Axiomatic Bargaining on Economic Environments with Lotteries*

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Abstract

Most contributions in axiomatic bargaining are phrased in the space of utilities. This comes in sharp contrast with standards in most other branches of economic theory. The present paper shows how Nash's original axiomatic system can be rephrased in a natural class of economic environments with lotteries, and how his uniqueness result can be recovered, provided one completes the system with a property of independence with respect to preferences over unfeasible alternatives. A similar result can be derived for the Kalai-Smorodinsky solution if and only if bargaining may involve multiple goods. The paper also introduces a distinction between welfarism and cardinal welfarism, and emphasizes that the Nash solution is *ordinally* invariant on the class of von Neumann-Morgensterm preferences.

Keywords: Bargaining, Welfarism, Nash, Kalai-Smorodinsky, Expected Utility

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1. INTRODUCTION

Nash's (1950) bargaining model and the solution he derived axiomatically have had a great impact on economic theory. Yet his contribution is at odd with the major trend of the field in the second half of the twentieth century because his arguments are phrased in the space of utilities. The standard, instead, has been to work with more primitive concepts such as strategies, economic or social outcomes, and the participants' preferences. The present paper shows how Nash's theory can be rephrased to meet these modern standards in a model where agents can choose lotteries over simple economic outcomes, while evaluating those lotteries with ordinal von Neumann-Morgenstern preferences (which, of course, are also a cornerstone of non-cooperative game theory and information economics).

Following Nash, a bargaining problem is a couple (U, d), where U is a compact convex subset of $\mathbb{R}^2$ that represents the set of utility pairs that are achievable through cooperation, and $d \in U$ is the utility pair that prevails in case of disagreement. A solution associates a feasible utility pair to each bargaining problem. Nash proved that there exists a unique solution that satisfies the properties of Efficiency, Symmetry, Scale Covariance, and Independence of Irrelevant Alternatives (IIA). It is obtained by maximizing over U the product of the participants' utility gains compared to d. A large literature ensued, some papers establishing alternative characterizations of the Nash bargaining solution, other introducing alternative axioms to characterize new solutions (see Thomson (1994) for a survey).

A bargaining problem in this sense is simply a representation of the underlying economic or social problem via some utility functions that encode the bargainers' preferences. As pointed out by Nash himself, the convexity of U follows from the idea that bargainers can agree on lotteries over basic outcomes, provided one restricts attention to utility functions that are linear in probabilities. Of course, there is no loss of generality in representing the options available to the bargainers and the final agreement in the space of utilities. One is also free to use linear utility functions if the bargainers have von Neumann-Morgenstern preferences. On the other hand, restricting bargaining solutions to take convex utility possibility sets as argument, instead of the underlying economic or social environments, is a significant assumption. Indeed, it presupposes that the image of the economic or social problems through linear representations of the bargainers' von Neumann-Morgenstern preferences is sufficiently informative to determine the solution. I will call this a property of 'Cardinal Welfarism' (C-WELF). It is related to the notion of welfarism introduced by Roemer (1986; 1988), except that it emphasizes in addition the key role that linear representations play in Nash's theory.[1]

C-WELF is not appealing as a postulate, because it is hard to understand what it entails in terms of the primitives, namely the set of available agreements and the bargainers' preferences. As an illustration, consider the five following simple problems. They will be constructed on different sets of economic outcomes, two of which being represented in Figure 1.[2] Each point on the graph specifies the monetary profit that

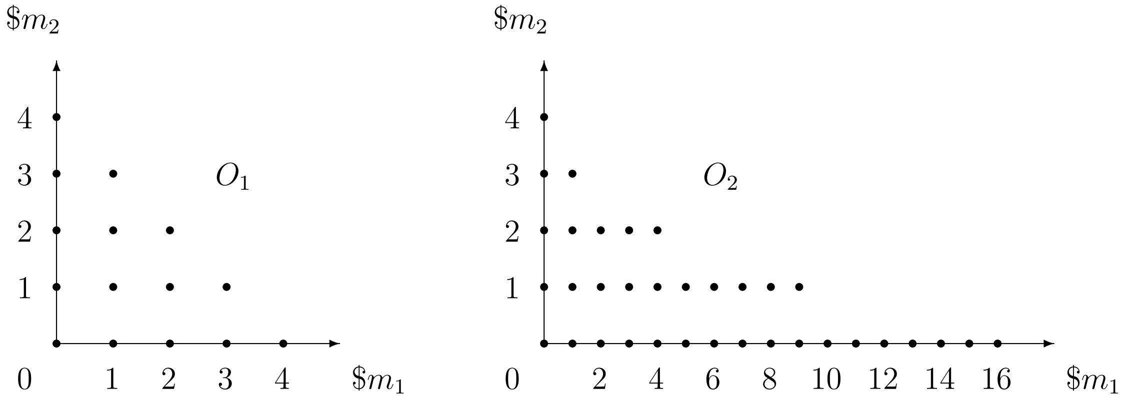

Figure 1

both bargainers will receive if they agree on that option, while they receive nothing in case of disagreement. In the first problem, the two bargainers can agree on any lottery over O_1, and are expected utility maximizers with Bernoulli function $u(m) = m$ (risk-neutral). In the second problem, the two bargainers can agree on any lottery over O_2, and are expected utility maximizers with Bernoulli functions $u_1(m) = m/4$ and $u_2(m) = m$ (same preferences as before, but with a different utility representation for the first bargainer). In the third problem, the two bargainers can agree on any lottery over O_1, and are expected utility maximizers with Bernoulli functions $u_1(m) = m^2/4$ (risk-loving) and u_2 being a concave function such that $u_2(0) = 0$, $u_2(1) = 7/4$, $u_2(2) = 3$, $u_2(3) = 15/4$, and $u_2(4) = 4$ (risk-averse). In the fourth problem, the two bargainers can agree on any lottery over O_2, and are expected utility maximizers with Bernoulli functions $u_1(m) = \sqrt{m}$ (risk-averse) and $u_2(m) = m$ (risk-neutral). In the fifth problem, the two bargainers must allocate four apples and four bananas (without cutting the fruits, but perhaps using a lottery), with Bernoulli functions $u_1(x_1, x_2) = (x_1 + x_2)/2$ and $u_2(x_1, x_2) = \sqrt{x_1 x_2}$, where x_1 is a

[1] C-WELF, on the other hand, is different from Sen's (1977, 1979) notion of Welfarism in social choice theory - see the discussion of the axiom of 'exhaustivity' in Section 3.

[2] The examples of the Introduction involve problems with a finite set of outcomes. The axiomatic results in the main text hold indeed on that class of problems, but also on the larger class of problems constructed over any compact set.

quantity of apples and x_2 is a quantity of bananas. These five problems (to list only a few) are very different in their economic description, but they happen to have the same image in the space of utilities. The resulting agreements must thus be related (in the sense of coinciding in utilities) in Nash's cardinal welfarist framework. I do not argue that a solution that satisfies this property is necessarily unappealing. On the other hand, there is no straightforward argument in its favor, and it is difficult to fully grasp what it entails. This is why C-WELF should not be accepted as a postulate.

Pushing the reasoning further, observe that conducting the analysis of bargaining problems in the space of utilities, that is assuming C-WELF *a priori*, is at best confusing and at worst misleading, because it creates a mismatch between the axioms' interpretation and their real content. The first bargaining problem described in the previous paragraph is clearly symmetric. Assuming that both bargainers have equal bargaining abilities, it is reasonable to anticipate that they will agree on a contract that gives \$2 to each of them (or a lottery that leaves them indifferent to that contract). This is consistent with the usual interpretation of Nash's symmetry axiom. Notice that the four other problems are also symmetric in Nash's sense (indeed they all have the same image in the utility space), but there is no clear relation between the players' bargaining positions in their economic description. A similar point can be made for his axioms of scale covariance and IIA. The information retained when representing bargaining problems in the space of utilities is just too scarce to determine whether what looks like a scale transformation, or a reduction in the set of available contracts is not actually obtained by considering completely unrelated problems.

Nash's theory seems to be build on some cardinal notion of utilities. Indeed, his solution is at best scale covariant in his model, and the convexity of the sets of feasible utilities is justified only when applying linear utility functions. Nash's appeal to von Neumann-Morgenstern's expected utiltiy theory to justify these assumptions is slightly misleading. Indeed, while traditional axioms guarantee the *existence* of linear representations, a preference relation in that theory is an *ordinal* concept. von Neumann and Morgenstern's and subsequent arguments do not favor linear over any other form of representation: any increasing transformation of a linear representation of a von Neumann-Morgenstern preference is another valid representation of the same ordering, though usually not linear. One comes to wonder then whether the axiomatic justification behind the Nash bargaining solution relies crucially on the possibility of measuring intensities of preferences. It does seem so in a welfarist context. Indeed, we know since Shapley (1969) that it is impossible to find a solution in the space of utilities that is efficient, strictly individually rational, and ordinally invariant (Theorem 3 in Section 5 below will make a similar point in my specific model). Therefore, if one believes in welfarism (which is required to accept Nash' theory in its original and since then standard formulation), then one must necessarily rely on other theories of preferences that are not ordinally invariant. While there are some interesting theoretical foundations of utility functions that are truly cardinal in that sense (see chapter 6 of Fishburn, 1970, for a survey), the consensus so far in mainstream Economics is that they have no practical meaning because they cannot be deduced by observing individual choices.

On the other hand, I already argued that welfarism is not appealing as a postulate. So I propose to re-phrase Nash's ideas in an explicit economic environment, allowing (selfish) bargainers with von Neumann-Morgenstern preferences to agree on lotteries. As expected, numerous non-welfarist solutions also satisfy his axioms in that setting (see Example 1, for instance). One of the main contributions of the present paper is to show that all these alternative solutions violate a simple property of independence with respect to preferences between unfeasible alternatives (IPUA). This provides an axiomatic justification for the Nash bargaining solution that is based on the ordinal concept of von Neumann-Morgenstern preferences, the use of linear representations coming now as a consequence of the axioms, instead of being a prerequisite. The result also deepens our understanding of the Nash bargaining solution, IPUA being logically weaker, and more straightforward to understand than C-WELF (see Section 5).

