Online Appendix for "The Recommendation Principle in Information Design"

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To capture the fact that the receiver's beliefs may depend on her type, let us now denote as μ a posterior function $\mu : \Theta \to \Delta(\Omega)$ telling us what the posterior belief is for every type of the receiver. We similarly redefine B to be a set of such posterior functions, i.e. $B \subseteq \Delta(\Omega)^{\Theta}$. We assume B to be convex, i.e. that $\mu, \mu' \in B$ implies $\lambda\mu + (1-\lambda)\mu' \in B$ for all $\lambda \in [0, 1]$.

To be general, we allow the experiment to condition on the type of the receiver to some extent. Let $\mathcal{P}$ be a partition of Θ with typical element $\hat{\Theta}$. We now define an experiment $\pi = (S, \sigma)$, where $\sigma = (\sigma_{\hat{\Theta}})_{\hat{\Theta}\in\mathcal{P}}$, $\sigma_{\hat{\Theta}} : \Omega \to \Delta(S)$. If $\hat{\Theta}$ is the coarsest partition of Θ, the communication strategy σ does not depend in any way on the type of the receiver. If we instead assume $\hat{\Theta}$ is the finest partition of Θ, this means the sender can condition the message sent on both the state and the private type of the receiver.

First of all, we need to characterize the class of distributions over posterior functions that each possible experiment π can generate. As the prior may depend on the type of the receiver, these distributions will be, in a sense, "subjective" and depend on θ. We denote a distribution over posterior functions as a collection $G = (G_\theta)_{\theta\in\Theta}$, where $G_\theta \in \Delta(B)$ for all $\theta \in \Theta$. We slightly abuse notation by denoting $\cup_{\theta\in\Theta}\text{supp}(G_\theta)$ as $\text{supp}(G)$.

Denote as $\sigma_{\hat{\Theta}}(s|\theta) = \sum_{\omega\in\Omega} \sigma_{\hat{\Theta}}(s|\omega)\mu_0(\theta)[\omega]$ the probability that type $\theta \in \hat{\Theta}$ associates to Receiving message $s \in S$ according to communication strategy $\sigma_{\hat{\Theta}}$. Let $\mathcal{G}(\mathcal{P})$ denote the set of distributions over posterior functions G that can be induced by an experiment $\pi = (S, \sigma)$, i.e.:

$$\mathcal{G}(\mathcal{P}) = \{G \in \Delta(B)^{\Theta} : \exists\, \pi \text{ s.t.} \sum_{s\in S} \mathbf{1}_{\{\mu^{\pi}(s,\cdot)=\mu\}}\sigma_{\hat{\Theta}}(s|\theta) = G_\theta(\mu) \text{ for all } \mu \in B \text{ and } \theta \in \hat{\Theta}\}$$

Theorem 1. *$G \in \mathcal{G}(\mathcal{P})$ if and only if:*

1. *$\mu_0(\theta) = \sum_{\mu \in supp(G)} G_\theta(\mu)\mu(\theta)$ for all $\theta \in \Theta$*
2. *$\frac{\mu(\theta)[\omega]G_\theta(\mu)}{\mu_0(\theta)[\omega]} = L_{\mu,\hat{\Theta}}(\omega)$ for all $\theta \in \hat{\Theta}$ and $\hat{\Theta} \in \mathcal{P}$, $\omega \in supp(\mu_0(\theta))$ and $\mu \in supp(G)$.*

For any $G = (G_\theta)_{\theta \in \Theta}$ and $\bar{a} \in A^\Theta$, let $B_{\bar{a}}^G$ be the set of posteriors for which action plan $\bar{a}$ is optimal for the receiver:

$$B_{\bar{a}}^G = \{\mu \in B : \bar{a}(\theta) \in \arg\max_{a' \in A} v(G_\theta, \mu(\theta), a', \theta) \text{ for all } \theta \in \Theta\}$$

