{C}$, the positive quantity

$$\Delta(w, \mathcal{C}') = \frac{\sum_{j=1}^{J} \delta_j w(S_j) - \sum_{k=1}^{K} \delta'_k w(T_k)}{\alpha}.$$

Notice that $\Delta(w, \mathcal{C}')$ does not depend on α. From now on we will assume that α is also larger than $\max_{\mathcal{C}'} \frac{\sum_k \delta'_k v(T_k) - v(N) + \gamma}{\Delta(w, \mathcal{C}')}$.

We end the proof by showing that $v + w$ is balanced, checking the relevant inequality for each proper minimal balanced collection of coalitions. For the collection $\mathcal{C}$, we have:

$$\sum_{j=1}^{J} \delta_j (v(S_j) + w(S_j)) < v(N) - \gamma + \alpha n = v(N) + w(N),$$

since $\gamma < v(N) - \sum_{j=1}^{J} \delta_j v(S_j)$, $\sum_{j=1}^{J} \delta_j w(S_j) = \alpha n$, and $w(N) = \alpha n - \gamma$. Next, consider any other proper minimal collection $\mathcal{C}'$. Then,

$$\begin{aligned} \sum_{k=1}^{K} \delta'_k (v(T_k) + w(T_k)) &= \sum_{k=1}^{K} \delta'_k v(T_k) + \sum_{j=1}^{J} \delta_j w(S_j) - \alpha \Delta(w, \mathcal{C}') \\ &< v(N) - \gamma + \alpha n \\ &= v(N) + w(N). \end{aligned}$$

The first equality follows from the definition of Δ, and the inequality follows from the additional restriction imposed on α at the end of last paragraph. We conclude that $v + w$ is balanced, and thus has a nonempty core. ■

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