No Profitable Decompositions in Quasi-Linear Allocation Problems*

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Abstract

We study the problem of allocating a bundle of perfectly divisible private goods from an axiomatic point of view, in situations where compensations can be made through monetary transfers. The key property we impose on the allocation rule requires that no agent should be able to gain by decomposing the problem into sequences of subproblems. Combined with additional standard properties, it leads to a characterization of the rule that shares the total surplus equally. Hence a traditional welfarist rule emerges as the unique consequence of our axioms phrased in a natural economic environment.

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1 Introduction

We consider situations where a group of people have to share a bundle of perfectly divisible private goods. We assume that compensations can be achieved through monetary transfers (quasi-linear framework). As often, instead of solving each specific problem in isolation, we study allocation rules that may be applied in many different instances. For

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most allocation problems and most rules, some participants can gain by decomposing the stakes in some way, requesting for instance to allocate good l before l', or to share a proportion of the total amount of goods available before allocating what remains. Of course, such decompositions often lead to an efficiency loss, which is not desirable. Even when there is no efficiency loss, a gain for one participant must result in a loss for another one when the allocation rule selects efficient outcomes. Hence the normative appeal of a rule may be lost if stakes are decomposed when implementing it. Finally, one advantage of agreeing on an allocation rule is to reduce conflict when it comes to solving particular problems. This advantage may be limited when implementing rules that are subject to such profitable decompositions, as participants will have conflicting preferences when it comes to setting the agenda. For all these reasons, we are interested in studying rules that satisfy a property of "No Profitable Decompositions" (NPD), requiring that no individual can gain by decomposing the problem into sequences of subproblems.

The main result of the paper establishes that NPD, once combined with other standard axioms, characterizes the allocation rule that corresponds to an equal split of the maximal total surplus among the participants. Equal surplus sharing being probably the simplest notion of microeconomic justice, one would think that there exist numerous axiomatic characterizations of this solution in bargaining and social choice theory. In reality there are only relatively few such results. The reason is that most contributions in axiomatic bargaining and social choice are phrased while taking utilities as primitive. Equal surplus sharing follows trivially from the properties of anonymity and efficiency in quasi-linear environments under this welfarist assumption. Most of the literature focuses instead on finding extensions of the equal surplus sharing solution to environments that are not quasi-linear. Unfortunately, the welfarist assumption lacks a clear normative and/or positive content, and is thus hard to accept as an axiom or postulate (see Roemer (1986, 1988)). The existence of appealing contextual solutions (e.g. egalitarian equivalence, or competitive equilibrium with equal income) also shows that the welfarist assumption is far from being innocuous. To be precise, we are not arguing that a solution is unappealing because it is welfarist. Instead, we suggest that the axiomatic approach should be applied more systematically to explicit economic and social environments. Some properties that were incompatible in the utility space may lead to the characterization of new (necessarily contextual) solutions. In other cases, welfarism will come as a consequence of axioms, hence giving us a deeper understanding of classical solutions. Our main result belongs to this second category. It is worth noting that NPD cannot even be phrased under the welfarist assumption, since the set of utilities that are feasible in the subsequent step of a decomposition depends on the economic description of the problem. This set may be strictly smaller than, and unrelated to, the set of utilities that are achievable when solving the problem in its entirety.

Beyond usual properties of anonymity, efficiency, and continuity, the result requires an axiom of independence with respect to preferences over non-feasible allocations (IND). As hinted by its name, IND requires that the solution of two allocation problems that differ only in the participants' perferences over outcomes that are not feasible coincide. As far as we can tell, this type of property was first mentionned explicitly in Karni and Schmeidler (1975).[1] It has been invoked on various occasions since then.[2] Though IND may appear completely innocuous at first sight, we must point out that it rules out solutions such as Pazner and Schmeidler's (1978) egalitarian equivalence.

We can now provide some intuition for our main characterization result. Consider various countries that have an equal claim over a newly-discovered field of natural gas. A total quantity Q is available to share. Let v_i be the function that measures the net social surplus for country i, as a function of the share it receives.[3] These functions are most likely to vary across countries because of different transportation costs and different needs (e.g. existence of alternative sources, and use of different technologies that make the resource more or less productive). NPD is more restrictive when it applies to many decompositions of the original problem. Consider for instance the case where the division of Q is tested against the iteration cubic meter by cubic meter of the solution. Suppose that $Q' < Q$ cubic meters have already been shared (combined with some monetary transfers). Given the possibility of monetary compensations, the efficient allocation of Q' prescribed by the solution must equalize the marginal social surplus across countries (assuming for simplicity that we have an interior solution). When considering the additional cubic

[1]Karni and Schmeidler themselves refer to a 1969 *mimeo* written by A. Gibbard.

[2]Here are a few references: Plott (1976), Grether and Plott (1982), Campbell (1992), Dutta et al. (2001), Ehlers and Weymark (2003), Fleurbaey (2003), Chambers (2005a), Fleurbaey and Tadenuma (2007), de Clippel (2009), and de Clippel and Eliaz (2009). The list is not exhaustive, but it illustrates well the various contexts where a property in the spirit of IND has been used, and the various formulations that have been proposed.

[3]The story is of course rather stylized, the objective being to emphasize the argument behind the main result of our paper. Still, the model is more general than it may seem at first sight. For instance, the costs of extraction seem to be overlooked, but they can possibly be expressed in terms of the energy required to extract the gas, which itself can be obtained from a fraction of the natural gas extracted. Q can then be interpreted as the *net* quantity available in the field. Also, our story does not incorporate time explicitly, but the functions v_i can be reinterpreted as the net present value of streams of resources to be extracted.

meter to be shared in the next iteration of the decomposition, all the countries look identical, because a cubic meter is essentially an infinitesimal quantity when compared to Q, and the countries' social surplus functions over quantities that are larger than this infinitesimal amount must be irrelevant under IND. In order to be anonymous (a minimal requirement for equitability), the solution should give an equal share to each country of the additional total surplus generated by the additional cubic meter to allocate. Iterating the process, it follows that the total surplus associated to Q should be shared equally across countries. The formal reasoning is more general (e.g. allowing for multiple goods, and without restricting attention to functions v_i that guarantee interior solutions), but also requires to focus on solutions that are regular (formalized in an axiom of continuity) in order to make the argument at the margin complete.

The paper unfolds as follows. Section 2 presents the model. The axioms and the main result are included in Section 3, while its proof is postponed to Section 5. Section 4 offers a review of the related literature.

2 Model

A set $\mathcal{I}$ of $I \geq 2$ individuals have to allocate a bundle ω of L perfectly divisible goods ($\omega \in \mathbb{R}^L_+$). Some compensation can be achieved through monetary transfers. An *allocation* is a couple $(x,t) \in \mathbb{R}^{IL}_+ \times \mathbb{R}^I$ where, for each $i \in \mathcal{I}$, t_i (resp. x_i) represents the net amount of money (resp. bundle of goods) that individual i receives. It is *feasible* if $\sum_{i \in I} x_i \leq \omega$ and $\sum_{i \in I} t_i \leq 0$. The set of feasible allocations will be denoted by $\mathcal{F}(\omega)$.