We then denote as $\lambda_G : \text{supp}(G) \to \Delta(A^\theta)$ any function such that $\bar{a} \in \text{supp}(\lambda_G(\mu))$ implies $\mu \in B_{\bar{a}}^G$. The average posterior inducing $\bar{a}$ (according to λ_G) is defined as:

$$\bar{\mu}_{\bar{a}}^{\lambda_G}(\theta) = \sum_{\mu \in \text{supp}(G)} \left(\frac{\lambda_G(\mu)[\bar{a}]G_\theta(\mu)}{\sum_{\mu \in \text{supp}(G_\theta)} \lambda_G(\mu)[\bar{a}]G_\theta(\mu)} \right) \mu(\theta)$$

The subjective distribution $H_\theta^{\lambda_G}$ induced by the direct mechanism is:

$$H_\theta^{\lambda_G}(\mu) = \sum_{\bar{a} \in A^\Theta} \mathbf{1}_{\{\mu = \bar{\mu}'^{\lambda_G}_{\bar{a},\theta}\}} \left(\sum_{\mu' \in \text{supp}(G)} \lambda_G(\mu')[\bar{a}]G_\theta(\mu') \right)$$

We then say that a function v satisfies *convex posterior pooling* (CPP) whenever for all $G \in \mathcal{G}$, and $\bar{a}$, $a' \in A$, θ and λ_G:

$$v(H_\theta^{\lambda_G}, \bar{\mu}_{\bar{a}}^{\lambda_G}(\theta), \bar{a}(\theta), \theta) \geq v(H_\theta^{\lambda_G}, \bar{\mu}_{\bar{a}}^{\lambda_G}(\theta), a', \theta)$$

Theorem 2. *Suppose the receiver's optimal action depends on her posterior, her type, and the posterior distributions generated by the experiment. The recommendation principle holds if and only if v satisfies CPP.*

Proofs

Proof of Theorem 1. Suppose $G \in \mathcal{G}(\mathcal{P})$. Therefore, there exists an experiment $\pi = (S, \sigma)$ such that for all $\theta \in \Theta$:

$$\sum_{s \in S} \mathbf{1}_{\{\mu^{\pi}(s,\cdot)=\mu\}} \sigma_{\hat{\Theta}}(s|\theta) = G_{\theta}(\mu)$$

Where $\eta^{\pi}(\mu)$ denotes the set of all $s \in S$ such that $\mu^{\pi}(s, \cdot) = \mu$.

The first item follows from properties of Bayesian updating, as for all $\omega \in \Omega$, $\theta \in \hat{\Theta}$, and $\hat{\Theta} \in \mathcal{P}$:

$$\begin{aligned}
\sum_{\mu \in \text{supp}(G_{\theta})} G_{\theta}(\mu)\mu(\theta)[\omega] &= \sum_{\mu \in B} G_{\theta}(\mu)\mu(\theta)[\omega] \\
&= \sum_{\mu \in B} \left(\sum_{s \in \eta^{\pi}(\mu)} \sigma_{\hat{\Theta}}(s|\theta) \right) \mu(\theta)[\omega] \\
&= \sum_{\mu \in B} \mu_0(\theta)[\omega] \left(\sum_{s \in \eta^{\pi}(\mu)} \sigma_{\hat{\Theta}}(s|\omega) \right) \\
&= \mu_0(\theta)[\omega] \left(\sum_{\mu \in B} \sum_{s \in \eta^{\pi}(\mu)} \sigma_{\hat{\Theta}}(s|\omega) \right) \\
&= \mu_0(\theta)[\omega] \left(\sum_{s \in S} \sigma_{\hat{\Theta}}(s|\omega) \right) \\
&= \mu_0(\theta)[\omega]
\end{aligned}$$

Where the second equality follows from $G \in \mathcal{G}(\mathcal{P})$, the third equality follows from the definition of a Bayesian posterior, the fourth equality follows from the fact that $S = \cup_{\mu \in B} \eta^{\pi}(\mu)$, and the last equality follows from the definition of σ.

