BEYOND NASH BARGAINING THEORY: THE NASH SET[1]

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and

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Running title: The Nash Set

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Abstract

We extend Nash's bargaining theory to non-convex and coalitional problems. This paper investigates the implications of Nash-like axioms for bilateral problems and the properties of consistency and converse consistency over multilateral settings. The result is a characterization of the Nash set of NTU games, defined as the solution concept where each pair of players is splitting the gains from trade at a point where the Nash product of their utilities, subject to efficiency, is critical. The intersection of the Nash set and the core is also characterized with the same axioms for the class of games where the core is non-empty. *Journal of Economic Literature* classification numbers: C71, C78.

1 Introduction

Applications of the Nash bargaining solution (Nash [22]) feature prominently in economic analysis. Nash first presented his solution for bilateral problems where the feasible possibilities are described by a convex set in the utility space. Subsequently, Harsanyi [7] extended it to multilateral convex problems where coalitions are powerless (the so-called pure bargaining problems). Harsanyi's starting point is a "consistency" condition, i.e., he argues that a bargaining solution should be "consistent" when ruling over bargaining problems with a different number of players. As is well known, though, problems that involve more than two players are substantially more complex than those of pure bargaining, since coalitions may often play a major role.

The theory should also aim to drop the convexity assumption, either because one may be interested in extending solution concepts beyond expected utility (see Rubinstein, Safra and Thomson [27], and Grant and Kajii [6]) or because, even within expected utility, randomizations may sometimes be inappropriate. See in this context Conley and Wilkie [3] and Zhou [34], who present single-valued extensions of the Nash solution to non-convex pure bargaining problems. Without convexity, it is difficult to maintain single-valuedness: closer to our work are Kaneko [13], and especially Maschler, Owen and Peleg [18] and Herrero [9], who allow set-valued solutions. Kaneko [13] selects the global maxima of the Nash product, and Maschler, Owen and Peleg [18] and Herrero [9] concentrate on the constrained critical points of the Nash product, i.e., local maxima and minima subject to efficiency.

Believed by many to lie quite apart from Nash's bargaining theory, the prekernel is a solution that was introduced by Davis and Maschler [4] for the class of coalitional games with transferable utility (TU games). It was usually explained as a solution where each player's surplus against any other was equal (see, for example, Maschler [17] and Osborne and Rubinstein [24]). Because player i's surplus is measured in i's utility scale and j's surplus in j's scale, it appeared that the solution were based on interpersonal comparisons of utility, thereby rendering it useless to positive applications of the theory. As a consequence, the extension of the prekernel to non-transferable (NTU) games was perceived as an uninteresting problem.[1] The TU prekernel has been misunderstood and unfairly criticized on these grounds for too long (see again references 17 and 24 on this point). Indeed, using the non-cooperative approach, Serrano [29] dispells these criticisms.

This paper's main actor is the Nash set.[2] The Nash set was first uncovered for NTU games in Serrano's [29] non-cooperative analysis. It is the set of payoffs where each pair of players splits the available surplus at a point where their Nash product of utility differences (with respect to the "threat point") is critical when the optimum is subject to efficiency. The induced bilateral problems (when keeping the utilities of the remaining agents fixed) have a "threat point" where each player threatens the other in the pair with forming a coalition that does not contain her. In particular, (1) for convex pure bargaining problems, these conditions yield precisely one payoff: the Nash solution; and (2) for TU games the same conditions yield a set-valued solution: the prekernel.

In this paper we follow an axiomatic approach. In the spirit of Harsanyi [7], we investigate the implications in the model of NTU games of using Nash-like axioms for bilateral problems in conjunction with a property of internal consistency and its converse. These two are the key axioms used by Peleg [26] in order to characterize the prekernel of TU games.[3]

Our main theorem is a pure axiomatization, in the sense that it is not restricted to a class where a certain solution concept is non-vacuous. We regard the existence problem and the axiomatic characterization as two completely separate issues. Theorem 1 says that, for the class of smooth NTU games, the Nash set is the only solution that satisfies consistency, converse consistency, and a set of five axioms of the Nash type imposed on the subclass of two-person smooth problems: non-emptiness, scale invariance, equal treatment if the game is TU, Pareto efficiency, and local independence.[4]

Non-emptiness, Pareto efficiency and scale invariance are basic axioms and are already used by Nash [22]. The equal treatment property for TU games is a weaker requirement than Nash's original symmetry axiom since it only applies when utility is transferable.