Utilities are quasi-linear. The *utility function* $u_i : \mathbb{R}^L_+ \to \mathbb{R}_+$ determines the maximal (or reservation) price $u_i(x)$ that individual i is ready to pay to consume each bundle $x \in \mathbb{R}^L_+$. The utility functions are assumed to be non-decreasing, continuous and such that $u(0) = 0$.[4] The set of all such functions is denoted by $\mathcal{U}$. Agent i's total utility associated to the allocation (x,t) is $u_i(x_i) + t_i$. A *utility profile* is a vector $\mathfrak{u}$ in $\mathbb{R}^I$. It is *feasible* if there exists a feasible allocation (x,t) such that $\mathfrak{u}_i = u_i(x_i) + t_i$, for each $i \in \mathcal{I}$.

An *allocation problem* P is a couple (ω, u), where ω is the bundle of L goods to share, and $u = (u_i)_{i \in \mathcal{I}} \in \mathcal{U}^I$ is the list of utility functions. The set of all allocation problems is denoted by $\mathcal{P}$.

An *allocation rule* is a correspondence $\mathcal{R} : \mathcal{P} \to \mathbb{R}^{IL}_+ \times \mathbb{R}^I$, which associates to each

[4]It is natural to assume that an individual's reservation price for consuming nothing is zero. Dropping this assumption would require to change some notations, but not the substance of our argument.

allocation problem a nonempty set of feasible allocations. We will assume throughout the paper that the allocation rules determine a single utility profile:

$$\{(x,t) \in \mathcal{R}(P) \text{ and } (x',t') \in \mathcal{R}(P)\} \Rightarrow \{u_i(x_i) + t_i = u_i(x'_i) + t'_i,\ \forall i \in \mathcal{I}\}, \qquad (1)$$

for each $P \in \mathcal{P}$, and each pair $((x,t),(x',t'))$ of allocations.

A *solution* is a function $\sigma : \mathcal{P} \to \mathbb{R}^I$ that associates a utility profile to each allocation problem. Condition (1) makes it meaningful to consider the solution associated to an allocation rule $\mathcal{R}$ that is defined as follows:

$$\sigma_i^{\mathcal{R}}(P) = u_i(x_i) + t_i,\ \forall i,$$

for some (or each, by (1)) $(x,t) \in \mathcal{R}(P)$, and each $P \in \mathcal{P}$.

For each allocation problem $P = (\omega, u)$,

$$s(P) = \max_{x \in \mathbb{R}_+^{IL}} \left\{ \sum_{i \in \mathcal{I}} u_i(x_i) \mid \sum_{i \in \mathcal{I}} x_i \leq \omega \right\}.$$

denotes the maximal total surplus achievable. The *equal surplus sharing solution*[5] σ^{ESS} is then given by:

$$\sigma_i^{ESS}(P) = \frac{s(P)}{I},$$

for each $i \in \mathcal{I}$, and each $P = (\omega, u) \in \mathcal{P}$. The *equal surplus sharing allocation rule* $\mathcal{R}^{ESS}$ is then naturally defined as follows:

$$\mathcal{R}^{ESS}(P) = \{(x,t) \in \mathcal{F}(\omega) \mid u_i(x_i) + t_i = \sigma_i^{ESS}(P),\ \forall i \in \mathcal{I}\},$$

for each $P = (\omega, u) \in \mathcal{P}$.

Finally, an allocation rule $\mathcal{R}$ is *welfarist* if $\sigma^{\mathcal{R}}(P) = \sigma^{\mathcal{R}}(P')$, for each pair (P, P') of allocation problems with $s(P) = s(P')$. This definition should make precise the discussion we had in the Introduction and that we will pursue in Section 4.

[5] One could argue that σ^{ESS} is actually the egalitarian solution. We refrain from using this terminology, because it also coincides with many other solutions such as the Nash or the Kalai-Smorodinsky solutions applied to the bargaining problem $(U(P), d(P))$, where $d(P) = 0$ and $U(P) = \{\mathfrak{u} \in \mathbb{R}^I \mid \sum_{i \in \mathcal{I}} \mathfrak{u}_i \leq s(P)\}$, for each $P \in \mathcal{P}$. The problems being quasi-linear, σ^{ESS} actually coincides with any solution that is welfarist, and satisfies the properties of "Efficiency" and "Equal Treatment of Equals" (cf. definitions below in the main text).

3 Main Result

Here are the axioms that we will impose on the allocation rule.

Efficiency (EFF) $\sum_{i\in\mathcal{I}} \sigma_i^{\mathcal{R}}(P) = s(P)$, *for each* $P \in \mathcal{P}$.

Equal Treatment of Equals (ETE) $\sigma_i^{\mathcal{R}}(P) = \sigma_j^{\mathcal{R}}(P)$, *for each* $P = (\omega, u) \in \mathcal{P}$, *and each* i, j *in* $\mathcal{I}$ *such that* $u_i = u_j$.

No Profitable Decompositions (NPD) *Let* $\tilde{P} = (\widetilde{\omega}, u) \in \mathcal{P}$, *let* $\omega \in \mathbb{R}_+^L$ *be such that* $\omega \leq \widetilde{\omega}$, *let* $P = (\omega, u)$, *let* $i \in \mathcal{I}$, *and let* $(\tilde{x}, \tilde{t}) \in \mathcal{R}(\tilde{P})$. *Then, there exists* $(x, t) \in \mathcal{R}(P)$ *and* $(y, r) \in \mathcal{R}(P_x)$ *such that*

$$u_i(x_i + y_i) + t_i + r_i \leq u_i(\tilde{x}_i) + \tilde{t}_i,$$

where $P_x = (\widetilde{\omega} - \omega, u^x)$ *is the "residual problem" obtained after distributing* (x, t), *i.e. with* $u_i^x(y_i) = u_i(x_i + y_i) - u_i(x_i)$, *for each* $y_i \in \mathbb{R}_+^L$ *and each* $i \in \mathcal{I}$.

Independence of Preferences over Non-Feasible Allocations (IND) *Let* $P = (\omega, u) \in \mathcal{P}$ *and* $\tilde{P} = (\omega, \tilde{u}) \in \mathcal{P}$ *be such that* $u_i(x) = \tilde{u}_i(x)$, *for each* $i \in \mathcal{I}$ *and each* $x \in \mathbb{R}_+^L$ *with* $x \leq \omega$. *Then* $\sigma_i^{\mathcal{R}}(P) = \sigma_i^{\mathcal{R}}(\tilde{P})$.