As hinted by its name, IPUA requires that the solution of two problems that differ only in the bargainers' perferences over outcomes that are not feasible coincide. Interestingly the axiom is not really new, but I believe it is the first time that it is explicitly applied to bargaining environments with lotteries. As far as I can tell, the first mention of a similar property can be found in Karni and Schmeidler (1975).[3] They show its close relation (together with IIA) to the maximization of a social welfare ordering that satisfies Arrow's independence property. This type of independence property has been rather often invoked in the social choice literature since then.[4] Other authors impose IPUA implicitly by defining a bargaining problem as a set of outcomes, and von Neumann-Morgenstern preferences over *feasible* lotteries (see e.g. Rubinstein et al., 1992; Valenciano and Zarzuelo, 1997). Notice also that any solution that coincides with the

[3]Karni and Schmeidler themselves refer to a 1969 *mimeo* written by A. Gibbard.

[4]Here are some relevant references: Plott (1976), Grether and Plott (1982), Campbell (1992), Dutta et al. (2001), Ehlers and Weymark (2003), Fleurbaey (2003), Chambers (2005), Fleurbaey and Tadenuma (2007), and de Clippel and Bejan (2009). The list aims at illustrating various formulations of the same idea, and various contexts where the property has been applied, but it is far from being exhaustive.

subgame-perfect Nash equilibrium outcome of some non-cooperative bargaining procedure whose outcomes involve only feasible agreements,[5] must necessarily satisfy IPUA. Solutions that can be derived via the Nash program must thus satisfy IPUA.[6] While IPUA may thus seem completely innocuous, I must also emphasize what it entails so that the reader can decide whether he/she finds it acceptable. One key consequence is that the bargainers' risk attitudes play no role in determining the solution of elementary problems where the bargainers can agree on lotteries over only three outcomes of the form $(0,0)$, $(x,0)$, and $(0,y)$, with x and y being economic bundles.[7] Indeed, there is only one possible preference relation when comparing lotteries over two bundles (0 and x for the first bargainer, and 0 and y for the second bargainer), where more probability on the most-preferred bundle is always better.

Roemer (1988) was first to emphasize the welfarist postulate that underlies axiomatic bargaining theory. His reconstruction of the theory on economic environments is not that informative, because the CONRAD property that he introduces to that effect turns out to be equivalent to, and probably even more difficult to interpret than, the property of welfarism itself. Roemer concludes that his reconstruction of bargaining theory demonstrates "*the lengths to which one must go to preserve the axiomatic characterization of the standard bargaining mechanisms on economic environments*" (Roemer (1988), page 30), but the complexity of one reconstruction does not necessarily imply that there are no more straightforward alternative routes. A major difference between our two approaches is that IPUA does not characterize C-WELF. Instead it is weaker, and easier to interpret. It is only when combined with Nash's other axioms (especially IIA) that one obtains C-WELF. As a corollary, my reconstruction is also less ambitious, since it is not guaranteed that every axiomatic result in bargaining theory can be recovered in economic environments by adding IPUA. Example 2 in Section 4, for instance, shows that replacing Nash's independence property by Kalai and Smorodinsky's (1975) restricted monotonicity property does not characterize their cardinal welfarist solution when bargaining over only one good. On the other hand, Theorem 2 proves that the characterization result holds when

[5]Implementation results that involve nonfeasible outcomes out of equilibrium are often seen as unappealing (see for instance Demange's (1984) criticism of Crawford (1979) procedure for generating Pareto efficient egalitarian allocations.

[6]The Nash program is sometimes understood in a broader sense, allowing the equilibrium of the non-cooperative procedure to depend on some additional features such as the bargainers' discount factors. Though I am not aware of a specific example, I suppose it is possible to implement in this broader sense a solution that does not satisfy IPUA.

[7]On the other hand, the bargainers' risk attitudes play a crucial role in determining the Nash bargaining solution in non-elementary problems.

allowing for multiple goods. My results also rely heavily on the use of lotteries and von Neumann-Morgenstern preferences, while Roemer worked with concave utility functions defined over deterministic outcomes.

Rubinstein et al. (1992) is another important paper on axiomatic bargaining in explicit environments. They emphasized the ordinal character of the Nash bargaining solution, as I do here. There are two main differences though. First, the corner-stone of their characterization is an axiom of independence with respect to changes in preferences that is far stronger than IPUA. They call it a property of IIA, because the changes in preferences involved do imply a contraction of the set of utilities as in Nash's IIA, but obvioulsy it is different from the usual interpretation of Nash's IIA axiom which involves changes in the set of available agreements (the set of feasible agreements is indeed fixed throughout Rubinstien et al.'s paper).[8] Their result develops an interesting alternative interpretation and characterization of the Nash bargaining solution, in the tradition of Zeuthen's (1930) idea of concession (see also Harsanyi, 1956) and Aumann and Kurz's (1977) notion of boldness (see also Burgos et al., 2002). My paper, on the other hand, shows that Nash's original result can be accomodated to some natural economic environments, provided one accepts IPUA. Second, Rubinstein et al.'s axiomatic result involves a strong symmetry axiom that is harder to interpret than, and that does not follow from, a principle of anonymity (or equal bargaining abilities). Their symmetry axiom constitutes in fact a first step towards C-WELF, as it implies, in their model, that the image of the solution in the utility space lies on the 45^o-line whenever the image of the bargaining problem is symmetric (in Nash's sense) for some linear representation of the players von Neumann-Morgenstern preferences. I conclude this introduction by observing that Rubinstein et al.'s axiomatic characterization is also valid on larger classes of preferences, allowing for some forms of non-expected utility (see also Grant and Kajii, 1995). I have not been able so far to adapt my arguments to this more general framework. Investigating the possibility of such extensions remains an interesting topic for future research. The related literature will be further discussed in Section 5.

2. DEFINITIONS

Let L be the set of goods. A *bargaining problem* is a triple $(O, \succeq_1, \succeq_2)$, where $O \subseteq \mathbb{R}^L_+ \times \mathbb{R}^L_+$ is the compact[9] set of available outcomes that specify a bundle for each bargainer, and

[8] This illustrates again that working in the space of utilities may be misleading when it comes to interpreting the axioms.

[9] O can be finite or infinite. All the results of the paper remain true if one restricts the domain to bargaining problems with a finite set O of outcomes.

$\succeq_i$ is player i's preference relation defined over $\Delta(\mathbb{R}^L_+)$,[10] the set of simple[11] lotteries over $\mathbb{R}^L_+$ ($i = 1, 2$). The bargainers receive no good if they do not reach an agreement, and I assume throughout the paper that the bargainers can agree to implement the disagreement outcome, i.e. $(0, 0) \in O$. I also assume that there exists $\mu \in \Delta(O)$ such that $\mu \succ_i 0$, for both $i = 1, 2$. Otherwise, the problem is easy to solve by applying an argument of efficiency. The bargainers' preferences are assumed to be strictly increasing (i.e. $o \succ_i o'$ whenever $o \geq o'$ and $o \neq o'$), and of the *von Neumann-Morgenstern* (vN-M) type, meaning that they are complete, transitive, continuous, and satisfy the usual independence axiom (see e.g. Fishburn (1970, chapter 8) and references therein).

A *solution* Σ associates to each bargaining problem $(O, \succeq_1, \succeq_2)$ a nonempty subset of lotteries defined over O. Nash's axioms can easily be rephrased in this economic context. Observe how their formal statement matches well their usual interpretation, contrarily to their phrasing in Nash's cardinal welfarist formulation (as already argued more in detail in Section 1). The following axioms are assumed to hold for each bargaining problem $(O, \succeq_1, \succeq_2)$.

Pareto Indifference (PI) *If μ and ν both belong to $\Sigma(O, \succeq_1, \succeq_2)$, then $\mu \sim_1 \nu$ and $\mu \sim_2 \nu$.*

Efficiency (EFF) *If $\mu \in \Sigma(O, \succeq_1, \succeq_2)$, then there does not exist $\nu \in \Delta(O)$ such that $\nu \succeq_i \mu$ for both $i \in \{1, 2\}$, and $\nu \succ_i \mu$ for some $i \in \{1, 2\}$.*

Anonymity (AN) *Let $O^* = \{(x, y) \in \mathbb{R}^L_+ \times \mathbb{R}^L_+ | (y, x) \in O\}$. For each $\mu \in \Delta(O)$, let $\mu^* \in \Delta(O^*)$ be the lottery defined as follows: $\mu^*(x, y) = \mu(y, x)$, for each $(x, y) \in O^*$. Then $\Sigma(O, \succeq_2, \succeq_1) = \{\mu^* | \mu \in \Sigma(O, \succeq_1, \succeq_2)\}$.*

Independence of Irrelevant Alternatives (IIA) *If $\mu \in \Sigma(O, \succeq_1, \succeq_2)$, $O' \subseteq O$, and $\mu \in \Delta(O')$, then $\mu \in \Sigma(O', \succeq_1, \succeq_2)$.*

PI requires that the solution provides a unique answer to each bargaining problem, meaning formally that the solution is essentially single-valued. There can be multiple lotteries in the solution of a problem only if both bargainers are indifferent between all of them. Nash imposed this type of restriction in the very definition of a solution, by focusing on functions instead of correspondences. I prefer to formulate it as an explicit axiom, drawing attention to the fact that unicity in the space of utilities does not necessarily imply unicity in the underlying economic environment. EFF guarantees that the solution makes the most out of the feasibility constraints faced by the two bargainers: there is no alternative contract that would make both of them at least as well off, and at least one

[10] Bargainers are thus assumed to be selfish, caring only about the bundle they will consume.