On the other hand, local independence is stronger than Nash's "independence of irrelevant alternatives" (IIA) axiom.[5] The original formulation of the condition says that "if at a commodity allocation all agents have a common marginal rate of substitution under preference profiles u and u', then the allocation should be chosen as a socially optimal outcome for u' whenever it is selected for u." The version in this paper expresses essentially the same concept in the payoff space. One justification of local independence is based on informational efficiency: in order to pin down the solution, local information (about the gradient of the Pareto frontier) suffices. If the problem looks the same locally, the solution will not change. Essentially, there is no "action at a distance" in the influence of the shape of the feasible set on the location of the solution. (See Nash [23], p. 138). A key property of the Nash solution is that, at it, the utility elasticity of the Pareto frontier of the feasible set is 1 for every pair of agents: denoting by du_i/du_j the slope of the Pareto frontier and (to simplify) normalizing the disagreement point to 0, we have that at the Nash solution $du_i/du_j = u_i/u_j$. Thus, the Nash solution picks the point where, for every $i \neq j$, a 1% of agent i's utility is traded off efficiently for a 1% of agent j's utility. This *local* property is the way the Nash solution captures a concept of "fairness," and contrasts openly with bargaining solutions based on *global* factors in the bargaining problem (such as bargainers' aspiration levels, like in Kalai and Smorodinsky [12]). Local independence builds in these local considerations and serves to extend naturally Nash's bargaining theory.

The axiom of consistency, first found in Harsanyi's [7] early work, was crucial in characterizing the Nash solution without the IIA axiom (Lensberg [16]).[6] Interestingly, we will need both (consistency and the IIA-like axiom of local independence) to extend the result to non-convex and coalitional domains. In order to understand consistency, one should think of a solution as a "court" ruling over problems that, in principle, may involve a variable number of interested parties. Consider a pure bargaining problem. If the solution is consistent, there is no room for profitable appeal before the court. That is, suppose the original problem involves three parties and the court solves this problem by recommending the utility vector (u_1, u_2, u_3). Suppose further that agent 3 is happy with this arrangement, but the other two would like to appeal the court's decision: "we know that agent 3 accepts your ruling, but we do not. We wish to resubmit the bilateral problem for your new consideration." If the court is consistent, it will respond by saying: "given that agent 3 was awarded u_3, my recommendation for the bilateral problem involving agents 1 and 2 is (u_1, u_2)." If coalitions have power, consistency captures the same idea, but in addition, it takes into account that those who appeal may interact with those who are happy and use their resources, as long as they are "bought off" at the awarded utilities in the multilateral problem. See also Krishna and Serrano [14] for a non-cooperative use of consistency.

We also show that our seven axioms are logically independent, so that the characterization provided is tight. We think that it is good news that we are able to find a solution that satisfies (and that it is the only one to do so) all these important seven principles. The axioms are also conceptually independent, as they express quite distinct *desiderata* from one another.

In Theorem 2, we show that for the class of games with non-empty cores, the same axioms as in our main theorem characterize the intersection of the core and the Nash set. This is related to Moldovanu's [19] partial axiomatization of this intersection for convex assignment problems.

The Nash set should be helpful in applications, because it is a natural generalization of Nash's bargaining theory to contexts where convexity is not assumed and coalitional interaction is a relevant consideration. The paper is organized as follows. Section 2 presents the model, while section 3 is devoted to the consistency properties of the Nash set. Section 4 contains our main result, as well as examples to show that the axioms are independent. Our result on the intersection of the core and the Nash set is the subject of Section 5. Section 6 presents examples that motivate the solution concept characterized in this paper and section 7 concludes by briefly comparing our work to the relevant literature.

2 The Model

Denote by $\mathbb{R}$ the set of the real numbers. If N is a non-empty finite set, denote by $|N|$ the cardinality of N, and by $\mathbb{R}^N$ the set of all functions from N to $\mathbb{R}$. We identify an element $x \in \mathbb{R}^N$ with an $|N|$-dimensional vector whose components are indexed by members of N; thus we write x_i for $x(i)$. If $x \in \mathbb{R}^N$ and $S \subseteq N$, we write x_S for the restriction of x to S, which is the element of $\mathbb{R}^S$ that associates x_i with each $i \in S$.