Continuity (CONT) a) Let $\omega \in \mathbb{R}_+^L$ and let $(\omega_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+^L$ that converges to ω. Then the sequence $(\sigma^{\mathcal{R}}(\omega_k, u))_{k\in\mathbb{N}}$ converges to $\sigma^{\mathcal{R}}(\omega, u)$, for each $u \in \mathcal{U}^I$.
b) For every compact set $K \subseteq \mathbb{R}_+^L$, there exist $M > 0$ such that[6]

$$\|\sigma^{\mathcal{R}}(\omega, u) - \sigma^{\mathcal{R}}(\omega, \tilde{u})\| \leq M d(u, \tilde{u}),$$

for every $\omega \in K$ and $u, \tilde{u} \in \mathcal{U}^I$.

EFF simply imposes on the rule to specify allocations that are Pareto efficient. It should not be possible to find another feasible allocation that would make all the individuals happier. ETE guarantees some minimal form of equity, in that two individuals with the same utility functions are treated identically. NPD guarantees that no participant can have an interest in manipulating the allocation rule through some decomposion of the stakes. As explained in the Introduction, a violation of that property may lead to conflict and inefficiency when it comes to implement the rule, as well as a violation of

[6] $d(u, \tilde{u}) = \max_{i\in\mathcal{I}} \sup_{x\in\mathbb{R}_+^L} |u_i(x) - \tilde{u}_i(x)|$.

the equity principles that motivated the solution in the first place. Different people may have different opinions regarding what is the right way of formalizing NPD depending on the agents relative optimism/pessimism when decomposing the stakes (given that allocation rules can be multi-valued). Our formulation presumes that each agent is most pessimistic, making a rule robust to profitable decompositions as soon as the combination of some element $(x, t) \in \mathcal{R}(P)$ and $(y, r) \in \mathcal{R}(P_x)$ makes him no better than the solution of the original problem $\tilde{P}$. The property is thus the weakest version one can think of,[7] making the uniqueness result in the next Theorem only more interesting. On the other hand, observe that Σ^{ESS} does satisfy the stronger version of NPD, where they are most optimistic.[8]

Strong NPD *Let $\tilde{P} = (\widetilde{\omega}, u) \in \mathcal{P}$, let $\omega \in \mathbb{R}^L_+$ be such that $\omega \leq \widetilde{\omega}$, let $P = (\omega, u)$, let $i \in \mathcal{I}$, and let $(\tilde{x}, \tilde{t}) \in \mathcal{R}(\tilde{P})$. Then, for all $(x, t) \in \mathcal{R}(P)$, and all $(y, r) \in \mathcal{R}(P_x)$, we have:*

$$u_i(x_i + y_i) + t_i + r_i \leq u_i(\tilde{x}_i) + \tilde{t}_i,$$

where $P_x = (\widetilde{\omega} - \omega, u^x)$ is the "residual problem" obtained after distributing (x, t), i.e. with $u_i^x(y_i) = u_i(x_i + y_i) - u_i(x_i)$, for each $y_i \in \mathbb{R}^L_+$ and each $i \in \mathcal{I}$.

While discussing the notion of exhaustivity towards the end of this Section, we will encounter a simple solution that satisfies NPD, but not its stronger version. Before stating our main result, let us motivate the last two axioms. An allocation rule must specify feasible allocations, and hence no individual can ever receive more than the amounts that are available for division. It is then natural to assume that the individuals' reservation prices for bundles that are not feasible should be irrelevant in the determination of the final allocation, as required by IND (see references in the introduction). It is also meaningful to require some form of continuity on the allocation rule. CONT formalizes the idea that small measurement mistakes should not trigger a major difference when computing the solution. Part (a) applies this principle to the total resources available, while part (b) requires the stronger property of Lipschitz continuity with respect to the utility functions.

Theorem *$\mathcal{R}^{ESS}$ satisfies EFF, ETE, NPD, IND and CONT. Conversely, any allocation*

[7] It is not even required for the agents' expectations to be consistent, in that (x, t) and (y, r) may very with i in the definition of NPD.

[8] Observe that the strong version of NPD actually implies condition (1).

rule that satisfies the axioms must be such that $\sigma^{\mathcal{R}} = \sigma^{ESS}$.

We already gave some intuition for this Theorem in the Introduction, and we defer the complete proof to Section 5. We now discuss the independence of the axioms. The equal split allocation rule, σ^{ES}, that shares ω equally among all the individuals without making any monetary compensation, satisfies all the axioms except EFF. An allocation rule that selects feasible allocations that split the total surplus in some fixed proportions which are not the same for all the individuals (same as Kalai's (1977) proportional solutions) clearly satisfies all our axioms, except ETE. Consider next the solution proposed by Moulin (1992). For each $P = (\omega, u) \in \mathcal{P}$, let

$$\sigma^M(P) = \{(x,t) \in \mathcal{F}(\omega) \mid u_i(x_i) + t_i = Sh_i(v^{(\omega,u)}), \forall i\},$$

where Sh denotes the Shapley value, and $v^{(\omega,u)}$ is the characteristic function defined as follows:

$$v^{(\omega,u)}(\mathcal{S}) = \max_{x \in \mathbb{R}_+^{IS}} \left\{ \sum_{i \in \mathcal{S}} u_i(x_i) \mid \sum_{i \in \mathcal{S}} x_i \leq \omega \right\},$$

for each coalition $\mathcal{S} \subseteq \mathcal{I}$ (i.e. the maximal surplus that members of $\mathcal{S}$ could share if they were free to distribute ω among themselves). σ^M satisfies EFF (resp. ETE; resp. CONT a); resp. b)) because the Shapley value is efficient (resp. symmetric; resp. continuous; resp.[9] linear). It obviously satisfies IND, given the way $v^{(\omega,u)}$ is defined. The Theorem thereby implies that it violates NPD. More explicitly, consider for instance the allocation problem $\tilde{P}$ with $L = 1$, $\mathcal{I} = \{1, 2\}$, $\widetilde{\omega} = 2$, $u_1(x) = 2x$ if $x \leq 1$ (resp. $1 + x$ if $x \geq 1$), and $u_2(x) = \min\{x, 1\}$, for each $x \in \mathbb{R}_+$. Any element of σ^M gives a utility of 2.5 to the first agent and 0.5 to the second agent. Even a pessimistic agent 2 (as in NPD) would want to decompose the stakes, starting for instance by allocating a single unit of the good. Indeed, the Moulin solution of that problem contains a unique allocation, with the first agent receiving the good and paying a half dollar to agent 2. Solving the residual problem, we conclude that the second agent can guarantee himself a utility of at least \$1 via this decomposition. To conclude, we have unfortunately not been able to prove separately the independence of IND and CONT from the rest of the axioms. While we clearly use both axioms in the proof in Section 5, it remains a possibility (and would

[9] To show that σ^M satisfies CONT b), one also needs to observe that $|v^{(\omega,u)}(\mathcal{S}) - v^{(\omega,\tilde{u})}(\mathcal{S})| \leq Sd(u, \tilde{u})$, which is shown explicitly in the Appendix for the special case $\mathcal{S} = \mathcal{I}$ (when checking that σ^{ESS} satisfies CONT).