[11] i.e. with finite support.

of them strictly better off. Anonymity requires that the solution does not depend on the identity of the bargainers. Formally, if we exchange the identity of the two bargainers, then the solution should change accordingly. It is thus assumed, as a benchmark, that the two players have equal bargaining abilities. Nash imposed a slightly weaker property of symmetry, namely that the solution should be symmetric if the problem itself is symmetric (i.e. $O^* = O$ and $\succeq_1 = \succeq_2$ in the present framework). The motivation behind the two axioms is the same though. As for IIA, suppose that the bargainers recognize that the lottery $\mu \in \Delta(O)$ is a reasonable agreement for the problem $(O, \succeq_1, \succeq_2)$. Suppose now they learn that less alternatives are available, in that they must agree on a lottery over $O' \subseteq O$, but that μ is still feasible, i.e. $\mu \in \Delta(O')$. It is then assumed that the bargainers will recognize that μ' is a reasonable agreement for the problem $(O', \succeq_1, \succeq_2)$ as well.

3. MAIN RESULT

PI, EFF, AN, and IIA are far from characterizing a unique solution. By PI, a solution determines a unique indifference class for each bargaining problem, but one may construct many solutions by selecting different subsets of feasible lotteries that belong to that class. The next axiom rules out such multiplicity, by requiring the solution to be exhaustive.

Exhaustivity (EX) *Let $\nu \in \Delta(O)$ and $\mu \in \Sigma(O, \succeq_1, \succeq_2)$. If $\mu \sim_1 \nu$ and $\mu \sim_2 \nu$, then $\nu \in \Sigma(O, \succeq_1, \succeq_2)$.*

The axiom is very convincing if the bargainers care only about fulfilling their preferences, as usually assumed in economics. Suppose that the bargainers recognize that the lottery $\mu \in \Delta(O)$ is a reasonable compromise for the problem $(O, \succeq_1, \succeq_2)$. Suppose also that both bargainers are indifferent between μ and another lottery $\nu \in \Delta(O)$. EX then assumes that the bargainers would recognize ν as another reasonable compromise for that problem.

EX is reminiscent of Sen's (1977, 1979) notion of welfarism (even though such statement is necessarily fuzzy, since Sen is concerned with social welfare orderings, while I study bargaining solutions). Sen argues that indifference on the part of all the society members does not necessarily imply social indifference when comparing alternatives. In such cases, the social comparison is based on some information that goes beyond the mere definition of individuals' preferences, and *a fortiori* utilities. One could reject EX on similar ground. For instance, two bargainers may be indifferent between a non-degenerate lottery μ and a deterministic outcome in the solution of a problem, but nevertheless consider that μ is not a reasonable compromise, on the basis that lotteries should be avoided whenever possible. While it is possible to adapt the axioms appearing in Theorem 1 below to accomodate this kind of solution, I will simply rule them out by assuming EX. First I think that EX is more reasonable in bargaining theory than in social choice. Second, the main focus of the present paper is on Roemer's notion of welfarism, as discussed in the Introduction and formalized by C-WELF in Section 5, which deserves careful attention even when assuming EX.

There are still many solutions that satisfy PI, EFF, AN, IIA, and EX. Here is a class of such solutions that share some similarity with the concept of egalitarian equivalence introduced by Pazner and Schmeidler (1978).

Example 1 *Let $(O, \succeq_1, \succeq_2)$ be a bargaining problem, and let $d \in \mathbb{R}^L_{++}$. For each $\mu \in \Delta(O)$, let $\alpha^d_i(\mu)$ be the unique real number such that i is indifferent between the lottery μ and receiving $\alpha^d_i(\mu)d_l$ units of each good l, for sure. Let $\hat{\alpha}^d(\mu)$ be the vector in $\mathbb{R}^2_+$ obtained by rearranging the components of $\alpha^d(\mu)$ increasingly. The egalitarian equivalent solution Σ^d_{EE} is obtained by maximizing $\hat{\alpha}^d$ according to the lexicographic order. It is not difficult to check that Σ^d_{EE} satisfies PI, EX, AN, EFF, and IIA. Notice that multiplying d by a scalar does not change the solution. There is thus a unique egalitarian equivalent solution when $L = 1$, the vector $\alpha(\mu)$ determining the certainty equivalent of μ for both bargainers. The solution varies with the direction d when $L \geq 2$.*

I now introduce the property that will play a key role in establishing the main result.

Independence of Preference between Unfeasible Alternatives (IPUA) *Let $(O, \succeq_1, \succeq_2))$ and $(O, \succeq'_1, \succeq'_2))$ be two bargaining problems. If $\mu \succeq_i \nu$ if and only if $\mu \succeq'_i \nu$, for each $\mu, \nu \in \Delta(\{x \in \mathbb{R}^L_+ | (\exists o \in O) : o_i = x\})$ and each $i \in \{1, 2\}$, then $\Sigma(O, \succeq_1, \succeq_2) = \Sigma(O, \succeq'_1, \succeq'_2)$.*

The two bargaining problems in IPUA differ only in the bargainers' perferences over options that are not available. It is then required that the solutions of these two problems coincide. IPUA has already been discussed in the Introduction, and will be further discussed in Section 5.

The proof of the main result requires one last axiom.

Strong Individual Rationality (SIR) *Let $(O, \succeq_1, \succeq_2)$ be a bargaining problem. Then $\mu \succ_1 0$ and $\mu \succ_2 0$, for each $\mu \in \Sigma(O, \succeq_1, \succeq_2)$.*

Agreements are always individually rational in the model I consider, since the disagreement outcome is the worst alternative. EFF already guarantees that the solution does not include the disagreement outcome (remember that there are some agreements that can make both bargainers better off than with the disagreement outcome). SIR rules out in addition solutions that give the disagreement outcome to one bargainer and a lottery that is strictly better than the disagreement outcome to the other bargainer. Indeed, this would clearly not constitute a compromise of the bargainers' opposing interests.

Theorem 1 *There exists a unique solution Σ that satisfies PI, EFF, AN, IIA, EX, IPUA, and SIR. It is computed as follows:*

$$\Sigma(O, \succeq_1, \succeq_2) = \arg \max_{\mu \in \Delta(O)} (U_1(\mu) - U_1(0))(U_2(\mu) - U_2(0)),$$

where $U_i : \Delta(\mathbb{R}^L_+) \to \mathbb{R}$ is any[12] *linear representation of the vN-M preferences $\succeq_i$ ($i = 1, 2$).*

The solution derived from the axioms in Theorem 1 is simply the reformulation of the Nash bargaining solution in our economic environment. It will thus be denoted by Σ_N in the remainder of the paper. Observe that the use of linear representations of the preferences follows from the axioms, instead of being assumed in the model and the axioms themselves. Theorem 1 will appear to be a consequence of two lemmas. I need to introduce one last axiom to state them.

Equal Probabilities in Elementary Problems (EPEP) *Let $O = \{(0, 0), (x, 0), (0, y)\}$, for some bundles x, y different from 0. Then $\Sigma(O, \succeq_1, \succeq_2) = \{\frac{1}{2}(x, 0) \oplus \frac{1}{2}(0, y)\}$, for each preference profile $(\succeq_1, \succeq_2)$.*

A bargaining problem is elementary if the feasible outcomes are either full disagreement, a bundle x for the first bargainer and the disagreement outcome for the second bargainer, or a bundle y for the second bargainer and the disagreement outcome for the first bargainer. EPEP requires that the solution places equal probabilities on $(x, 0)$ and $(0, y)$, independently of what x, y and the bargainers' preferences are. We will see in Lemma 2 that this property that may seem a bit arbitrary for the moment is in fact implied by the axioms listed in Theorem 1. First I show that it is sufficient, in combination with PI, IIA, and EX, to characterize the Nash bargaining solution.

Lemma 1 *Σ_N is the only solution that satisfies PI, IIA, EX, and EPEP.*

Proof: The fact that Σ_N satisfies the axioms follows from the usual properties of the Nash bargaining solution defined in the space of utilities. I will thus focus on proving uniqueness. Let Σ be a solution that satisfies the axioms, let $(O, \succeq_1, \succeq_2)$ be a bargaining

[12] If two different sets of linear utility functions (U_1, U_2) and (V_1, V_2) represent $(\succeq_1, \succeq_2)$, then there exists $\alpha \in \mathbb{R}^2_+$ and $\beta \in \mathbb{R}^2$ such that $U_i = \alpha_i V_i + \beta_i$, for $i = 1, 2$. Hence $\arg\max_{\mu \in \Delta(O)}(U_1(\mu) - U_1(0))(U_2(\mu) - U_2(0)) = \arg\max_{\mu \in \Delta(O)}(V_1(\mu) - V_1(0))(V_2(\mu) - V_2(0))$, and the solution is thus well-defined.

problem, and let (U_1, U_2) be two linear representations of $(\succeq_1, \succeq_2)$, and let

$$(\sigma_1^*, \sigma_2^*) = (U_1(\mu), U_2(\mu)),$$

for some (or each, by PI) $\mu \in \Sigma_N(O, \succeq_1, \succeq_2)$. Notice that both σ_1^* and σ_2^* must be strictly positive. Let

$$\mathcal{U} = \{(U_1(\mu), U_2(\mu)) | \mu \in \Delta(O)\}, \text{ and}$$

$$\mathcal{V} = \{x \in \mathbb{R}^2 | x \geq (U_1(0), U_2(0)) \text{ and } (x_1 - U_1(0))(x_2 - U_2(0)) \geq (\sigma_1^* - U_1(0))(\sigma_2^* - U_2(0))\}.$$