Let $S \subseteq N$ and $Y \subseteq \mathbb{R}^S$. A representation for Y is a function $g : \mathbb{R}^S \to \mathbb{R}$ such that

$$Y = \{x \in \mathbb{R}^S | \ \ g(x) \leq 0\}$$

and the interior of Y is the set

$$\text{Int}Y = \{x \in \mathbb{R}^S | \ \ g(x) < 0\}.$$

We also write

$$\partial Y = \{y \in Y | \ \ x_i > y_i \quad \forall i \in S \quad \text{implies} \quad x \notin Y\},$$

write $g_i(x)$ for the partial derivative of g at $x \in I\!R^S$ with respect to component $i \in S$, and $\nabla g(x)$ for the gradient vector of g at $x \in I\!R^S$.

The pair (N, V) is a coalitional game, or simply a game, if V is a correspondence that associates with every $S \subseteq N$ a subset $V(S) \subseteq I\!R^S$ such that

(1) $V(S)$ is non-empty and closed;

(2) For each $x_S \in I\!R^S$,

$$\partial V(S) \cap (\{x_S\} + I\!R^S_+)$$

and

$$\partial V(S) \cap (\{x_S\} - I\!R^S_+)$$

are bounded;

(3) If $(x_S, y_S) \in V(S) \times \partial V(S)$ and $x_S \geq y_S$, $x_S = y_S$.[7]

For example, a pure bargaining problem and an exchange economy where agents have strictly monotone preferences fit the above assumptions.

A player is a member of N, and a non-empty subset of N is a coalition. A payoff to player i is a point of $I\!R^{\{i\}}$, and a payoff profile on coalition S is a point of $I\!R^S$.

The game (N, V) is smooth if there is a differentiable representation g for $V(N)$ with positive gradients on $\partial V(N)$; namely for each $i \in N$, $g_i(x) > 0$ at any $x \in \partial V(N)$.

A class of games is rich if it contains all two-person smooth games.

A transferable utility game, or a TU game, is a smooth game (N, V) which is defined by a function v that associates with every coalition S a real number $v(S)$ such that

$$V(S) = \{x_S \in I\!R^S | \sum_{i \in S} x_i \leq v(S)\}.$$

We abuse the notation, and use (N, v) to denote the associated coalitional game.

Let Γ be a non-empty class of games. A solution on Γ is a relation σ which associates with every $(N, V) \in \Gamma$ a (possibly empty) subset $\sigma(N, V)$ of $I\!R^N$ such that $\sigma(N, V)$ is a subset of $V(N)$ for every $(N, V) \in \Gamma$.

Definition. Let $(\{i,j\},V)$ be a two-person smooth game. The *Nash set* of $(\{i,j\},V)$ is:

$$\mathcal{N}(\{i,j\},V) = \{x \in \partial V(\{i,j\}) | \quad g_i(x)(x_i - v_i) = g_j(x)(x_j - v_j)\},$$

where g is a representation for $V(\{i,j\})$, and

$$(v_i, v_j) = (\max V(\{i\}), \max V(\{j\})).$$

That is, the solution $\mathcal{N}$ consists of the critical points of the function $(z_i - v_i)(z_j - v_j)$ subject to $z \in \partial V(\{i,j\})$. The expression in the definition is just a reduced form of the first order conditions of these optimization programs.

Remark 2.1. The solution $\mathcal{N}$ reduces to the Nash bargaining solution on the class of two-person smooth games $(\{i,j\},V)$ such that $V(\{i,j\})$ is a convex set containing (v_i, v_j).

Definition. Let Γ be a non-empty class of games. Then a solution σ on Γ satisfies *non-emptiness* if $\sigma(N,V) \neq \emptyset$ for all $(N,V) \in \Gamma$, and satisfies *Pareto efficiency* if $\sigma(N,V) \subseteq \partial V(N)$ for each $(N,V) \in \Gamma$.

Remark 2.2. On the class of two-person smooth games, $\mathcal{N}$ satisfies *non-emptiness* and *Pareto efficiency*.