make the Theorem only even more interesting) that one of them might be dropped, or at least weakened. We will therefore only show that they cannot both be dropped and, for notational simplicity, we will do so only when $L = 1$. Let $\hat{\mathcal{P}}$ be the set of problems (ω, u) for which there exist $\alpha \in \mathbb{R}^I_{++}$ and $x^* \geq 0$ such that u_i is differentiable and $u_i'(x) = \alpha_i$, for all $x \geq x^*$ and all $i \in \mathcal{I}$. Consider then the following allocation rule:

$$\mathcal{R}(P) = \mathcal{R}^{ESS}(P), \text{ for all } P \in \mathcal{P} \setminus \hat{\mathcal{P}}, \text{ and}$$

$$\mathcal{R}(P) = \{(x,t) \in \mathcal{F}(\omega) \mid u_i(x_i) + t_i = Sh_i(v^u)s(\omega, u)\ \forall i \in \mathcal{I}\},$$

for each $P = (\omega, u) \in \hat{\mathcal{P}}$, where v^u is the characteristic function defined as follows:

$$v^u(\mathcal{S}) = \frac{\max_{i \in \mathcal{S}} \alpha_i}{\max_{i \in \mathcal{I}} \alpha_i},$$

for each coalition $\mathcal{S} \subseteq \mathcal{I}$. It is not difficult to check that $\mathcal{R}$ satisfies EFF, ETE, and NPD (because $\hat{\mathcal{P}}$ is closed under decompositions, and $s(\omega, u)$ is additive, as shown in the Appendix when checking that σ^{ESS} satisfies NPD), but violates both CONT and IND.

It is worthwhile to note that our characterization of the egalitarian solution does not require exhaustivity of the allocation rule, an assumption imposed by most of the papers that characterize classical welfarist solutions in non-welfarist environments. An allocation rule σ is *exhaustive* if, for each $P \in \mathcal{P}$, an allocation $(x', t') \in \sigma(P)$ whenever it is feasible and it generates the same utility profile as an allocation $(x, t) \in \sigma(P)$. Observe that σ^{ESS} is exhaustive, while σ^{ES} is not. The "exhaustive extension" of σ^{ES} is defined as follows:

$$\bar{\sigma}^{ES}(P) = \{(x,t) \in \mathcal{F}(\omega) | u_i(x_i) + t_i = u_i(\frac{\omega}{I}), \forall i\},$$

for each $P \in \mathcal{P}$. While σ^{ES} satisfies also the stronger version of NPD, $\bar{\sigma}^{ES}$ satisfies only NPD. To see that, consider for instance the allocation problem $\tilde{P}$ with $L = 1$, $\mathcal{I} = \{1, 2\}$, $\widetilde{\omega} = 4$, $u_1(x) = x$, and $u_2(x) = \min\{x, 1\}$, for each $x \in \mathbb{R}_+$. Any element of $\bar{\sigma}^{ES}$ gives a utility of 2 to the first agent and 1 to the second agent. An optimistic agent 2 may hope to be better off with first receiving nothing of the good plus a compensation of one dollar when allocating the first two units, and then getting 1 unit of the good with no compensation in the residual problem. A pessimistic agent 2, on the other hand, would have no strict incentive to decompose the stakes when $\bar{\sigma}^{ES}$ is used (any allocation rule that contains an allocation that satisfies the strong NPD must necessarily satisfy NPD). The property of exhaustivity may thus be restrictive in that it rules out reasonable allocation rules when used in conjunction with other axioms (strong NPD in this example).

We conclude this section by arguing that the natural analogues of EFF, ETE, NPD, and IND are likely to be incompatible when monetary compensations are not available. When there is a single good to be allocated, the equal split solution is the only solution that satisfies the axioms, at least if preferences are strictly increasing. This is a direct consequence of ETE, since there is only one possible such ordinal preference - the more the better. Moving to two goods or more leads to an impossibility. This follows from Moulin and Thomson's (1988, Theorem 1) impossibility result. Indeed, the natural extension of NPD in a framework without monetary compensations will imply their property of "Resource Monotonicity." At the same time, IND and NPD will imply their "Individual Rationality" axiom, which requires that each individual prefers the final outcome to an equal split of the total endowment. Notice that applying the natural extension of NPD good by good will imply that property, since IND imply that the solution of each smaller problem (focus on one good) depends only on the individuals' preferences for that good, and as before, there is only one such preference (restricting attention to preferences that are strictly monotonic). Moulin and Thomson's (1988) two-good two-individual counter-examples therefore apply, and it is not difficult to extend them to counter-examples with any number of goods and individuals. It remains an interesting question to find restricted domains that are different from the quasi-linear case, and where the axioms would be compatible again (see comment (D) Section 4 of Moulin and Thomson (1988)).

4 Related Literature

Graham et al. (1990, Section II) characterized the equilibrium allocation rule that prevails in single-unit second-price auctions in the presence of nested buyer rings. Its computation is reminiscent of the principle of serial cost sharing (Littlechild and Owen (1973)), and each resulting allocation happens to coincide with the Shapley value of some characteristic function derived from the buyers' reservation prices. Indeed, the payoffs have a strong normative appeal as well (see Moulin (1992, Section 5)). There seems to be a natural procedure to adapt this allocation rule to problems that involve a quantity Q of a *divisible* good: decompose the problem into a sequence of allocation problems with infinitesimal quantities, solve each infinitesimal problem via the previous solution (treating each infinitesimal quantity as indivisible), and integrate in order to obtain a solution for the original problem. Of course, the procedure works well only for problems with decreasing marginal utilities, as otherwise the resulting allocations are not necessarily efficient. Suppose also that the utility functions are regular, that is differentiable and such that the efficient allocation of any positive Q gives a positive amount of the good to each participant (interior solutions). It turns out that the resulting solution then coincides with equal surplus sharing. This is true not only when applying the constructive procedure to the Graham et al./Moulin allocation rule, but also to any solution that guarantees to each agent a payoff that is larger than or equal to his valuation for the indivisible good to be allocated divided by the number of participants, a rather weak equity property first introduced by Moulin and Thomson (1988) and that plays a central role in Moulin (1992). The proof of this new result is very similar to Step 1 in the proof of our Theorem (see Section 5). The general idea is that, at every step of the continuous summation, the lower bound on the participant's final utility is binding, and equal to the common marginal utility (which is also equal to the derivative of the total surplus, by the envelope theorem) divided by the number of participants.[10] The details for the full proof are left to the dedicated reader. It is interesting to note that the Graham et al./Moulin allocation rule, as well as many of the rules that meet Moulin and Thomson's lower-bound requirement, are not welfarist. Yet, once iterated to obtain a solution for the divisible case, they all result in the same welfarist solution.