The sets $\mathcal{U}$ and $\mathcal{V}$ are both convex, and $\mathcal{U} \cap \mathcal{V} = \{\sigma^*\}$. The separating hyperplane theorem implies that

$$\mathcal{U} \subseteq \{x \in \mathbb{R}^2 | \frac{x_1 - U_1(0)}{\sigma_1^* - U_1(0)} + \frac{x_2 - U_2(0)}{\sigma_2^* - U_2(0)} \leq 2\},$$

because the gradient of the function $(x_1 - U_1(0))(x_2 - U_2(0))$ at σ^* is proportional to $(\sigma_2^* - U_2(0), \sigma_1^* - U_1(0))$. Hence $\mathcal{U}$ is included in the triangle with extreme points $(U_1(0), U_2(0))$, $(2\sigma_1^* - U_1(0), U_2(0))$, and $(U_1(0), 2\sigma_2^* - U_2(0))$. Let finally $x^i \in \mathbb{R}_+^L$ be such that $U_i(x^i) = 2\sigma_i^* - U_i(0)$, and let $O' = O \cup \{(x^1, 0), (0, x^2)\}$. Notice that $\mathcal{U}' = \{(U_1(\mu), U_2(\mu)) | \mu \in \Delta(O')\}$ coincides with the triangle whose extreme points are $(U_1(0), U_2(0))$, $(U_1(x^1), U_2(0))$, and $(U_1(0), U_2(x^2))$. Let $\mu' \in \Sigma(O', \succeq_1, \succeq_2)$. There must exist $\alpha, \beta \in [0, 1]$ such that $(U_1(\mu'), U_2(\mu')) = \alpha(U_1(0), U_2(0)) + \beta(U_1(x^1), U_1(0)) + (1 - \alpha - \beta)(U_2(0), U_2(x^2))$. EX implies that the lottery that gives $(0, 0)$ with probability α, $(x^1, 0)$ with probability β, and $(0, x^2)$ with probability $1 - \alpha - \beta$ belongs to $\Sigma(O', \succeq_1, \succeq_2)$. IIA implies that this lottery also belongs to $\Sigma(\{(0, 0), (0, x^2), (x^1, 0)\}, \succeq_1, \succeq_2)$. EPEP implies that $\alpha = 0$ and $\beta = 1/2$. Hence $U_i(\mu') = \sigma_i^*$, for both $i = 1, 2$. This implies that $\mu \in \Sigma(O', \succeq_1, \succeq_2)$, by EX, and that $\mu \in \Sigma(O, \succeq_1, \succeq_2)$, by IIA. I just proved that $\Sigma_N \subseteq \Sigma$. PI implies that $\Sigma = \Sigma_N$. ■

Lemma 1 is not without interest on its own. First, it offers a characterization of the Nash bargaining solution for a *fixed* profile of vN-M preferences. IPUA and the main axiom of Rubinstein and al. (1992), on the contrary, involve comparisons of bargaining problems with different preferences. Second, while examples of elementary problems have been used in the past to discuss the general appeal of the Nash and the Kalai-Smorodinsky bargaining solutions (see Roth, 1979, pages 67-70; Roemer, 1996, Section 2.5), it has not been observed that the general property underlying these examples, as captured by EPEP, is in fact strong enough in itself to characterize the Nash bargaining solution when combined with the economic reformulation of Nash's axioms. Finally, observe that Lemma 1 is reminiscent of Myerson's (1984) characterization of a generalized Nash bargaining solution for environments with incomplete information, as his random dictatorship axiom is closely related to (and immediately implies) EPEP. I now prove that the axioms listed in Theorem 1 together imply EPEP.

Lemma 2 *PI, EFF, AN, IIA, EX, IPUA, and SIR together imply EPEP.*

Proof: By EFF, any lottery in the solution of the problem described in EPEP must give $(0, y)$ with some probability α, and $(x, 0)$ with probability $1-\alpha$. I prove now that $\alpha = 1/2$.

Suppose first that $y = x$. AN implies that $(1-\alpha)(0, x) \oplus \alpha(x, 0)$ belongs to $\Sigma(O^*, \succeq_2, \succeq_1)$. Notice that the preference orderings $\succeq_i$ and $\succeq_{-i}$ coincide on $\Delta(\{0, x\})$. Hence $(1-\alpha)(0, x) \oplus \alpha(x, 0)$ belongs to $\Sigma(O^*, \succeq_1, \succeq_2)$, by IPUA, or to $\Sigma(O, \succeq_1, \succeq_2)$, since $O^* = O$. PI implies that $\alpha = 1/2$.

Suppose now that $y \neq x$.[13] Hence there exists $\pi \in \mathbb{R}^L_{++}$ such that $\pi \cdot x > \pi \cdot y$ or $\pi \cdot y > \pi \cdot x$.[14] I will assume that the former inequality holds - a similar argument applies in the other case. SIR implies that $\alpha \in]0, 1[$. Let $\beta \in]0, \min\{\alpha, 1-\alpha\}[$. Let $u_2(z) = \pi \cdot z$, for all $z \in \mathbb{R}^L_{++}$, let $a \in \mathbb{R}^L_{++}$ be such that $\pi \cdot a = (1-\beta)\pi \cdot y$, and let $b \in \mathbb{R}^L_{++}$ be such that $\pi \cdot b = \beta\pi \cdot y$. Let then $u_1 : \mathbb{R}^L_{++} \to \mathbb{R}$ be the continuous and strictly increasing function defined as follows:

$$u_1(z) = \begin{cases} \pi \cdot z & \text{if } \pi \cdot z \leq \pi \cdot a \\ \pi \cdot a + \frac{\pi \cdot (y-a)}{\pi \cdot (x-a)} \pi \cdot (z-a) & \text{if } \pi \cdot z \geq \pi \cdot a \end{cases}$$

Let $\succeq'_i$ be the preference ordering on $\Delta(\mathbb{R}^L_+)$ derived through the expected utility criterion applied to the Bernoulli function u_i. Observe that $\succeq'_1$ coincides with $\succeq_1$ on $\Delta(\{0, x\})$, and that $\succeq'_2$ coincides with $\succeq_2$ on $\Delta(\{0, y\})$. IPUA implies that the lottery that gives $(x, 0)$ with probability $1-\alpha$ and $(0, y)$ with probability α belongs to $\Sigma(\{(0, 0), (x, 0), (0, y)\}, \succeq'_1, \succeq'_2)$. Let $\nu \in \Sigma(\{(0, 0), (x, 0), (0, y), (a, b), (b, a)\}, \succeq'_1, \succeq'_2)$. EFF implies that $\nu((0, 0)) = 0$. For any $\gamma \in [0, 1]$, both agents are indifferent given $(\succeq'_1, \succeq'_2)$ between the lottery that gives (a, b) with probability γ and (b, a) with probability $1-\gamma$, and the lottery that gives $(x, 0)$ with probability $\beta + \gamma - 2\beta\gamma$ and $(0, y)$ with probability $1 - \beta - \gamma + 2\beta\gamma$. Hence both bargainers must be indifferent between ν and some lottery in $\Delta(\{(x, 0), (0, y)\})$. This lottery must belong to $\Sigma(\{(0, 0), (x, 0), (0, y), (a, b), (b, a)\}, \succeq'_1, \succeq'_2)$, by EX, and IIA

[13] This is the difficult part of the proof, where one can fully appreciate the difference between anonymity and Nash's cardinal welfarist property of symmetry.

[14] The proof might be easier to understand at first when $L = 1$, in which case π can be normalized to 1.

implies that it also belong $\Sigma(\{(0,0),(x,0),(0,y)\},\succeq_1',\succeq_2')$. PI thus implies that the lottery that gives $(0,y)$ with probability α and $(x,0)$ with probability $1-\alpha$ belongs to $\Sigma(\{(0,0),(x,0),(0,y),(a,b),(b,a)\},\succeq_1',\succeq_2')$. Both bargainers are indifferent given ($\succeq_1',\succeq_2'$) between this last lottery and the lottery that gives (b,a) with probability $\frac{\alpha-\beta}{1-2\beta}$ and (a,b) with probability $\frac{1-\alpha-\beta}{1-2\beta}$ (these are well-defined probabilities because $\beta<\min\{\alpha,1-\alpha\}<1/2$). EX implies that this new lottery belongs to $\Sigma(\{(0,0),(x,0),(0,y),(a,b),(b,a)\},\succeq_1'$ $,\succeq_2')$, and hence also to $\Sigma(\{(0,0),(a,b),(b,a)\},\succeq_1',\succeq_2')$, by IIA. Notice that $\succeq_1'$ coincides with $\succeq_2'$ on $\Delta(\{0,a,b\})$ ($\pi\cdot b<\pi\cdot a$ because $\beta<1/2$). AN and PI imply that the only element of $\Sigma(\{(0,0),(a,b),(b,a)\},\succeq_1',\succeq_2')$ is the lottery that puts an equal weight on (a,b) and on (b,a) (similar to the argument that showed that $\alpha=1/2$ when $y=x$ in the previous paragraph). Hence $\frac{\alpha-\beta}{1-2\beta}=1/2$, or $\alpha=1/2$. ■

Proof of Theorem 1: The fact that Σ_N is the only candidate to satisfy the axioms follow from Lemmas 1 and 2. The well-known properties of the Nash bargaining solution in the space of utilities implies at once that Σ_N satisfies PI, EFF, AN, IIA, EX, and SIR. So all what remains to do is check that it also satisfies IPUA. Let U_i (resp. V_i) be a linear representation of $\succeq_i$ (resp. $\succeq_i'$). If $\succeq_i$ coincides with $\succeq_i'$ on $\Delta(O)$, then there exists $\lambda\in\mathbb{R}_+$ and $\alpha\in\mathbb{R}$ such that $U_i(\cdot)=\lambda V_i(\cdot)+\alpha$ on $\Delta(O)$ (usual argument on equivalent linear representations of vN-M preferences applied to $\Delta(O)$). Hence $\Sigma_N(O,\succeq_1,\succeq_2)=\Sigma_N(O,\succeq_1',\succeq_2')$, as in IPUA. ■

The rest of this section is devoted to showing the independence of the axioms appearing in Theorem 1 and Lemma 1. Consider first the solution that is determined by EPEP on elementary problems, and that associates to any other bargaining problem the set of lotteries that are Pareto optimal and strictly individually rational. It satisfies all the axioms listed in both Theorem 1 and Lemma 1, except PI. The Kalai-Smorodinsky solution, discussed at length in the next section, satisfies all the axioms listed in Theorem 1 and Lemma 1, except IIA. The solution(s) defined in Example 1 satisfies all the axioms listed in Theorem 1, except IPUA, and all the axioms from Lemma 1, except EPEP. The weighted Nash solutions satisfy all the axioms of Theorem 1, except AN. It also satisfies all the axioms of Lemma 1, except EPEP. The modified Nash solution that selects deterministic outcomes in Σ_N whenever possible, i.e. $\tilde{\Sigma}_N(O,\succeq_1,\succeq_2)=\Sigma_N(O,\succeq_1,\succeq_2)\cap O$, if this set is nonempty, and $\tilde{\Sigma}_N(O,\succeq_1,\succeq_2)=\Sigma_N(O,\succeq_1,\succeq_2)$, otherwise, satisfies all the axioms from both Theorem 1 and Lemma 1, except EX. I have not been able to show the independence of EFF and SIR from the other axioms in Theorem 1, but it is easy to see that at least one of them is needed since the solution that always selects the origin satisfies all the axioms listed in that theorem, except EFF and SIR.