Let (N,v) be a TU game, and i,j be two distinct players in N. Then i and j are substitutes in (N,v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i,j\}$.

Definition. Let Γ be a class of games. A solution σ on Γ satisfies *equal treatment for TU games* if $x_i = x_j$ for each $x \in \sigma(N,v)$ whenever (N,v) is a TU game in Γ and i and j are substitutes in (N,v).

Remark 2.3. On the class of two-person TU games, $\mathcal{N}$ satisfies *equal treatment*.

Let (N,V) be a game, $\alpha \in I\!R^N_{++}$, and $\beta \in I\!R^N$. For each coalition S, we define the function $\lambda_S^{\alpha,\beta}$ from $I\!R^S$ to itself by

$$\lambda_S^{\alpha,\beta}(x_S) = (\alpha_i x_i + \beta_i)_{i \in S}$$

for each $x_S \in I\!R^S$. We then define $\lambda^{\alpha,\beta}(V)$ as the correspondence that associates with every coalition S a set

$$\lambda^{\alpha,\beta}(V)(S) = \{y \in I\!R^N : \quad \exists x_S \in V(S) | \quad y_S = \lambda_S^{\alpha,\beta}(x_S)\}.$$

That is, these two definitions simply describe positive affine transformations of the utility scales.

Definition. Let Γ be a class of games. A solution σ on Γ satisfies *scale invariance* if for each $(N, V) \in \Gamma$, for each $\alpha \in \mathbb{R}^N_{++}$ and each $\beta \in \mathbb{R}^N$, $\sigma(N, \lambda^{\alpha,\beta}(V)) = \lambda^{\alpha,\beta}_N(\sigma(N, V))$.

Remark 2.4. On the class of two-person smooth games, $\mathcal{N}$ satisfies *scale invariance.*

Definition. Let Γ be a non-empty class of two-person smooth games. A solution σ on Γ satisfies *local independence* if for each $(\{i, j\}, V) \in \Gamma$, and each $x \in \sigma(\{i, j\}, V)$, we have that $x \in \sigma(N, V')$ whenever

1. $x \in \partial V(\{i, j\}) \cap \partial V'(\{i, j\})$,
2. $(v_i, v_j) = (v'_i, v'_j)$ and
3. $\nabla g(x)$ is proportional to $\nabla g'(x)$,

where $(v_i, v_j) = (\max V(\{i\}), \max V(\{j\}))$, $(v'_i, v'_j) = (\max V'(\{i\}), \max V'(\{j\}))$, and g and g' are representations for $V(\{i, j\})$ and $V'(\{i, j\})$, respectively.[8]

Local independence is stronger than Nash's axiom of "independence of irrelevant alternatives" (IIA). However, it is not harder to justify. In fact, Nash's motivation for his axiom that there should be "no action at a distance," as he phrased it, is really a justification for local independence, not for IIA. We need to work with local independence because our domain includes non-convex bargaining problems. Thus, Nash's "trick" of enclosing the bargaining feasible set in his famous right angle triangle would not help us. Note that we could formulate local independence for non-smooth problems by requiring the non-empty intersection of the sets of supporting lines. We choose to present our result in this section for smooth problems only to be consistent with the sequel (see Remark 4.2).

Remark 2.5. On the class of two-person smooth games, $\mathcal{N}$ satisfies *local independence.*

Proposition 1. Let $\Gamma^{\{i,j\}}$ be the class of two-person smooth games $(\{i, j\}, V)$. Then a solution on $\Gamma^{\{i,j\}}$ satisfies *non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance* and *local independence* if and only if it is $\mathcal{N}$.

Proof. The solution $\mathcal{N}$ on $\Gamma^{\{i,j\}}$ satisfies the five axioms listed. Now we prove the uniqueness part of the statement. Say $i = 1$ and $j = 2$. Let $(\{1, 2\}, V)$ be a two-person smooth game, and σ a solution on $\Gamma^{\{i,j\}}$ which satisfies the five axioms. We prove that $\sigma(\{1, 2\}, V) = \mathcal{N}(\{1, 2\}, V)$.