At first sight, NPD may seem very similar to Kalai's (1977) axiom of *step-by-step negotiations* (see also Myerson (1977), and Young's (1988) composition principle in taxation problems). In reality, the two axioms are rather different. Indeed, NPD cannot even be phrased in Kalai's welfarist framework, because the set of utility profiles that are feasible when sharing the bundle $\omega - \omega'$ after having solved for $\omega' < \omega$ may be strictly smaller than the set of utility profiles that are feasible when sharing the bundle ω. Kalai assumes instead that the solution for the problem of dividing ω' is a partial agreement that serves as a disagreement point in a new bargaining problem where any division of the bundle ω can still be agreed upon. NPD, on the contrary, assumes that any partial agreement is final and non-renegotiable.[11] Kalai's arguments in support of the egalitarian solution

[10] Notice that requiring the efficient allocations to be interior is important. If the first participant's utility function equals the quantity he consumes, while the second participant's utility function equals twice the quantity she consumes, then the solution obtained by iterating the Graham et al./Moulin allocation rule does not coincide with an equal split of the total surplus.

[11] Kalai himself (page 1627) offers a very clear discussion of his axiom of step-by-step negotiations, emphasizing that, although a property in the spirit of NPD would be a natural formulation of the general principle, his axiom must have an alternative interpretation in terms of partial agreements because NPD

are not very informative for the quasi-linear case that we focus on. Indeed, equal sharing of the surplus follows immediately from the properties of efficiency and anonymity when one is ready to work in the space of utilities. The purpose of Kalai's argument instead is to characterize proportional solutions in a welfarist framework when utilities are non-transferable. It may be interesting to test the robustness of Kalai's result, by trying to rephrase it in explicit economic environments. As has been showed on different occasions, and most forcefully by Roemer (1988), axioms that characterize a solution in the space of utilities are usually satisfied by other non-welfarist solutions as well.

The additivity/super-additivity property[12] that plays a key role in various axiomatic results of social choice and cooperative game theory is often motivated by referring to multiple issues (see e.g. Shapley (1953), Peters (1986), Ponsati and Watson (1997)).[13] The story behind the axiom is that the participants' payoffs when bargaining over all the issues at once should be larger than or equal to the sum of their payoffs when bargaining over the different issues separately. A difficulty though is that all the papers in that vein are written in welfarist frameworks. Yet it is usually impossible to derive the utility possibility set when bargaining over two issues simultaneously, from the two sets of the utilities that are feasible when bargaining over each issue separately. The usual motivation behind the additivity/superadditivity property is thus meaningful only when utility functions are assumed to be additively separable across issues, in which case the former set is indeed the sum of the other two.[14] So, while applying NPD to decompositions good by good is reminiscent of these ideas on multi-issue bargaining, we believe that

cannot be phrased in the space of utilities. It is thus surprising that, to the best of our knowledge, the property of NPD has not been studied sooner in non-welfarist environments.

[12]Quasi-linear problems lead to utility possibility sets that are half-spaces, and super-additivity is then equivalent to additivity.

[13]The additivity/super-additivity property is sometimes given an alternative interpretation in terms of a preference to agree before the resolution of some uncertain events, see e.g. Myerson (1981) and Perles and Maschler (1981). This kind of argument is unrelated to our analysis, since there is no uncertainty in our framework, and utility functions do not contain any information regarding risk attitudes.

[14]Green (1983) has taken a first step away from welfarism in quasi-linear problems, by dissociating monetary compensations from the set of utilities that are achievable in the absence of transfers (see also Green (2005), Chambers (2005b), and Chambers and Green (2005) for more recent results). The additivity/superadditivity property is subject to the same limitation as far as its interpretation is concerned, but it is worth noting that these authors do obtain interesting solutions that are both anonymous and efficient, while different from the equal surplus sharing rule. These solutions would trivially satisfy IND if they were rephrased in our explicit economic environments, because the utility possibility set obtained in the absence of monetary transfers does not change when one modifies the utility function of any participant over bundles that involve more goods than available in the total endowment. Those solutions must therefore violate NPD and/or CONT.

NPD is a more appropriate formulation. Arguing in a non-welfarist framework, we are indeed able to treat problems with no underlying restriction on utility functions. NPD also highlights another class of multi-issue problems that arise from alternative decompositions. Indeed, a participant may insist, for instance, on sharing first a fraction of the total endowment, before sharing what remains. The two issues that this decomposition generates are inter-dependent, even if the utility functions are additively separable (or even if $L = 1$), and therefore cannot be phrased in any welfarist model. As for Kalai (1977), the proof of our result has no analogue in the literature on multi-issue bargaining, since the equal surplus sharing solution follows trivially from the axioms of efficiency and anonimity when working exclusively in the space of utilities.

O'Neill et al. (2004) introduce a new welfarist model of bargaining, where the set of feasible utility profiles expands over time according to a differentiable function. Our two papers thus share a common line of argument, in that a solution is ultimately characterized by integrating its local behavior, which can be determined by imposing rather weak axioms. A first obvious difference is that there is no given bargaining agenda in our model. The integration step follows from the NPD property instead. More importantly, the arguments bear on different objects in our two papers. Working in the space of utilities, equal surplus sharing is not derived in O' Neill et al., but instead assumed by their symmetry property. The key ingredient in their result is that the efficient frontier of the expanded set of feasible utilities at time $t + \Delta t$ that lies above the agreement reached at time t is essentially linear when Δt is infinitesimal. Scale covariance then leads to a problem in the space of utilities that can be solved by direct application of the symmetry axiom. The key ingredient in our result is that the participants' preferences are essentially identical when an infinitesimal quantity $\Delta\omega$ has to be divided after a strictly positive quantity ω has already been distributed (assuming that we have an interior solution). Notice how the set of feasible utilities at time t does not depend on previous agreements in O'Neill et al.'s model. Rephrased in an economic environment like ours, this implies that the whole quantity of all the goods that have been bargained in the past must be renegotiated at every t, as in Kalai's interpretation of the property of step-by-step decomposition. In our case, to integrate the solution of local problems that follow a path from 0 to ω often leads to an inefficient solution because past agreements are assumed to be non-renegotiable (except when $L = 1$ and marginal utilities are decreasing, as in the first paragraph of the present section).

NPD is related to the CONRAD property that Roemer (1988) introduced to recover most classical results in bargaining theory with axioms phrased in economic environments. Though weaker, the CONRAD property is far more cumbersome than NPD, because it restricts in a rather ad-hoc way the set of decompositions over which it applies (adding goods in which at most one agent is interested, provided the set of feasible utility profiles remains the same). If a person likes Roemer's idea of consistency in CONRAD, then we think that he or she will prefer to go all the way to NPD. Notice that Roemer's proof cannot be adapted to our framework because he makes crucial use of preferences that are not quasi-linear. Our result has also the advantage of holding for any fixed number of goods, while Roemer works with a variable and potentially infinite number of goods.