Suppose now $\omega \in \text{supp}(\mu_0(\theta))$ for $\theta \in \hat{\Theta}$ and $\hat{\Theta} \in \mathcal{P}$. Notice that, for all $s \in \eta^{\pi}(\mu)$, $\omega \in \Omega$, $\theta \in \hat{\Theta}$, and $\hat{\Theta} \in \mathcal{P}$:

$$\mu(\theta)[\omega] = \mu^{\pi}(s, \theta)[\omega] = \frac{\sigma_{\hat{\Theta}}(s|\omega)}{\sigma_{\hat{\Theta}}(s|\theta)} \mu_0(\theta)[\omega]$$

It follows that, for all $\omega \in \Omega$, $\theta \in \hat{\Theta}$, $\hat{\Theta} \in \mathcal{P}$, and $\mu \in \text{supp}(G_\theta)$:

$$\begin{aligned}
\mu(\theta)[\omega] &= \sum_{s \in \eta^\pi(\mu)} \frac{\sigma_{\hat{\Theta}}(s|\theta)}{\sum_{s \in \eta^\pi(\mu)} \sigma_{\hat{\Theta}}(s|\theta)} \left(\frac{\sigma_{\hat{\Theta}}(s|\omega)}{\sigma_{\hat{\Theta}}(s|\theta)} \mu_0(\theta)[\omega] \right) \\
&= \sum_{s \in \eta^\pi(\mu)} \frac{\sigma_{\hat{\Theta}}(s|\theta)}{G_\theta(\mu)} \left(\frac{\sigma_{\hat{\Theta}}(s|\omega)}{\sigma_{\hat{\Theta}}(s|\theta)} \mu_0(\theta)[\omega] \right) \\
&= \frac{\mu_0(\theta)[\omega]}{G_\theta(\mu)} \left(\sum_{s \in \eta^\pi(\mu)} \sigma_{\hat{\Theta}}(s|\omega) \right)
\end{aligned}$$

Then, for all $\omega \in \text{supp}(\mu_0(\theta))$, $\theta \in \hat{\Theta}$, $\hat{\Theta} \in \mathcal{P}$, and $\mu \in \text{supp}(G_\theta)$ we can set $L_{\hat{\Theta}}$ such that:

$$L_{\mu,\hat{\Theta}}(\omega) = \sum_{s \in \eta^\pi(\mu)} \sigma_{\hat{\Theta}}(s|\omega) = \frac{G_\theta(\mu)\mu(\theta)[\omega]}{\mu_0(\theta)[\omega]}$$

Concluding the first part of the proof.

As for the converse statement, define an experiment $\pi = (S, \sigma)$ such that $S = B$ for all $\hat{\Theta} \in \mathcal{P}$ and $\omega \in \text{supp}(\mu_0(\theta))$ for at least one $\theta \in \hat{\Theta}$:

$$\sigma_{\hat{\Theta}}(\mu|\omega) = L_{\mu,\hat{\Theta}}(\omega)$$

If $\omega \notin \text{supp}(\mu_0(\theta))$ for all $\theta \in \hat{\Theta}$, let $\sigma_{\hat{\Theta}}(\mu|\omega) \in \Delta(B)$ be arbitrary. Notice that if $\omega \in \text{supp}(\mu_0(\theta))$ for at least one $\theta \in \hat{\Theta}$, then $\sigma_{\hat{\Theta}} \in \Delta(B)$ as:

$$\sum_{\mu \in B} \sigma_{\hat{\Theta}}(\mu|\omega) = \sum_{\mu \in B} L_{\mu,\hat{\Theta}}(\omega) = \sum_{\mu \in B} \left(\frac{G_\theta(\mu)\mu(\theta)[\omega]}{\mu_0(\theta)[\omega]} \right) = \frac{\mu_0(\theta)[\omega]}{\mu_0(\theta)[\omega]} = 1$$

Moreover, the posterior after observing message $s = \mu$ is, for all $\theta \in \Theta$ and $\omega \in \text{supp}(\mu_0(\theta))$:

$$\begin{aligned}
\mu^\pi(\mu, \theta)[\omega] &= \frac{L_{\mu,\hat{\Theta}}(\omega)}{\sum_{\omega' \in \Omega} L_{\mu,\hat{\Theta}}(\omega')\mu_0(\theta)[\omega']} \mu_0(\theta)[\omega] \\
&= \frac{L_{\mu,\hat{\Theta}}(\omega)}{\sum_{\omega' \in \Omega} G_\theta(\mu)\mu(\theta)[\omega']} \mu_0(\theta)[\omega] \\
&= \frac{G_\theta(\mu)\mu(\theta)[\omega']}{G_\theta(\mu)} \\
&= \mu(\theta)[\omega]
\end{aligned}$$