4. THE KALAI-SMORODINSKY SOLUTION

IIA is the axiom that has most often been criticized in Nash's model. Although it is undeniable that arguments along the lines of IIA are heard in real-life bargaining, it is not clear that they are systematically followed. Suppose for instance that O' is obtained from O by removing exclusively alternatives that are very favorable to the first bargainer. In such cases, the second bargainer may have a valid argument against IIA, because the reduction from O to O' seems to place the first bargainer in a weaker position. The main alternative cardinal welfarist solution that emerged from this criticism was proposed by Kalai and Smorodinsky (1975). They propose to replace IIA by a property of monotonicity that applies only when the bargainers' utopia points remain unchanged.[15] In this section, I investigate whether EPEP or IPUA can be used in combination with the natural economic reformulation of Kalai and Smorodinsky's axioms to characterize their solution.

I start by redefining the Kalai-Smorodinsky solution and the property of conditional monotonicity in my economic framework:

$$\Sigma_{KS}(O, \succeq_1, \succeq_2) = \arg \max_{\mu \in \Delta(O)} \min_{i=1,2} \frac{U_i(\mu) - U_i(0)}{\max_{\nu \in \Delta(O)} U_i(\nu) - U_i(0)},$$

where (U_1, U_2) is any[16] linear representation of the vN-M preferences $(\succeq_1, \succeq_2)$.

Conditional Monotonicity (C-MON) *Let $(O, \succeq_1, \succeq_2)$ be a bargaining problem, and O' be a set larger than O. If there is no $\mu' \in \Delta(O')$ such that either $\mu' \succ_1 \mu$ or $\mu' \succ_2 \mu$ for all $\mu \in \Delta(O)$, then for all $\mu \in \Sigma(O, \succeq_1, \succeq_2)$ there exists $\mu' \in \Sigma(O', \succeq_1, \succeq_2)$ such that $\mu' \succeq_1 \mu$ and $\mu' \succeq_2 \mu$.*[17]

It is not difficult to check that the Kalai-Smorodinsky solution satisfies both EPEP and IPUA. On the other hand, the uniqueness result derived in Theorem 1 and Lemma 1 is not always guaranteed when IIA is replaced by C-MON, as the following example shows.

Example 2 *Consider first the economic reformulation of the solutions characterized by Peters and Tijs (1985). Let $\phi : [1, 2] \to conv\{(1, 0), (0, 1), (1, 1)\}$ that is continuous, non-decreasing and such that $\phi(1) = (1/2, 1/2)$ and $\phi(2) = (1, 1)$. The function ϕ thus*

[15] Kalai and Smorodinsky's property of monotonicity implies a weak form of IIA that applies only when the bargainers' utopia points remain unchanged, hence addressing to some extent the criticism formulated against IIA.

[16] See Footnote 12.

[17] The Kalai-Smorodinsky solution also satisfies a stronger monotonicity property requiring that $\mu' \succeq_1 \mu$ and $\mu' \succeq_2 \mu$, for all $\mu \in \Sigma(O, \succeq_1, \succeq_2)$ and all $\mu' \in \Sigma(O', \succeq_1, \succeq_2)$.

determines a monotonic curve in the subset conv$\{(1,0),(0,1),(1,1)\}$ *of the utility space. Let then* Σ^{ϕ} *be the solution that associates to each bargaining problem* $(O,\succeq_1,\succeq_2)$ *the set of lotteries* λ *that are Pareto optimal and such that* $(U_1^*(\lambda),U_2^*(\lambda)) = \phi(U_1^*(\lambda)+U_2^*(\lambda))$, *where* U_i^* *is the unique linear representation of* $\succeq_i$ *such that* $U_i^*(0)=0$ *and* $\max_{o\in O} U_i^*(o)=1$. *It is not difficult to check that* Σ^{ϕ} *is well-defined, and satisfies PI, EFF, C-MON, EX, and EPEP, but is different from* Σ_{KS} *as soon as* $\phi(t)\neq(t/2,t/2)$, *for some* $t\in]1,2[$. *This shows that the uniqueness result in Lemma 1 does not hold when IIA is replaced by C-MON. Still, all these solutions are cardinal welfarist, and the Kalai-Smorodinsky is the only member of that family that is anonymous. I now propose a more intricate example that shows that uniqueness cannot be recovered in Theorem 1, even if one adds the requirement of anonymity. Let* x *and* y *be two bundles on the diagonal of* $\mathbb{R}^L_+$ *such that* $U_1^*(x)=U_2^*(y)=1$. *Consider then the monotonic curve in the triangle conv*$\{(1,0),(0,1),(1,1)\}$ *that starts at* $(1/2,1/2)$, *and follows a direction parallel to the vector* $(\frac{x_1}{x_1+y_1},\frac{y_1}{x_1+y_1})$ *until it reaches an edge of the triangle, in which case it continues until* $(1,1)$ *on that edge. Let* ψ *be the functional description of that curve, i.e.* $\psi(t)$ *is the intersection of the curve with the line* $u_1+u_2=t$, *for each* $t\in[1,2]$. *It is not difficult to check that* Σ^{ψ} *is well-defined, and satisfies PI, EFF, C-MON, EX, EPEP, and AN, but is different from* Σ_{KS}. *The difference with the first part of the example is that the curve* ψ *is defined endogenously, as* x *and* y *may thus vary with the bargaining problem under consideration. The resulting solution is anonymous and non-welfarist. Notice that this new solution also satisfies IPUA and SIR when* $L=1$, *thereby showing that the uniqueness result of Theorem 1 does not systematically hold either when IIA is replaced by IPUA. On the other hand, the solution violates IPUA when* $L\geq 2$ *(because* x *and* y *do not need to be part of feasible alternatives). It turns out that this is not a coincidence: replacing IIA by C-MON characterizes the Kalai-Smorodinsky solution when* $L\geq 2$, *as the following theorem shows. The theorem also shows that the content of Lemma 1 remains true if IIA is replaced by C-MON and* $L\geq 2$. *On the other hand, this variant of Lemma 1 would now be useless to prove the next theorem, since there is no equivalent to Lemma 2 with C-MON instead of IIA, as illustrated by* Σ^{ψ} *when* $L\geq 2$.

Theorem 2 Σ_{KS} *is the only solution that satisfies PI, EFF, AN, C-MON, EX, IPUA, and SIR if* $L\geq 2$.

Proof: The fact that Σ_{KS} satisfies the axioms follows from the usual properties of the Kalai-Smorodinsky solution defined in the space of utilities. I will thus focus on proving uniqueness. Let Σ be a solution that satisfies the axioms, let $(O,\succeq_1,\succeq_2)$ be a bargaining problem, let $\lambda\in\Sigma_{KS}(O,\succeq_1,\succeq_2)$, and let (U_1,U_2) be two linear representations of $(\succeq_1$

$,\succeq_2)$. Notice that

$$\frac{U_1(\lambda) - U_1(0)}{\max_{\mu\in\Delta(O)} U_1(\mu) - U_1(0)} = \frac{U_2(\lambda) - U_2(0)}{\max_{\mu\in\Delta(O)} U_2(\mu) - U_2(0)}. \tag{1}$$

Suppose on the contrary that one of the two ratios, let's say the one on the left-hand side, is strictly smaller than the other one. Let x be an element of O such that $U_1(x) = \max_{\mu\in\Delta(O)} U_1(\mu)$. Then the lottery that picks x with probability ϵ, and λ with probability $1-\epsilon$, guarantees a larger minimal ratio if ϵ is small enough, thereby contradicting the fact that $\lambda \in \Sigma_{KS}(O, \succeq_1, \succeq_2)$. This establishes equation (1). Let ρ be this common number.

Let $x, y, \bar{x}, \bar{y}$ be strictly positive bundles (not necessarily in O) such that $U_1(\bar{x}) = U_1(\lambda)$, $U_2(\bar{y}) = U_2(\lambda)$, $U_1(x) = \max_{\mu\in\Delta(O)} U_1(\mu)$, $U_2(y) = \max_{\mu\in\Delta(O)} U_2(\mu)$, $x >> \bar{x}$, and $y >> \bar{y}$. The four vectors can, for instance, be taken on the diagonal of $\mathbb{R}^L_+$. Let $o \in O$ be such that $U_1(o) = \max_{\mu\in\Delta(O)} U_1(\mu)$. Let x' be an element on the diagonal that falls above o. Monotonicity implies that $x' \succeq_1 \mu$, for all $\mu \in \Delta(O)$. Continuity implies that there exist a convex combinations between 0 (the worst element of O) and x' that will leave the first bargainer indifferent when comparing it to o. Let's call x this new bundle. Since $U_1(\lambda) \leq U_1(x)$, there exists a convex combination between x and 0 that will leave the first bargainer indifferent when comparing it to λ. Let's call $\bar{x}$ this new bundle. A similar construction leads to y and $\bar{y}$. Clearly, $x \leq \bar{x}$ and $y \leq \bar{y}$. If both inequalities are binding, then any efficient lottery in $\Delta(O)$ gives the same expected utility as λ, and we are done proving that $\Sigma(O, \succeq_1, \succeq_2) = \Sigma_{KS}(O, \succeq_1, \succeq_2)$. On the other hand, it cannot be that only one of the two inequalities are binding, by (1), and hence I can assume in the rest of the proof that both inequalities are strict.