The present paper studies the exact same problem as Moulin (1992), but from a different perspective. Moulin introduces four new properties: resource monotonicity, population solidarity, (weak and strong) individual rationality, and the stand-alone test. He then shows that these four properties, as well as most possible combinations of two or three properties out of the list, are incompatible both on the general domains and when restricting attention to concave utility functions. On the other hand, there exists a solution that satisfies the four axioms simultaneously (using the weaker version of individual rationality) on the restricted domain where goods are substitutes, that is when restricting attention to utility functions that are concave in each good, as well as submodular. We have already discussed Moulin's solution when checking the independence of our axioms at the end of the previous section. There we noted that it satisfies all our axioms, except NPD. It is thus subject to strategic manipulations of the agenda, leading, as we argued earlier, to possible conflict, inefficiency, and violation of the equity principles that motivated the solution in the first place. It is not difficult to check, on the other hand, that the equal surplus sharing solution that we characterized, satisfies all of Moulin's axioms except the stand-alone test (not only on the restricted domain where goods are substitute, but over the all domain). Let us thus explain briefly the content of that test, and why, though interesting, we do not see it as an uncontroversial principle of equity. A solution passes the stand-alone test if no coalition of agents receives a higher aggregated payoff than the maximal surplus that its members could achieve if they were free to share the whole total endowment, giving nothing to non-members. The solution must thus belong to the anti-core of the fictitious characteristic function used to compute the Moulin solution. It implies for instance that an agent on his own cannot get a payoff that is larger than his reservation price for consuming the total endowment. The equal surplus sharing solution takes a different standpoint on equity. Even in the limit case where an agent does not care for the goods being shared, we think that he should not be treated as irrelevant because he is a member of the group that collectively owns the total endowment. More generally, it is true that efficiency requires that an agent should not consume much of the goods being shared when others have higher marginal utilities, but this does not mean that there should be no or little monetary compensations in order to reach an equitable outcome. It remains a fact that consuming less is a favor to other agents, insofar as it lets them consume more, and it seems fair to compensate agents on the basis of that criterion as well.

We close this literature review by briefly discussing two alternative axiomatic characterizations of the equal surplus sharing solution in non-welfarist environments. Moulin (1985, Theorem 2) provides one such result when selecting a public decision, together with monetary compensations, when there are at least three participants. Interestingly, his key axiom, No Advantageous Reallocations (NAR), is another property of robustness against some class of potential manipulations of the solution to be implemented. Indeed it requires that no coalition of individuals can be better off by publicly changing their utility functions via contingent monetary transfers. NPD on the other hand operates through decompositions of the total endowment, while the participants' utility functions are fixed. Ginés and Marhuenda (2000) study economies where money is used to produce multiple public goods. They succeed in characterizing the equal surplus sharing solution by giving some economic content to Kalai's (1977) monotonicity property. The axiom restricts the behavior of the solution when the individuals' satisfaction from consuming the public goods increase. This kind of principle has nothing to do with the axioms we discussed in Section 3. Ginés and Marhuenda also show that their result does not extend to the production of private goods. This confirms that there is no connection between our result and theirs.

5 Proof

It is clear that $\mathcal{R}^{ESS}$ satisfies EFF, ETE and IND. Part (a) of CONT is an immediate consequence of Berge's (1959) maximum theorem. As for part (b) of CONT, let $x \in \mathbb{R}_+^{IL}$ be such that $\sum_{i \in \mathcal{I}} x_i \leq \omega$ and $\sum_{i \in \mathcal{I}} u_i(x_i) = s(\omega, u)$. Then $\sum_{i \in \mathcal{I}} \tilde{u}_i(x_i) \leq s(\omega, \tilde{u})$, and hence $s(\omega, u) - s(\omega, \tilde{u}) \leq Id(u, \tilde{u})$. A similar argument also implies that $s(\omega, \tilde{u}) - s(\omega, u) \leq Id(u, \tilde{u})$. Hence $|\sigma_i^{ESS}(\omega, u) - \sigma_i^{ESS}(\omega, \tilde{u})| \leq d(u, \tilde{u})$, for every $u, \widetilde{u} \in \mathcal{U}^I$ (independently of ω), and thus $\mathcal{R}^{ESS}$ satisfies CONT. Finally, to check that it satisfies NPD (or even its strong version), it is enough to observe that

$$s(\omega', u) \geq \max \left\{ \sum_{i \in \mathcal{I}} u_i(x_i^* + y_i) \mid y \in \mathbb{R}_+^{IL} \text{ and } \sum_{i \in \mathcal{I}} y_i \leq \omega' - \omega \right\},$$

for any x^* that maximizes $\sum_{i \in \mathcal{I}} u_i(x_i)$ over the set of vectors $x \in \mathbb{R}_+^{IL}$ such that $\sum_{i \in \mathcal{I}} x_i \leq \omega$.

Let's prove now the second part of the Theorem. Let thus $\mathcal{R}$ be a rule that satisfies the five axioms, and let $(\bar{\omega}, u)$ be an allocation problem. We have to prove that $\sigma^{\mathcal{R}}(\bar{\omega}, u) = \sigma^{ESS}(\bar{\omega}, u)$. For each $\bar{\omega} \in \mathbb{R}_{++}^L$, let $X(\bar{\omega}) := \{x \in \mathbb{R}_+^L | x \leq \bar{\omega}\}$, and let $\mathcal{V}(\bar{\omega})$ be the following set of functions:

$$\mathcal{V}(\bar{\omega}) = \{u \in \mathcal{C}^2(intX(\bar{\omega})) \mid \forall d \in \partial \mathbb{R}_+^L,\ \forall l \in \{1, \ldots, L\} : d_l = 0 \Rightarrow \lim_{x \to d, x \in int(X(\bar{\omega}))} \frac{\partial u}{\partial x_l}(x) = +\infty\},$$

where $\mathcal{C}^2(int(X(\bar{\omega})))$ denotes the set of functions that are twice continuously differentiable on the interior of $X(\bar{\omega})$. We are now ready to proceed with the proof in four steps.

Step 1. *Suppose that $\bar{\omega} \in \mathbb{R}_{++}$. If $u \in \mathcal{V}(\bar{\omega})^I$, then $\sigma_i^{\mathcal{R}}(\cdot, u)$ admits a right directional derivative along any vector $d \in \mathbb{R}_+^L \setminus \{0\}$, at any point ω in the interior of $X(\bar{\omega})$. In addition, this derivative is equal to $\frac{1}{I} \nabla_\omega s(\cdot, u) \cdot d$. Formally,*

$$\lim_{\varepsilon \to 0^+} \frac{\sigma_i^{\mathcal{R}}(\omega + \varepsilon d, u) - \sigma_i^{\mathcal{R}}(\omega, u)}{\varepsilon} = \frac{1}{I} \nabla_\omega s(\omega, u) \cdot d,$$

for each $\omega \in int(X(\bar{\omega}))$ and each $d \in \mathbb{R}_+^L \setminus \{0\}$.