It follows that π induces the posterior distribution

$$\begin{aligned}\sum_{s\in\eta^{\pi}(\mu)}\sigma_{\hat{\Theta}}(s|\theta) &= \sum_{\omega\in\Omega}\sigma_{\hat{\Theta}}(\mu|\omega)\mu_0(\theta)[\omega] \\ &= \sum_{\omega\in\Omega}\left(\frac{G_\theta(\mu)\mu(\theta)[\omega]}{\mu_0(\theta)[\omega]}\right)\mu_0(\theta)[\omega] \\ &= \sum_{\omega\in\Omega}G_\theta(\mu)\mu(\theta)[\omega] \\ &= G_\theta(\mu)\end{aligned}$$

Completing the proof. □

Proof of Theorem 2. Suppose v satisfies CPP and that π implements the distribution $d^\pi : \Omega \times \Theta \to \Delta(A)$. Denote as $G = (G_\theta)_{\theta\in\Theta}$ the collection of distributions over posterior functions induced by π.

As π implements d^π, there exists $\alpha^\pi : S \times \Theta \to A$ that maximizes v for each message s and type θ of the receiver when the distribution of posteriors is G_θ, and that is such that for all $s \in S$ and $\theta \in \Theta$:

$$d(\omega,\theta)[a] = \sum_{s\in S}\mathbf{1}_{\{\alpha^\pi(s,\theta)=a\}}(s)\sigma_{\hat{\Theta}}(s|\omega)$$

For any $\bar{a} : \Theta \to A$, let $\eta(\bar{a})$ denote the set of all messages s that induce the receiver to play according to $\bar{a}$. Let $\eta(\bar{a}) = \{s \in S : \alpha^\pi(s,\cdot) = \bar{a}\}$ and consider now experiment $\pi^* = (A^\Theta, \sigma^{\pi^*})$, where for all $\omega \in \Omega$ and $\bar{a} \in A^\Theta$:

$$\sigma^{\pi^*}_{\hat{\Theta}}(\bar{a}|\omega) = \sum_{s\in\eta(\bar{a})}\sigma_{\hat{\Theta}}(s|\omega)$$

Notice that for all $\theta \in \Theta$ and $\bar{a}$ such that $\eta(\bar{a}) \neq \emptyset$:

$$\mu^{\pi^*}(\bar{a},\theta) = \frac{\sum_{s\in\eta(\bar{a})}\sigma_{\hat{\Theta}}(s|\theta)\mu^\pi(s,\theta)}{\sum_{s\in\eta(\bar{a})}\sigma_{\hat{\Theta}}(s|\theta)}$$

Moreover, π^* induces distribution over posteriors $H_\theta^{\lambda_G}$ such that, for all $\theta \in \hat{\Theta}$ and $\tilde{\mu} \in B$:

$$\begin{aligned}H_\theta^{\lambda_G}(\tilde{\mu}) &= \sum_{\bar{a}\in A^\theta} \mathbf{1}_{\{\mu^{\pi^*}(\bar{a},\cdot)=\tilde{\mu}\}}\sigma_{\hat{\Theta}}^{\pi^*}(\bar{a}) \\ &= \sum_{\bar{a}\in A^\theta} \mathbf{1}_{\{\mu^{\pi^*}(\bar{a},\cdot)=\tilde{\mu}\}} \left(\sum_{s\in\eta(\bar{a})} \sigma_{\hat{\Theta}}(s|\theta)\right) \\ &= \sum_{\bar{a}\in A^\theta} \mathbf{1}_{\{\mu^{\pi^*}(\bar{a},\cdot)=\tilde{\mu}\}} \left(\sum_{\mu\in\text{supp}(G)} \lambda_G(\mu)[\bar{a}]G_\theta(\mu)\right)\end{aligned}$$

where for all $\mu \in \text{supp}(G)$ and $\bar{a} \in A^\Theta$:

$$\lambda_G(\mu)[\bar{a}] = \frac{1}{G_\theta(\mu)} \sum_{s\in\eta(\bar{a})} \mathbf{1}_{\{\mu^\pi(s,\cdot)=\tilde{\mu}\}}\sigma_{\hat{\Theta}}(s|\theta).$$