By non-emptiness, there exists $x = (x_1, x_2) \in \mathcal{N}(\{1,2\}, V)$. By Pareto efficiency, $x \in \partial V(\{1,2\})$. By differentiability, there is a unique tangent line to the curve $\partial V(\{1,2\})$ at x:

$$\nabla g(x) \cdot (z - x) = g_1(x)(z_1 - x_1) + g_2(x)(z_2 - x_2) = 0.$$

Define the two-person smooth game $(\{1,2\}, V')$ by $V'(\{1\}) = V(\{1\})$, $V'(\{2\}) = V(\{2\})$, and

$$V'(\{1,2\}) = \{z \in I\!R^{\{1,2\}} | \nabla g(x) \cdot (z - x) \leq 0\}.$$

Then, since the Nash set satisfies local independence, $x \in \mathcal{N}(\{1,2\}, V')$. Note that

$$\mathcal{N}(\{1,2\}, V') = \quad \{(1/2)[\frac{g_2(x)}{g_1(x)}(x_2 - v_2') + x_1 + v_1'], \;\; (1/2)[\frac{g_1(x)}{g_2(x)}(x_1 - v_1') + x_2 + v_2']\},$$

which is the midpoint of the segment on the line $\partial V'(\{1,2\})$ truncated by the coordinates

$$(v_1', v_2') = (\max V'(\{1\}), \max V'(\{2\})).$$

Hence, $\{x\} = \mathcal{N}(\{1,2\}, V')$ (see Figure 1).

Define the TU game $(\{1,2\}, w)$ by $w(\{1\}) = w(\{2\}) = 0$, and

$$w(\{1,2\}) = g_1(x)(x_1 - v_1') + g_2(x)(x_2 - v_2').$$

By non-emptiness, Pareto efficiency and equal treatment for TU games, we have

$$\sigma(\{1,2\}, w) = \{(1/2)w(\{1,2\}), (1/2)w(\{1,2\})\}.$$

Let $\alpha = (1/g_1(x), 1/g_2(x))$, and $\beta = (v_1', v_2')$. By scale invariance,

$$\sigma(\{1,2\}, \lambda^{\alpha,\beta}(w)) = \lambda_N^{\alpha,\beta}(\sigma(\{1,2\}, w))$$

$$= \{(1/2g_1(x))w(\{1,2\}) + v_1', (1/2g_2(x))w(\{1,2\}) + v_2'\} = \mathcal{N}(\{1,2\}, V') = \{x\}.$$

Note that $(\{1,2\}, \lambda^{\alpha,\beta}(w))$ is the game $(\{1,2\}, V')$. Hence, $\sigma(\{1,2\}, V') = \sigma(\{1,2\}, \lambda^{\alpha,\beta}(w)) = \{x\}$, so that $x \in \sigma(\{1,2\}, V')$. By local independence, $x \in \sigma(\{1,2\}, V)$. Thus, $\mathcal{N}(\{1,2\}, V) \subseteq \sigma(\{1,2\}, V)$.

In exactly the same way, we can show $\sigma(\{1,2\}, V) \subseteq \mathcal{N}(\{1,2\}, V)$. Hence, $\sigma(\{1,2\}, V) = \mathcal{N}(\{1,2\}, V)$. Q.E.D.

3 Reduced Game Properties of the Nash Set

We introduce now a two-person "reduced game" studied by Peleg [25]. Let $\Pi^N = \{P \subseteq N : |P| = 2\}$, which is the set of two-person coalitions in N.

Definition. Let (N, V) be a game, $x \in V(N)$, and $P \in \Pi^N$. The two-person reduced game of (N, V) with respect to P given x_{-P} is the pair $(P, V_{x,P})$, consisting of the set P and the correspondence $V_{x,P}$ that associates with every $S \subseteq P$ a subset $V_{x,P}(S)$ of $\mathbb{R}^P$, where

$$V_{x,P}(\{i\}) = \{y_i \in \mathbb{R}^{\{i\}} | (y_i, x_Q) \in V(\{i\} \cup Q),\ Q \subseteq N \backslash P\}$$

for each $i \in P$, and

$$V_{x,P}(P) = \{y_P \in \mathbb{R}^P | (y_P, x_{-P}) \in V(N)\},$$

where we denote by $x_{-S} = x_{N \backslash S}$ for every $S \subseteq N$.