Proof: Let ω, d as above, let $\varepsilon \in (0, 1]$, and let $i \in \mathcal{I}$. NPD applied with $\widetilde{\omega} = \omega + \varepsilon d$ implies that there exists $(x^i(\varepsilon), t^i(\varepsilon)) \in \mathcal{R}(\omega, u)$ such that

$$\sigma_i^{\mathcal{R}}(\omega + \varepsilon d, u) - \sigma_i^{\mathcal{R}}(\omega, u) \geq \sigma_i^{\mathcal{R}}(\varepsilon d, u^{x^i(\varepsilon)}). \quad (2)$$

For each $j \in \mathcal{I}$, let

$$\alpha_j^i(\varepsilon) = \max_{0 \leq y \leq \varepsilon d} |\nabla u_j(x_j^i(\varepsilon)) \cdot y - u_j^{x^i(\varepsilon)}(y)|,$$

and let $u_j^{i,\varepsilon}$ be the utility function defined as follows:

$$u_j^{i,\varepsilon}(y) = \begin{cases} \nabla u_j(x_j^i(\varepsilon)) \cdot y & \text{if } |\nabla u_j(x_j^i(\varepsilon)) \cdot y - u_j^{x^i(\varepsilon)}(y)| \le \alpha_j^i(\varepsilon) \\ u_j^{x^i(\varepsilon)}(y) + \alpha_j^i(\varepsilon) & \text{if } \nabla u_j(x_j^i(\varepsilon)) \cdot y - u_j^{x^i(\epsilon)}(y) > \alpha_j^i(\varepsilon) \\ u_j^{x^i(\varepsilon)}(y) - \alpha_j^i(\varepsilon) & \text{if } u_j^{x^i(\varepsilon)}(y) - \nabla u_j(x_j^i(\varepsilon)) \cdot y > \alpha_j^i(\varepsilon) \end{cases}$$

for each $y \in \mathbb{R}_+^L$. It is easy to check that $u_j^{i,\epsilon} \in \mathcal{U}$, for each $j \in \mathcal{I}$.

Let K be the compact set $K = \{y \in \mathbb{R}_+^L | y \le d\}$. Part (b) of CONT implies that there exists $M > 0$ such that

$$\sigma_i^{\mathcal{R}}(\varepsilon d, u^{x^i(\varepsilon)}) \ge \sigma_i^{\mathcal{R}}(\varepsilon d, u^{i,\varepsilon}) - M d(u^{x^i(\varepsilon)}, u^{i,\varepsilon}) \tag{3}$$

for each $\varepsilon \in (0,1]$. Since $u \in \mathcal{V}(\bar{\omega})^I$ and $x^i(\varepsilon)$ is an efficient split of ω, it must be interior, and thus $\nabla u_j(x_j^i(\varepsilon)) = \nabla u_k(x_k^i(\varepsilon))$, for every $j \ne k$. Then IND, ETE, and EFF imply $\sigma_i(\varepsilon d, u^{i,\varepsilon}) = \frac{\varepsilon}{I} \nabla u_i(x_i^i(\varepsilon)) \cdot d$. This in turn equals $\frac{\varepsilon}{I} \nabla_\omega s(\omega, u) \cdot d$, by the envelope theorem. Observe also that the uniform distance between $u^{x^i(\varepsilon)}$ and $u^{i,\varepsilon}$ is equal to $\alpha^i(\varepsilon) = \max_{j \in \mathcal{I}} \alpha_j^i(\varepsilon)$. Hence (2) and (3) imply that

$$\frac{\sigma_i^{\mathcal{R}}(\omega + \varepsilon d, u) - \sigma_i^{\mathcal{R}}(\omega, u)}{\varepsilon} \ge \frac{1}{I} \nabla_\omega s(\omega, u) \cdot d - M \frac{\alpha^i(\varepsilon)}{\varepsilon}. \tag{4}$$

We are now ready to prove by contradiction that the ratio on the left-hand side of (4) converges to $\frac{1}{I} \nabla_\omega s(\omega, u) \cdot d$ when ε converges to 0, for each $i \in \mathcal{I}$. For simplicity, let's refer to this ratio as $r_i(\varepsilon)$. If the property is not true, then we can find $j \in \mathcal{I}$, $\beta > 0$, and a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ of strictly positive numbers that converges to 0 such that

$$|r_j(\varepsilon_k) - \frac{1}{I} \nabla_\omega s(\omega, u) \cdot d| \ge \beta, \tag{5}$$

for all k. Taylor's theorem implies[15] that $M \frac{\alpha^j(\varepsilon_k)}{\varepsilon_k}$ converges to 0 when k goes to infinity, and hence there exists $k_0 \in \mathbb{N}$ such that $M \frac{\alpha^j(\varepsilon_k)}{\varepsilon_k} < \beta$, for all $k \ge k_0$. Combining this with (5), we must have $r_j(\varepsilon_k) - \frac{1}{I} \nabla_\omega s(\omega, u) \cdot d \ge \beta$, for all those k's. Combining this with (4)

[15] Taylor's theorem implies indeed that, for each $\varepsilon > 0$, $\alpha_m^j(\varepsilon)$ ($m \in \mathcal{I}$) is equal to the absolute value of the remainder term, which is smaller than $\varepsilon^2 \cdot L^2 \cdot \|d\|^2$ times the supremum of the absolute value of the elements of the Hessian matrix, $\nabla^2 u_j(x_m^j(\varepsilon) + y)$, over all vectors y between 0 and εd, and all vectors of bundles $x^j(\varepsilon)$ that are part of an element in $\mathcal{R}(\omega, u)$. EFF and the definition of $\mathcal{V}(\bar{\omega})$ guarantee the set of all such $x^j(\varepsilon)$ is contained in a compact subset of $(intX(\bar{\omega}))^I$ (closedness of the set of efficient vectors of bundles for u follows from Berge's (1959) maximum theorem), and hence the supremum is finite.

for $i \in \mathcal{I} \setminus \{j\}$, we obtain:

$$\frac{s(\omega + \varepsilon_k d, u) - s(\omega, u)}{\varepsilon_k} = \sum_{i \in \mathcal{I}} r_i(\varepsilon_k) \geq \nabla_\omega s(\omega, u) \cdot d + \beta + M \sum_{i \in \mathcal{I} \setminus \{j\}} \frac{\alpha^i(\varepsilon_k)}{\varepsilon_k},$$

for all $k \geq k_0$. Taking the limit when k tends to infinity, we get a contradiction: $\nabla_\omega s(\omega, u) \cdot d \geq \nabla_\omega s(\omega, u) \cdot d + \beta$. ■

Step 2. *Suppose that $\bar{\omega} \in \mathbb{R}^L_{++}$. Let $u \in (\mathcal{V}(\bar{\omega}) \cap \mathcal{U})^I$. Then $\sigma_i^{\mathcal{R}}(\bar{\omega}, u) = \sigma_i^{ESS}(\bar{\omega}, u)$, for all $i \in \mathcal{I}$.*