Let ϵ be a strictly positive number, and let ξ be the vector in $\mathbb{R}^L$ defined as follows:

$$(\forall l \geq 3) : \xi_l = \min\{x_l, y_l\},$$

$$\xi_1 = \frac{1}{1-\epsilon^2}[y_1 - \epsilon^2 x_1 + \epsilon(y_2 - x_2) + \epsilon \sum_{i=3}^{L}(y_i - \xi_i) - \epsilon^2 \sum_{i=3}^{L}(x_i - \xi_i)],$$

$$\xi_2 = \frac{1}{1-\epsilon^2}[\epsilon(x_1 - y_1) + x_2 - \epsilon^2 y_2 + \epsilon \sum_{i=3}^{L}(x_i - \xi_i) - \epsilon^2 \sum_{i=3}^{L}(y_i - \xi_i)].$$

Let $\bar{\xi}$ be the vector derived by applying the same equations to $(\bar{x}, \bar{y})$. It is easy to check that all the components of both ξ and $\bar{\xi}$ are strictly positive and that $\xi > \bar{\xi}$, if ϵ is chosen small enough. Indeed, the limit of $\bar{\xi}_1$ (resp. $\bar{\xi}_2$) when ϵ tends to zero is $\bar{y}_1 > 0$

(resp. $\bar{x}_2 > 0$), the limit of $\xi_1 - \bar{\xi}_1$ (resp. $\xi_2 - \bar{\xi}_2$) when ϵ tends to zero is $y_1 - \bar{y}_1 > 0$ (resp. $x_2 - \bar{x}_2 > 0$), and the inequalities regarding the other components are obvious. Straightforward algebra also allows to show that

$$\xi_1 + \epsilon \sum_{l=2}^{L} \xi_l = y_1 + \epsilon \sum_{l=2}^{L} y_l \text{ and } \bar{\xi}_1 + \epsilon \sum_{l=2}^{L} \bar{\xi}_l = \bar{y}_1 + \epsilon \sum_{l=2}^{L} \bar{y}_l$$

$$\xi_2 + \epsilon \sum_{l=1, l\neq 2}^{L} \xi_l = x_2 + \epsilon \sum_{l=1, l\neq 2}^{L} x_l \text{ and } \bar{\xi}_2 + \epsilon \sum_{l=1, l\neq 2}^{L} \bar{\xi}_l = \bar{x}_2 + \epsilon \sum_{l=1, l\neq 2}^{L} \bar{x}_l.$$

In other words, ξ (resp. $\bar{\xi}$) has been chosen in the intersection of the hyperplane of normal $(1, \epsilon, \ldots, \epsilon)$ that goes through y (resp. $\bar{y}$) and the hyperplane of normal $(\epsilon, 1, \epsilon, \ldots, \epsilon)$ that goes through x (resp. $\bar{x}$).[18]

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous, unbounded and strictly increasing function such that $f(0) = 0$, $f(\sum_{l=1}^{L} \bar{\xi}_l) = \rho$, and $f(\sum_{l=1}^{L} \xi_l) = 1$. Let then $v(z) = f(\sum_{l=1}^{L} z_l)$, for each $z \in \mathbb{R}_+^L$, and $\succeq'$ be the vN-M preference on $\Delta(\mathbb{R}_+^L)$ derived by applying the expected utility criterion to v. I now prove that AN and EFF imply that $(\bar{\xi}, \bar{\xi}) \in \Sigma(O', \succeq', \succeq')$, where $O' = \{(0,0), (\xi, 0), (0, \xi), (\bar{\xi}, \bar{\xi})\}$. Let $\mu \in \Sigma(O', \succeq', \succeq')$. EFF implies that $\mu((0,0)) = 0$. AN implies that $\mu^* \in \Sigma(O', \succeq', \succeq')$. PI implies that $\mu((\xi, 0)) = \mu((0, \xi))$. Notice that $\rho \geq 1/2$. If $\rho = 1/2$, then then both bargainers are indifferent between $(\bar{\xi}, \bar{\xi})$ and the lottery that gives $(\xi, 0)$ and $(0, \xi)$ with equal probabilities. There are multiple lotteries in $\Sigma(O', \succeq', \succeq')$, and EX implies that $(\bar{\xi}, \bar{\xi}) \in \Sigma(O', \succeq', \succeq')$. If $\rho > 1/2$, then EFF implies that $\mu((\xi, 0)) = \mu((0, \xi)) = 0$, and one concludes again that $(\bar{\xi}, \bar{\xi}) \in \Sigma(O', \succeq', \succeq')$, as desired.

Let now $g_1 : \mathbb{R}_+ \to \mathbb{R}$ and $g_2 : \mathbb{R}_+ \to \mathbb{R}$ be two continuous, unbounded and strictly increasing functions such that $g_1(0) = g_2(0) = 0$, $g_1(\bar{x}_2 + \epsilon \sum_{l=1, l\neq 2}^{L} \bar{x}_l) = g_2(\bar{y}_1 + \epsilon \sum_{l=2}^{L} \bar{y}_l) = \rho$, and $g_1(x_2 + \epsilon \sum_{l=1, l\neq 2}^{L} x_l) = g_2(y_1 + \epsilon \sum_{l=2}^{L} y_l) = 1$. Let also $w_1(z) = g_1(z_2 + \epsilon \sum_{l=1, l\neq 2}^{L} z_l)$ and $w_2(z) = g_2(z_1 + \epsilon \sum_{l=2}^{L} z_l)$, for each $z \in \mathbb{R}_+^L$. Let finally $\succeq_i''$ be the vN-M preference on $\Delta(\mathbb{R}_+^L)$ derived by applying the expected utility criterion to w_i ($i = 1, 2$). In other words, $\succeq_1''$ (resp. $\succeq_2''$) has indifference curves along the hyperplanes of normal $(\epsilon, 1, \epsilon, \ldots, \epsilon)$ (resp. $(1, \epsilon, \ldots, \epsilon)$), and is such that agent 1 (resp. 2) remains indifferent between getting $\bar{\xi}$ for sure and playing the lottery that gives him ξ, with probability ρ, and 0, with probability $1 - \rho$. Notice that $\succeq_i''$ coincides with $\succeq'$ on $\Delta(\{0, \xi, \bar{\xi}\})$, because $w_i(o) = v(o)$, for each

[18] It might be easier to focus on the case $L = 2$ when reading the proof for the first time. Notice that the construction of ξ and $\bar{\xi}$ necessitates at least two goods. We know from the Example 2 that the Theorem itself does not hold when $L = 1$.

$i \in \{1,2\}$ and each $o \in \{0, \xi, \bar{\xi}\}$, and hence $(\bar{\xi}, \bar{\xi}) \in \Sigma(O', \succeq_1'', \succeq_2'')$, by IPUA.

Let $O'' = \{(0,0), (x,0), (0,y), (\bar{x}, \bar{y})\}$. Notice that if $\mu \succeq_1'' \bar{\xi}$ and $\mu \succeq_2'' \bar{\xi}$, then $\mu \sim_1'' \bar{\xi}$ and $\mu \sim_2'' \bar{\xi}$, for each $\mu \in \Delta(O' \cup O'')$. We also have that $\bar{x} \sim_1'' \bar{\xi}$ and $\bar{y} \sim_2'' \bar{\xi}$. C-MON and EX thus imply that $(\bar{x}, \bar{y}) \in \Sigma(O' \cup O'', \succeq_1'', \succeq_2'')$. A similar argument also implies that $(\bar{x}, \bar{y}) \in \Sigma(O'', \succeq_1'', \succeq_2'')$.[19] Observe that $\succeq_1''$ coincides with $\succeq_1$ on $\Delta(\{0, x, \bar{x}\})$. Indeed, renormalizing U_1 by substracting $U_i(0)$ and rescaling it by dividing by $\max_{\mu \in \Delta(O)} U_1(\mu) - U_1(0)$, one obtains an alternative Bernoulli function that also represents $\succeq_1$, and that coincides with w_1 on $\{0, x, \bar{x}\}$. Similarly, $\succeq_2''$ coincides with $\succeq_2$ on $\Delta(\{0, y, \bar{y}\})$. IPUA thus implies that $(\bar{x}, \bar{y}) \in \Sigma(O'', \succeq_1, \succeq_2)$. Notice now that if $\mu \succeq_1 \bar{x}$ and $\mu \succeq_2 \bar{y}$, then $\mu \sim_1 \bar{x}$ and $\mu \sim_2 \bar{y}$, for each $\mu \in \Delta(O \cup O'')$. We also have that $\lambda \sim_1 \bar{x}$ and $\lambda \sim_2 \bar{y}$. C-MON and EX thus imply that $\lambda \in \Sigma(O \cup O'', \succeq_1, \succeq_2)$. A similar argument also implies that $\lambda \in \Sigma(O, \succeq_1, \succeq_2)$. Hence $\Sigma_{KS} \subseteq \Sigma$, and PI implies that Σ in fact coincides with Σ_{KS}. ■

I now discuss the independence of the axioms appearing in Theorem 2. Let $(O, \succeq_1, \succeq_2)$ be a bargaining problem, and let o^i be an element of O such that $U_i(o^i) = \max_{\mu \in \Delta(O)} U_1(\mu)$. The solution that selects all the lotteries that are Pareto efficient and weakly preferred to the lottery that selects o^1 and o^2 with equal probabilities satisfies all the axioms of Theorem 2, except PI. The solutions Σ^{ϕ} introduced at the beginning of Example 2 satisfy all the axioms, except AN. The Nash bargaining solution studied in the previous section satisfies all the axioms, except C-MON. The modified Kalai-Smorodinsky solution that selects deterministic outcomes in Σ_{KS} whenever possible, i.e. $\tilde{\Sigma}_{KS}(O, \succeq_1, \succeq_2) = \Sigma_{KS}(O, \succeq_1, \succeq_2) \cap O$, if this set is nonempty, and $\tilde{\Sigma}_{KS}(O, \succeq_1, \succeq_2) = \Sigma_{KS}(O, \succeq_1, \succeq_2)$, otherwise, satisfies all the axioms, except EX. Finally, the solution that always selects the origin satisfies all the axioms, except EFF and SIR.