We now show that the right-hand side of this equality does not depend on θ whenever it is non-zero. If μ and θ are such that $\mu(\theta)[\omega] > 0$ for at least one $\omega \in \Omega$:

$$\begin{aligned}\frac{1}{G_\theta(\mu)} \sum_{s\in\eta(\bar{a})} \mathbf{1}_{\{\mu^\pi(s,\cdot)=\mu\}}\sigma_{\hat{\Theta}}(s|\theta) &= \frac{\sum_{s\in\eta(\bar{a})} \mathbf{1}_{\{\mu^\pi(s,\cdot)=\mu\}}\sigma_{\hat{\Theta}}(s|\theta)}{G_\theta(\mu)} \\ &= \frac{\sum_{s\in\eta(\bar{a})} \mathbf{1}_{\{\mu^\pi(s,\cdot)=\mu\}}\sigma_{\hat{\Theta}}(s|\theta)}{\sum_{s\in S} \mathbf{1}_{\{\mu^\pi(s,\cdot)=\mu\}}\sigma_{\hat{\Theta}}(s|\theta)} \\ &= \frac{\sum_{s\in\eta(\bar{a})} \mathbf{1}_{\{\mu^\pi(s,\cdot)=\mu\}}\left(\sigma_{\hat{\Theta}}(s|\omega)\frac{\mu_0(\theta)[\omega]}{\mu(\theta)[\omega]}\right)}{\sum_{s\in S} \mathbf{1}_{\{\mu^\pi(s,\cdot)=\mu\}}\left(\sigma_{\hat{\Theta}}(s|\omega)\frac{\mu_0(\theta)[\omega]}{\mu(\theta)[\omega]}\right)} \\ &= \frac{\sum_{s\in\eta(\bar{a})} \mathbf{1}_{\{\mu^\pi(s,\cdot)=\mu\}}\sigma_{\hat{\Theta}}(s|\omega)}{\sum_{s\in S} \mathbf{1}_{\{\mu^\pi(s,\cdot)=\mu\}}\sigma_{\hat{\Theta}}(s|\omega)}\end{aligned}$$

where the third equality follows from Bayes' rule. Otherwise, for all $\omega \in \Omega$ such that $\mu(\theta)[\omega] = 0$ for all $\omega \in \Omega$, we have either $\sigma_{\hat{\Theta}}(s|\omega) = 0$ or $\mu_0(\theta)[\omega] = 0$, so that $\sigma(s|\theta) = 0$.

We can then rewrite, for all $\bar{a}$ such that $\eta(\bar{a}) \neq \emptyset$ and $\theta \in \Theta$:

$$
\begin{aligned}
\mu^{\pi^*}(\bar{a},\theta) &= \frac{\sum_{s\in\eta(\bar{a})} \sigma_{\hat{\Theta}}(s|\theta)\mu^{\pi}(s,\theta)}{\sum_{s\in\eta(\bar{a})} \sigma_{\hat{\Theta}}(s|\theta)} \\
&= \frac{\sum_{s\in\eta(\bar{a})} \left(\sum_{\mu\in\operatorname{supp}(G)} \mathbf{1}_{\{\mu^{\pi}(s)=\mu\}}\sigma_{\hat{\Theta}}(s|\theta)\mu(\theta)\right)}{\sum_{s\in\eta(\bar{a})} \left(\sum_{\mu\in\operatorname{supp}(G)} \mathbf{1}_{\{\mu^{\pi}(s)=\mu\}}\sigma_{\hat{\Theta}}(s|\theta)\right)} \\
&= \frac{\sum_{\mu\in\operatorname{supp}(G)} \left(\sum_{s\in\eta(\bar{a})} \mathbf{1}_{\{\mu^{\pi}(s)=\mu\}}\sigma_{\hat{\Theta}}(s|\theta)\right)\mu(\theta)}{\sum_{\mu\in\operatorname{supp}(G)} \left(\sum_{s\in\eta(\bar{a})} \mathbf{1}_{\{\mu^{\pi}(s)=\mu\}}\sigma_{\hat{\Theta}}(s|\theta)\right)} \\
&= \frac{\sum_{\mu\in\operatorname{supp}(G)} \lambda_G(\mu)[\bar{a}]G(\mu)\mu(\theta)}{\sum_{\mu\in\operatorname{supp}(G)} \lambda_G(\mu)[\bar{a}]G(\mu)} \\
&= \bar{\mu}_{\bar{a}}^{\lambda_G}(\theta)
\end{aligned}
$$