Proof: Fix $i \in \mathcal{I}$, and define the function $f : [0,1] \to \mathbb{R}$ by $f(t) = \sigma_i^{\mathcal{R}}(t\bar{\omega}, u) - \frac{1}{I}s(t\bar{\omega}, u)$. Part (a) of CONT implies that f is continuous and, according to Step 1, f also has a right derivative with $f'_+(t) = 0$ for all $t \in (0, 1)$. Then f must be a constant function (for a proof, see for example Knight (1980)) and thus, $\sigma_i^{\mathcal{R}}(\bar{\omega}, u) - \frac{1}{I}s(\bar{\omega}, u) = \sigma_i^{\mathcal{R}}(0, u)$. IND implies that $\sigma_i^{\mathcal{R}}(0, u) = \sigma_i^{\mathcal{R}}(0, v)$ for any utility profile v. In particular, one can take a utility profile in which all agents are identical. Then ETE together with $s(0, u) = 0$ implies that $\sigma_i^{\mathcal{R}}(0, u) = 0$ and thus $\sigma_i^{\mathcal{R}}(\bar{\omega}, u) = \frac{1}{I}s(\bar{\omega}, u)$. ■

Step 3. *For each $i \in \mathcal{I}$, there exists a sequence $(u_i^n)_{n \in \mathbb{N}}$ of functions in $\mathcal{V}(\bar{\omega}) \cap \mathcal{U}$ that converges uniformly to u_i.*

Proof: Let $(Q_i^n)_{n \in \mathbb{N}}$ be the sequence of multivariate Bernstein polynomials derived from u_i on $X(\bar{\omega})$ (a definition can be found in Lorentz (1953), for instance). It is well-known that it converges uniformly to u_i on $X(\bar{\omega})$. Also, the elements of the sequence are smooth and non-decreasing on $X(\bar{\omega})$ (because u_i is non-decreasing). Unfortunately, they will typically be decreasing in some regions out of $X(\bar{\omega})$. Let then $\tilde{Q}_i^n : \mathbb{R}^L_+ \to \mathbb{R}$ be the function obtained by projecting bundles on $X(\bar{\omega})$ before applying the polynomial Q_i^n, i.e. $\tilde{Q}_i^n(x) := Q_i^n((\min\{x^l, \bar{\omega}^l\})_{l \in L})$, for each $x \in \mathbb{R}^L_+$. These functions are continuous and non-decreasing on the whole domain, by construction. They coincide with the underlying polynomials on $X(\bar{\omega})$, and hence are smooth on the interior of that domain. Yet, they do not belong to $\mathcal{V}(\bar{\omega})$, because they do not satisfy the limit conditions on partial derivatives. For each $n \in \mathbb{N}$, let then $v_i^n : \mathbb{R}^L_+ \to \mathbb{R}$ be the function defined as follows:

$$v_i^n(x) = (1 - \frac{1}{n})(\tilde{Q}_i^n(x) - \tilde{Q}_i^n(0)) + \frac{1}{n}(e^{\sum_{l=1}^L \sqrt{x_l}} - 1),$$

for each $x \in \mathbb{R}^L_+$. It is now easy to check that $v_i^n \in \mathcal{V}(\bar{\omega}) \cap \mathcal{U}$. Observe also that

$$\max_{x \in X(\bar{\omega})} |u_i(x) - v_i^n(x)| \leq (1 - \frac{1}{n})|\tilde{Q}_i^n(0)| + \max_{x \in X(\bar{\omega})} |u_i(x) - \tilde{Q}_i^n(x)| + \frac{1}{n} \max_{x \in X(\bar{\omega})} |\tilde{Q}_i^n(x) - e^{\sum_{l=1}^L \sqrt{x_l}} + 1|.$$

Each of the three terms on the right-hand side converges to 0 when n tends to infinity. Indeed, $\lim_{n\to\infty} \tilde{Q}_i^n(0) = u_i(0) = 0$, and the sequence $(\tilde{Q}_i^n)_n$ is uniformly bounded on $X(\bar{\omega})$, since it is uniformly convergent. This proves that $(v_i^n)_{n\in\mathbb{N}}$ is uniformly convergent to u on $X(\bar{\omega})$, but not necessarily on the whole domain. Hence we propose one last transformation of the sequence. For each $n \in \mathbb{N}$, let

$$\gamma(n) = \max_{x \in X(\bar{\omega})} |u_i(x) - v_i^n(x)|,$$

and let u_i^n be the utility function defined as follows:

$$u_i^n(x) = \begin{cases} v_i^n(x) & \text{if } |u_i(x) - v_i^n(x)| \leq \gamma_i(n) \\ u_i(x) - \gamma_i(n) & \text{if } u_i(x) - v_i^n(x) > \gamma_i(n) \\ u_i(x) + \gamma_i(n) & \text{if } v_i^n(x) - u_i(x) > \gamma_i(n) \end{cases}$$

for each $x \in \mathbb{R}^L_+$. It is easy to check that $u_i^n \in \mathcal{V}(\bar{\omega}) \cap \mathcal{U}$, for each $n \in \mathbb{N}$, and that the sequence converges uniformly to u_i on $\mathbb{R}^L_+$, as desired. ■

Step 4. $\sigma^{\mathcal{R}}(\bar{\omega}, u) = \sigma^{ESS}(\bar{\omega}, u)$.

Proof: Suppose first that $\bar{\omega} \in \mathbb{R}^L_{++}$. For each $i \in \mathcal{I}$, construct a sequence $(u_i^n)_{n\in\mathbb{N}}$ of functions in $\mathcal{V}(\bar{\omega}) \cap \mathcal{U}$ that converges uniformly to u_i, as in Step 3, and let $u^n = (u_1^n, ..., u_I^n)$. We have:

$$\sigma_i^{\mathcal{R}}(\bar{\omega}, u) = \lim_{n\to\infty} \sigma_i^{\mathcal{R}}(\bar{\omega}, u^n) = \lim_{n\to\infty} \frac{s(\bar{\omega}, u^n)}{I} = \frac{s(\bar{\omega}, u)}{I},$$

for each $i \in \mathcal{I}$, where the first equality follows from part (b) of CONT, the second equality follows from step 2, and the third equality follows from the fact that $\mathcal{R}^{ESS}$ satisfies part (b) of CONT.

Suppose finally that $\bar{\omega} \in \mathbb{R}^L_+$. We can construct a sequence $(\omega^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^L_{++}$ that converges to $\bar{\omega}$. We have:

$$\sigma_i^{\mathcal{R}}(\bar{\omega}, u) = \lim_{n\to\infty} \sigma_i^{\mathcal{R}}(\omega^n, u) = \lim_{n\to\infty} \frac{s(\omega^n, u)}{I} = \frac{s(\bar{\omega}, u)}{I},$$

for each $i \in \mathcal{I}$, where the first equality follows from part (a) of CONT, the second equality follows from the previous paragraph, and the third equality follows from the fact that $\mathcal{R}^{ESS}$ satisfies part (a) of CONT. ■

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