5. RELATED LITERATURE

Roemer's (1988) and Rubinstein et al.'s (1992) papers have already been partly discussed in the Introduction, and I will not repeat the points that I have already raised there. On the other hand, I want to make the comparison with Nash (1950) more formal, especially to understand precisely what is the difference between the properties of welfarism and cardinal welfarism.

[19] More generally, C-MON implies that IIA holds for any pair $(O, \succeq_1, \succeq_2)$ and $(O', \succeq_1, \succeq_2)$ of bargaining problems as in the statement of IIA, and such that there is no $\mu \in \Delta(O)$ for which either $\mu \succ_1 \mu'$ or $\mu \succ_2 \mu'$ for all $\mu' \in \Delta(O')$.

Comparison with Nash (1950)

Rephrasing Nash's original argument in my framework essentially amounts to replace EPEP or IPUA by an axiom of welfarism in both Theorem 1 and Lemma 1 (AN and EFF must be added in the former case, and SIR can be deleted in the latter). Generally speaking, welfarism is taken in Roemer's sense (not Sen's - see footnote 1) that the image of the bargaining problems in the space of utilities is sufficient to determine their solutions. I want to make this statement precise now.

Even though vN-M preferences can be represented by linear utility functions, there is no reason *a priori* to focus on such representations when formulating the welfarist property. Expected utility theory is indeed an ordinal theory of preference over lotteries. A utility function $U : \Delta(\mathbb{R}^L_+) \to \mathbb{R}$ *represents* a player's preference relation $\succeq$ over $\Delta(\mathbb{R}^L_+)$ if $\mu \succeq \nu$ is equivalent to $U(\mu) \geq U(\nu)$, for each μ and each ν in $\Delta(\mathbb{R}^L_+)$. The function obtained through a strictly increasing transformation of a utility function that represents a vN-M preference relation $\succeq$ of the expected utility type gives another utility function that represents it as well. Of course, the property of being linear in probabilities is usually not preserved under such transformations. The formal definition of welfarism in my framework should thus go as follows.

Welfarism (WELF) *Let $(O, \succeq_1, \succeq_2))$ and $(O', \succeq'_1, \succeq'_2))$ be two bargaining problems. Let U_1 (resp. U_2; V_1; V_2) be a utility function that represents $\succeq_1$ (resp. $\succeq_2$; $\succeq'_1$; $\succeq'_2$). If*

$$\{(U_1(\mu), U_2(\mu))|\mu \in \Delta(O)\} = \{(V_1(\mu), V_2(\mu))|\mu \in \Delta(O')\},$$

then

$$\{(U_1(\mu), U_2(\mu))|\mu \in \Sigma(O, \succeq_1, \succeq_2))\} = \{(V_1(\mu), V_2(\mu))|\mu \in \Sigma(O', \succeq'_1, \succeq'_2))\}.$$

Welfarism is a strong property, as the following impossibility result shows.

Theorem 3 *There is no solution that satisfies PI, SIR, and WELF.*

<u>Proof</u>: Consider the bargaining problem $(O, \succeq_1, \succeq_2)$, where $O = \{(0,0), (\bar{x}, 0), (0, \bar{x})\}$ with $\bar{x}$ being the bundle that contains $1/L$ units of each good, and $\succeq_i$ is the preference relation on $\Delta(\mathbb{R}^L_+)$ that is represented by the following utility function:

$$U_i(\mu) = \sum_{x \in supp(\mu)} \sum_{l \in L} \mu(x) x_l,$$

for each $\mu \in \Delta(\mathbb{R}^L_+)$ (i.e. expected utility with $u_i(x) = \sum_{l \in L} x_l$). Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ and

$g : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as follows:

$$f(y) = \frac{2y}{1+y}$$

$$g(y) = \begin{cases} \frac{y}{2-y} & \text{if } y \leq 1 \\ y & \text{if } y \geq 1 \end{cases}$$

for each $y \in \mathbb{R}_+$. Notice that both f and g are strictly increasing. Hence $V_1(\cdot) = f(U_1(\cdot))$ represents $\succsim_1$, and $V_2(\cdot) = g(U_2(\cdot))$ represents $\succsim_2$.

I now prove that

$$\{(U_1(\mu), U_2(\mu)) | \mu \in \Delta(O)\} = \{(V_1(\mu), V_2(\mu)) | \mu \in \Delta(O)\}. \tag{2}$$

Observe indeed that the set on the left-hand side is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. The set on the right-hand side is contained in that triangle, because $\frac{2\alpha}{1+\alpha} + \frac{\beta}{2-\beta} \leq 1$, for each pair α, β of non-negative numbers that sum up to no more than 1. Conversely, any element (x, y) of the triangle can be obtained as the utility pair under (V_1, V_2) of the lottery that gives $(\bar{x}, 0)$ with probability $x/(2-x)$, $(0, \bar{x})$ with probability $2y/(1+y)$, and $(0,0)$ with the remaining probability.

Given (2), WELF implies that

$$\{(U_1(\mu), U_2(\mu)) | \mu \in \Sigma(O, \succsim_1, \succsim_2)\} = \{(V_1(\mu), V_2(\mu)) | \mu \in \Sigma(O, \succsim_1, \succsim_2)\}.$$

Let $\mu \in \Sigma(O, \succsim_1, \succsim_2)$. PI implies that $U_1(\mu) = V_1(\mu)$. Since f has only two fixed points, $y = 0$ or 1, this is possible only if $\mu((\bar{x}, 0)) = 0$ or 1, which contradicts SIR. ■

In particular, the Nash and Kalai-Smorodinsky solutions do not satisfy WELF. This may sound surprising at first since these two solutions are often said to be welfarist. My point here is that they only satisfy a much weaker form of welfarism that holds only for those utility functions that are linear in probabilities.

Cardinal Welfarism (C-WELF) *Let $(O, \succsim_1, \succsim_2)$ and $(O', \succsim'_1, \succsim'_2)$ be two bargaining problems. Let u_1 (resp. u_2; v_1; v_2) be a continuous Bernoulli function that allows to represent $\succsim_1$ (resp. $\succsim_2$; $\succsim'_1$; $\succsim'_2$) via the expected utility criterion. If*

$$\{(\textstyle\sum_{x \in supp(\mu)} \mu(x) u_1(x), \sum_{x \in supp(\nu)} \mu(x) u_2(x)) | \mu \in \Delta(O)\}$$
$$= \{(\textstyle\sum_{x \in supp(\mu)} \mu(x) v_1(x), \sum_{x \in supp(\nu)} \mu(x) v_2(x)) | \mu \in \Delta(O)\},$$

then

$$\{(\textstyle\sum_{x \in supp(\mu)} \mu(x) u_1(x), \sum_{x \in supp(\nu)} \mu(x) u_2(x)) | \mu \in \Sigma(O, \succsim_1, \succsim_2)\} =$$

$$\{(\sum_{x \in supp(\mu)} \mu(x) v_1(x), \sum_{x \in supp(\nu)} \mu(x) v_2(x)) | \mu \in \Sigma(O', \succeq'_1, \succeq'_2))\}.$$

The key role of linear representations of vN-M preferences was already clear in Nash's paper itself. They are needed to justify the convexity of the utility possibility sets, and to motivate the scale covariance axiom (instead of covariance with respect to a larger class of utility transformations). One may thus wonder whether Nash's result is valid only when phrased with the help of linear representations, that have the advantage indeed to offer some cardinal measure of the bargainers' satisfaction (ratios of utility gains being scale invariant). If so, then Nash's claim that his bargaining theory builds on von Neumann and Morgenstern's notion of expected utility would be wrong. Instead one would have to resort to other theories of preferences that are truly cardinal (some such theories are surveyed in chapter 6 of Fishburn, 1970, but they have not had much of an impact in economics so far). Fortunately, Theorem 1 and Lemma 1 show on the contrary that the Nash bargaining solution and its characterization can be phrased in terms of ordinal vN-M preferences.

Theorem 3 is similar to Shapley's (1969) result showing that one cannot construct a single-valued solution in the space of utilities that would be single-valued, strictly individually rational and covariant with respect to any increasing transformations (as opposed to only affine transformations). Theorem 3 makes Shapley's welfarist postulate explicit, and shows that the result holds even if one restricts attention to vN-M preferences over lotteries in an explicit economic environment. Some authors have investigated ways to avoid Shapley's impossibility result when there are more than two bargainers (Kibris (2004, 2008), Safra and Samet (2004, 2005)), while keeping the welfarist assumption that solutions should be defined in the space of utilities. Theorems 1 and 2, as well as Lemma 1, show that ordinal invariance is not out of reach, even when there are only two bargainers, provided one drops WELF, a property that has no straightforward motivation anyway, and one restricts attention to vN-M preferences, which are indeed a cornerstone of non-cooperative game theory and information economics.

Other authors proposed welfarist extensions of the Nash bargaining solution and its axiomatic characterization to non-convex utility possibility sets (Conley and Wilkie (1996), Zhou (1997), and references therein), while imposing some form of scale covariance (instead of ordinal covariance as in the papers cited in the previous paragraph). The content given to the concept of utility in those papers then becomes even less obvious than in Nash's model. Recovering these solutions on some natural class of explicit economic environments, if possible, would clarify this. Notice that the economic problems I describe may have a non-convex image in the utility space even when the bargainers have vN-M preferences if one does not restrict attention to linear representations. Of course, the utility profile corresponding to the solution will usually not maximize the product of the bargainers' utility gains, and while different representations of the same bargaining problem will be related by an ordinal transformation, the solution will behave in a way that is usually not covariant.