where the fourth equality follows from the definition of λ_G. As α^{π} maximizes v for all types θ, we have that for all $\theta \in \Theta$, $s \in \eta(\bar{a})$, and $a' \in A$:

$$v(G,\mu^{\pi}(s,\theta),\alpha^{\pi}(s,\theta),\theta) \geq v(G,\mu^{\pi}(s,\theta),a',\theta)$$

so that $\mu^{\pi}(s) \in B_{\bar{a}}^{G}$ for all $s \in \eta(\bar{a})$. Moreover, as $B_{\bar{a}} \subseteq \operatorname{supp}(G)$, for all $\mu \in B_{\bar{a}}$, there exists $s \in S$ such that $\mu^{\pi}(s) = \mu$. By CPP, this implies that for all $\theta \in \Theta$ and $a' \in A$:

$$v(H_{\theta}^{\lambda_G},\mu^{\pi^*}(\bar{a},\theta),\bar{a}(\theta),\theta) = v(H_{\theta}^{\lambda_G},\mu^{\pi^*}(\bar{a},\theta),\alpha^{\pi}(s,\theta),\theta) \geq v(H_{\theta}^{\lambda_G},\mu^{\pi^*}(\bar{a},\theta),a',\theta).$$

Consider $\alpha^{\pi^*} : A^{\Theta} \times \Theta \to A$ is such that $\alpha^{\pi^*}(\bar{a},\theta) = \bar{a}(\theta)$, and for all $\theta \in \Theta$. As α^{π^*} still induces action distribution d^{π} given communication strategy $\sigma_{\hat{\Theta}}^{\pi^*}$, and the obedience constraint is satisfied, we conclude the proof.

As for the converse statement, suppose the recommendation principle holds. Take any G, $\lambda_G : \operatorname{supp}(G) \to \Delta(A^{\Theta})$ and $\bar{a} \in A^{\Theta}$, and prior $\mu_0(\theta) = \sum_{\mu\in\operatorname{supp}(G_\theta)} \mu(\theta)G_\theta(\mu)$ for all $\theta \in \Theta$. Notice that $\mu_0 \in B$ as B is convex and $\operatorname{supp}(G_\theta) \subseteq B$ for all $\theta \in \Theta$. Consider now experiment $\pi = (\operatorname{supp}(G) \times A^{\Theta}, \sigma)$ such that for all $(\mu,\bar{a}) \in (\operatorname{supp}(G) \times A^{\Theta})$ and $\omega \in \Omega$ with $\mu_0(\theta)[\omega] > 0$ for at least one $\theta \in \Theta$:

$$\sigma_{\hat{\Theta}}((\mu,\bar{a})|\omega) = \lambda(\mu)[\bar{a}]L_{\hat{\Theta}}(\omega)$$

For ω with $\mu_0(\theta)[\omega] = 0$ for all $\theta \in \Theta$, let instead $\sigma_{\hat{\Theta}}(\cdot|\omega)$ be any distribution over messages $\Delta(S)$.