IPUA is easier to interpret than C-WELF. First it is phrased directly in terms of preferences instead of their linear representation. Second, it relates the solutions of problems that differ only in the bargainers' preferences over unfeasible alternatives, and not in the set of available outcomes. IPUA is also logically strictly weaker than C-WELF. It is easy to check indeed that C-WELF implies IPUA, while the following example shows that the converse is not true.

Example 3 *For simplicity, I will describe a solution for the case $L = 1$, but there are multiple ways to extend it to any number of goods. Consider the following function f that associates an element of $\Delta(O)$ to each compact subset O of $\mathbb{R}^2_+$. For each $x \in O$, let x^* be the vector obtained by rearranging its components increasingly. There are at most two vectors $x \in O$ that maximize x^* according to the lexicographic order. The function f picks the optimal vector if it is unique, and picks the uniform lottery between the two optimal vectors otherwise. Let (U_1, U_2) be linear representations of $\succeq_1$ and $\succeq_2$. For each lottery $\mu \in \Delta(O)$ and each $i \in \{1, 2\}$, let $\alpha^f_i(\mu)$ be the ratio $\frac{U_i(\mu)-U_i(0)}{U_i(f(O))-U_i(0)}$. Let $\hat{\alpha}^f$ be the vector in $\mathbb{R}^2_+$ obtained by rearranging the components of $\alpha^f(\mu)$ increasingly. The solution Σ is then obtained by maximizing α^f according to the lexicographic order. It is not difficult to check that Σ satisfies IPUA, but not C-WELF, and that it is well-defined in that the solution does not change when one chooses a different linear representation of the bargainers' vN-M preferences. Observe that Σ also satisfies PI, EX, AN, and EFF. The way Σ is defined should make it clear that one can construct many alternative solutions that have similar properties.*

Further Comparison with Roemer's Reconstruction of Axiomatic Bargaining Theory

I here pursue the discussion of Roemer's work that I started in the Introduction. It is important to recognize the role played by the domain of definition of solutions when formulating axiomatic results. Particularly, one would not obtain a characterization of the Nash bargaining solution if CONRAD was replaced by IPUA in Roemer's paper. Here are the main differences between our two papers regarding the definition of a bargaining problem. First, instead of restricting attention to economic problems that result from all the possible reallocations of some collective endowment to be shared, my domain includes bargaining problems build on any finite set of bundles. A stark consequence of Roemer's assumption is that every solution satisfies IIA in his framework, since an efficient allocation cannot remain feasible if the total endowment to distribute diminishes. Second, bargainers can use lotteries to reach an agreement in my model, and these lotteries are evaluated via vN-M ordinal preferences. Roemer, instead, endows the bargainers with a concave utility function defined over a set of deterministic contracts. Thus his theory is still rooted in a notion of utility that is not ordinally invariant (as an increasing transformation of a concave function is not necessarily concave). Third, my reasoning works for any fixed number of goods (including the interesting case of only one good), while Roemer's argument depends crucially on the possibility of adding goods, thereby considering a framework with a variable (possibly infinite) number of goods. Overall, I believe that my framework is closer to Nash's (1950) original construction of a bargaining problem, starting with explicit economic bundles instead of his more abstract notion of 'anticipation.'

Working with utility functions instead of preferences, Roemer still requires an axiom of scale invariance to recover the Nash bargaining solution. I consider solutions that are directly defined in terms of the ordinal information encoded in vN-M preferences, observing in particular that the Nash solution is ordinally invariant. I also introduced the distinction between welfarism and cardinal welfarism, emphasizing that the Nash solution satisfies only the second (weaker) property. Indeed, welfarism and ordinal invariance are essentially incompatible (cf. Theorem 3).

Roemer (1996, Section 2.5) suggests the following example as a criticism of both Nash's and Kalai-Smorodinsky's solutions (a similar example was also discussed in Roth (1979, pages 67-70), but Roemer's description more directly fits my model). Consider two risk-neutral bargainers with the following options: either 1) the first bargainer gets a silver dollar, while the second receives nothing, or 2) the first bargainer gets nothing, while the second receives a Rolls Royce, or 3) both bargainers receive nothing. As in the present paper, Roemer assumes that they can agree on any lottery over these three basic outcomes, and he observes that any solution that is cardinal welfarist, scale covariant, efficient, and symmetric must lead to an agreement with equal probabilities on the two first options. EPEP extends this example into a formal testable implication, and Lemma 1 shows that it is in fact characteristic of the Nash bargaining solution under PI, IIA and EX. Roemer expects, on the other hand, that the agreement will place a much larger probability on the first option occuring in his example.[20] The violation of a

[20]The outcome would be that the second bargainer would get the Rolls Royce with probability 1 if

testable implication for a specific bargaining solution does not necessarily diminish the interest of its axiomatic characterization. Indeed, it forces us to think about which more fundamental property is not followed by the participants of the experiment.[21] Lemma 2 sheds new light on the possible causes for the violation of EPEP. Example 1 provides a class of solutions where the violation of EPEP goes in pair with a violation of IPUA. Perhaps more interestingly, if one accepts IPUA and one understands PI, EFF, AN, and EX as defining a benchmark, then a violation of EPEP must necessarily go in pair with a violation of IIA.[22] As an illustration, the next example provides a class of solutions that satisfy all the axioms of this paper, except both EPEP and IIA.

Example 4 *For each bargaining problem $P = (O, \succsim_1, \succsim_2)$, let*

$$w(P) = \frac{\max_{o \in O} \sum_{l \in L} o_2^l}{\max_{o \in O} \sum_{l \in L} o_1^l + \max_{o \in O} \sum_{l \in L} o_2^l}.$$

The solution that associates to any such bargaining problem P the set of lotteries that maximize the weighted Nash product with weight $w(P)$ on the first bargainer and $1 - w(P)$ on the second bargainer, satisfies all the axioms of the paper, except both EPEP and IIA.

Two Other Related Papers

Valenciano and Zarzuelo's (1997) preference-based bargaining problem is the combination of a set X of possible outcomes, a disagreement point, and vN-M preferences over $\Delta(X)$. Their main result (see their theorem 4) establishes that C-WELF is equivalent to a property of invariance to isomorphic transformations in the underlying situations (after having performed two operations on the basic problems, the first one that consists

I happened to play that game once in the shoes of the first bargainer. The reason, of course, is some form of altruism - knowing that somebody would get a free Rolls Royce thanks to my decision is worth more to me than a dollar. It is thus important to emphasize again that the model I study presumes that the bargainers are selfish, in that they only care about their own consumption. Extending the results of this paper to more general classes of preferences is an interesting topic for future research. In any case, Roemer's point sounds more plausible if the difference in value between the two prizes is less dramatic. Roth (1979) cites indeed some experimental evidence going in that direction.

[21] It is also possible that experimental subjects are inconsistent in their choices, in that they would accept the arguments presented in the axioms, but not comply with the predictions of the theory because they are not aware of the implication of those axioms. In such cases, axiomatic results may have the value of helping bargainers to solve their inconsistencies.

[22] It is important to emphasize again the role of the domain: the larger the more powerful IIA becomes. I already argued that the domain of economic outcomes I chose to work with is both natural and close to what Nash himself had in mind. Even if not every single bargaining problem in that domain would necessarily occur, they all remain conceivable and can thus be part of the bargainers' arguments to reach an agreement.

in adding an equivalent outcome for each lottery in the original problem, and the second one that consists in restricting attention to equivalence classes defined by the two players' indifference relations). Their framework is close to mine if X is always taken as a subset of $\mathbb{R}^L_+$, except for one crucial difference. Defining the preferences over $\Delta(X)$ instead of $\Delta(\mathbb{R}^L_+)$ amounts to implicitly assume IPUA in the model itself. Remaining abstract about what outcomes can be, it seems impractical to even phrase IPUA, as one would need to define *a priori* what are the players' preferences over any conceivable set of outcomes. In contrast, I restricted attention to lotteries over economic outcomes, one of the most natural class of explicit environments. IPUA can now be phrased meaningfully. Doing this, one is forced to recognize that imposing such a property is perhaps not that innocuous after all (e.g. ruling out the principle of egalitarian equivalence, and overlooking the bargainers' risk attitude in elementary problems), though some will find it acceptable (e.g. if one believes in the Nash program). IPUA is thus clearly a key property that provides a deeper understanding of classical axiomatic results in bargaining theory. Contrarily to Valenciano and Zarzuelo, my objective was not to characterize C-WELF in general, but instead to see how the reformulation of Nash's axioms can be complemented to recover his solutions. I proved that IPUA is strong enough in itself. Particularly, adding a property of 'invariance to isomorphic transformations' to the list of axioms appearing in Theorem 1 would be redundant.

Hanany (2007) revisits Rubinstein et al. (1992) from the perspective of the revealed group preferences framework of Peters and Wakker (1991). His definition of a bargaining problem is similar to mine, making the additional assumption that $L = 1$, but allowing on the other hand for a larger class of preferences. One might think that his second theorem is formally related to my Theorem 1 (the weak axiom of revealed preferences coincides with IIA under EX and PI). A major difference though is that Hanany restricts attention to deterministic agreements (while risk preferences play of course a role in determining this outcome). The difficulty is that such agreements may be dominated by a lottery (his axiom of efficiency rules out the use of lotteries when making Pareto comparisons). Even if maximizing the Nash product over deterministic outcomes happens to lead to an agreement that is fully efficient, using lotteries may still result in a better compromise (think of elementary problems, for instance). Actually the very bargaining problems that play a key role in Hanany's proof have this feature. Finally, Hanany relies on the strong symmetry axiom of Rubinstein et al. (1992) (or actually a small variant that was proposed by Grant and Kajii, 1995). As already explained in the Introduction, such property is significantly stronger than AN, and is in fact closely related to C-WELF itself.

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