We first show that $\sigma_{\hat{\Theta}}(\cdot|\omega) \in \Delta(S)$ for any ω such that $\mu_0(\theta)[\omega] > 0$ for some $\theta \in \Theta$. Notice that, by substituting for $L_{\hat{\Theta}}$:

$$\begin{aligned}\sum_{\mu \in \text{supp}(G)} \sum_{\bar{a} \in A^{\Theta}} \sigma_{\hat{\Theta}}((\mu, \bar{a})|\omega) &= \sum_{\mu \in \text{supp}(G)} \sum_{\bar{a} \in A^{\Theta}} \left(\lambda(\mu)[\bar{a}] \frac{\mu(\theta)[\omega] G_\theta(\mu)}{\mu_0(\theta)[\omega]} \right) \\ &= \sum_{\mu \in \text{supp}(G)} \left(\sum_{\bar{a} \in A^{\Theta}} \lambda(\mu)[\bar{a}] \right) \frac{\mu(\theta)[\omega] G_\theta(\mu)}{\mu_0(\theta)[\omega]} \\ &= \sum_{\mu \in \text{supp}(G)} \frac{\mu(\theta)[\omega] G_\theta(\mu)}{\mu_0(\theta)[\omega]} \\ &= \frac{1}{\mu_0(\theta)[\omega]} \left(\sum_{\mu \in \text{supp}(G_\theta)} G_\theta(\mu) \mu(\theta)[\omega] \right) \\ &= \frac{1}{\mu_0(\theta)[\omega]} \mu_0(\theta)[\omega] \\ &= 1,\end{aligned}$$

where the fourth equality follows from the fact that $G_\theta(\mu) = 0$ for $\mu \in \text{supp}(G)/\text{supp}(G_\theta)$ and the first statement of Theorem 1.A similar argument delivers $\sigma_{\hat{\Theta}}(\cdot|\omega) \in \Delta(S)$ for $\omega \in \Omega$ with $\mu_0(\theta)[\omega] = 0$ for all $\theta \in \Theta$.

Notice moreover that, for all $\theta \in \Theta$, $\mu \in \text{supp}(G)$, and $\bar{a} \in A^{\Theta}$:

$$\sigma_{\hat{\Theta}}((\mu, \bar{a})|\theta) = \sum_{\omega \in \Omega} \sigma_{\hat{\Theta}}((\mu, \bar{a})|\omega) \mu_0(\theta)[\omega] = \sum_{\omega \in \Omega} \mu(\theta)[\omega] \lambda(\mu)[\bar{a}] G_\theta(\mu) = \lambda(\mu)[\bar{a}] G_\theta(\mu),$$

where the third equality follows from the definition of $\sigma_{\hat{\theta}}$ and the fact that $\mu(\theta)[\omega] = 0$ whenever $\mu(\theta)[\omega] = 0$.

We now observe that the Bayesian posterior upon observing message $(\mu, \bar{a})$ with positive probability coincides with μ, as for all $\theta \in \Theta$, $\omega \in \Omega$, $\mu \in \text{supp}(G_\theta)$ and $\bar{a} \in \lambda(\mu)[\bar{a}]$, we have by Bayes' rule:

$$\mu^{\pi}((\mu, \bar{a}), \theta)[\omega] = \frac{\lambda(\mu)[\bar{a}] G_\theta(\mu) \mu(\theta)[\omega]}{\lambda(\mu)[\bar{a}] G_\theta(\mu)} = \mu(\theta)[\omega].$$

Therefore, for each $\theta \in \Theta$, π induces distribution over posteriors G_θ as:

$$\sum_{\bar{a} \in A^\Theta} \sigma_{\hat{\Theta}}(\mu, \bar{a}|\theta) = \sum_{\bar{a} \in A^\Theta} \lambda(\mu)[\bar{a}] G_\theta(\mu) = G_\theta(\mu).$$

Consider now α^π such that $\alpha^\pi((\mu, \bar{a}), \theta) = \bar{a}(\theta)$. As the recommendation principle holds, we can implement the action distribution d^π induced by π and α^π via a direct recommendation experiment π^*, pooling together all messages in S that induce the same action:

$$\sigma_{\hat{\Theta}}^{\pi^*}(\bar{a}|\omega) = \sum_{\mu \in \text{supp}(G)} \sigma_{\hat{\Theta}}((\mu, \bar{a})|\omega).$$

This implies:

$$\begin{aligned}
\sigma_{\hat{\Theta}}^{\pi^*}(\bar{a}|\theta) &= \sum_{\mu \in \text{supp}(G)} \sigma_{\hat{\Theta}}((\mu, \bar{a})|\theta) \\
&= \sum_{\mu \in \text{supp}(G)} \left(\sum_{\omega \in \Omega} \sigma_{\hat{\Theta}}((\mu, \bar{a})|\omega) \mu_0(\theta)[\omega] \right) \\
&= \sum_{\mu \in \text{supp}(G)} \lambda_G(\mu)[\bar{a}] G_\theta(\mu).
\end{aligned}$$

Therefore, π^* induces the posterior distribution

$$\begin{aligned}
H_\theta^{\lambda_G}(\mu) &= \sum_{\bar{a} \in A^\theta} \mathbf{1}_{\{\mu^{\pi^*}(\bar{a}, \theta) = \mu\}} \sigma_{\hat{\Theta}}^{\pi^*}(\bar{a}|\theta) \\
&= \sum_{\bar{a} \in A^\theta} \mathbf{1}_{\{\mu^{\pi^*}(\bar{a}, \cdot) = \mu\}} \left(\sum_{\mu' \in \text{supp}(G)} \lambda_G(\mu')[\bar{a}] G_\theta(\mu') \right) \\
&= \sum_{\bar{a} \in A^\theta} \mathbf{1}_{\{\bar{\mu}_{\bar{a}}^{\lambda_G} = \mu\}} \left(\sum_{\mu' \in \text{supp}(G)} \lambda_G(\mu')[\bar{a}] G_\theta(\mu') \right),
\end{aligned}$$

where the last equality follows from the fact that the posterior that the receiver holds after observing message $\bar{a}$ with positive probability is, for all $\omega \in \Omega$ and $\theta \in \Theta$:

$$\begin{aligned}
\mu^{\pi^*}(\bar{a},\theta)[\omega] &= \frac{\sigma_{\hat{\Theta}}^{\pi^*}(\bar{a}|\omega)}{\sigma_{\hat{\Theta}}^{\pi^*}(\bar{a})}\mu_0(\theta)[\omega] \\
&= \frac{\sum_{\mu\in\mathrm{supp}(G)}\sigma_{\hat{\Theta}}((\mu,\bar{a})|\omega)\mu_0(\theta)[\omega]}{\sum_{\mu\in\mathrm{supp}(G)}\lambda_G(\mu)[\bar{a}]G(\mu)} \\
&= \frac{\sum_{\mu\in\mathrm{supp}(G)}\lambda_G(\mu)[\bar{a}]G(\mu)\mu(\theta)[\omega]}{\sum_{\mu\in\mathrm{supp}(G)}\lambda_G(\mu)[\bar{a}]G(\mu)} \\
&= \bar{\mu}_{\bar{a}}^{\lambda_G}(\theta)[\omega],
\end{aligned}$$

where the fourth equality follows again from the fact that $\sigma_{\hat{\Theta}}((\mu,\bar{a})|\omega)\mu_0(\theta)[\omega] = \mu(\theta)[\omega]\sigma_{\hat{\Theta}}((\mu,\bar{a})) = \mu(\theta)[\omega]\lambda_G(\mu)[\bar{a}]G(\mu)$ for s such that $\mu^{\pi}(s,\cdot)=\mu$ and that $\sum_{s\in s}\mathbf{1}_{\{\mu^{\pi}(s,\cdot)=\mu\}}\sigma_{\hat{\Theta}}(s|\theta) = G_\theta(\mu)$ for all $\mu \in \mathrm{supp}(G_\theta)$.

As α^{π^*} is such that $\alpha^{\pi^*}(\bar{a},\cdot)=\bar{a}$, we then have that for all $a' \in A$ and $\theta \in \Theta$:

$$v(H_\theta^{\lambda_G},\bar{\mu}_{\bar{a}}^{\lambda_G}(\theta),\bar{a}(\theta),\theta) = v(H_\theta^{\lambda_G},\mu^{\pi^*}(\bar{a},\theta),\bar{a}(\theta),\theta) \geq v(H_\theta^{\lambda_G},\mu^{\pi^*}(\bar{a},\theta),a',\theta) = v(H_\theta^{\lambda_G},\bar{\mu}_{\bar{a}}^{\lambda_G}(\theta),a',\theta).$$

This conclades the proof. $\square$