Thus, given a payoff vector x for the grand coalition N, it is clear what is the feasible set for the pair P in the reduced game. In addition, each player in P expects to be able to cooperate with any of the players not in P provided they are paid their components of x.

Definition. Let Γ be a non-empty class of games. Then a solution σ on Γ satisfies *bilateral consistency* if for each $(N, V) \in \Gamma$, each $x \in \sigma(N, V)$, and each $P \in \Pi^N$, $(P, V_{x,P}) \in \Gamma$ and $x_P \in \sigma(P, V_{x,P})$.

Bilateral consistency says that the solution should be invariant to the number of players, provided they have the expectations embodied in the bilateral reduced game. A strengthening of this property is consistency. A solution σ on a class Γ satisfies *consistency* if the same condition as above holds, but for all subsets P, that is, not only restricted to those subsets P of N of cardinality 2.

Definition. Let Γ be a non-empty class of games. Then a solution σ on Γ satisfies *converse consistency* if for each $(N, V) \in \Gamma$, and each $x \in \partial V(N)$, $x \in \sigma(N, V)$ whenever $x_P \in \sigma(P, V_{x,P})$ for every $P \in \Pi^N$.

If a solution is converse consistent, then in order to impose it on a society, an arbitrator should simply make sure that it is imposed on every pair of agents. Consistency and its converse for NTU games were first presented in Peleg [25] in order to characterize the core. We use exactly the same axioms in this paper.

Definition. Let (N,V) be a game. The *Nash set* of (N,V) is:

$$\mathcal{N}(N,V) = \{x \in \partial V(N) |\ \ g_i(x)(x_i - v_i(x_{-\{i,j\}})) = g_j(x)(x_j - v_j(x_{-\{i,j\}}))\ \ \forall i,j \in N\},$$

where g is a representation for $V(N)$, and

$$(v_i(x_{-\{i,j\}}), v_j(x_{-\{i,j\}})) = (\max V_{x,\{i,j\}}(\{i\}), \max V_{x,\{i,j\}}(\{j\})).$$

As before for the case of two players, the Nash set is the set of payoffs where each pair of players receives a critical point of the function $(z_i - v_i(x_{-\{i,j\}}))(z_j - v_j(x_{-\{i,j\}}))$ subject to $(z_i, z_j, x_{-\{i,j\}}) \in \partial V(N)$. If the game is TU, the representation in the definition of the first order conditions of the optimization program yields precisely the "surplus equality" equations of the prekernel.

Note that in each bilateral problem the "threat point" $(v_i(x_{-\{i,j\}}), v_j(x_{-\{i,j\}}))$ consists of the maximum utilities that each of the two players could achieve without the cooperation of the other agent, provided that the payoff proposed to the grand coalition is x. This "threat point" can influence the pair's negotiations. Of course, if one is dealing with a pure bargaining problem, this point will end up being (v_i, v_j), the standard "disagreement point". In addition, the payoff x is required to be in $\partial V(N)$, that is, suppose that for some reason the grand coalition should stay together even if the "threat point" for some pair lies outside of $V(N)$ (this may be the case in a cost sharing problem with a decreasing returns technology, for example). This requirement, though, should not be argued as a drawback of the solution concept, as it can readily be adapted for any coalition structure other than N.

Remark 3.1. On a rich class of smooth games, $\mathcal{N}$ satisfies bilateral consistency and converse consistency. It also satisfies consistency, stronger than its bilateral version.

4 A Characterization of the Nash Set

We have checked that on the class of two person smooth games, the solution $\mathcal{N}$ satisfies non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence. We now show that for the class of games with more than two players it is uniquely characterized by the two-person versions of all these axioms together with bilateral consistency and its converse.

Theorem 1. Let Γ_0 be a rich class of smooth games. A solution on Γ_0 satisfies *bilateral consistency, converse consistency* and the following five axioms for two player games *non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence* if and only if it is $\mathcal{N}$.

Proof. The solution $\mathcal{N}$ on Γ_0 satisfies the seven axioms listed. Now we prove uniqueness.

Let $(N, V) \in \Gamma_0$, and σ a solution on Γ_0 that also satisfies the seven axioms of the Theorem. We prove that $\sigma(N, V) = \mathcal{N}(N, V)$. The proof for $|N| = 1$ is trivial. We have already proven the case of $|N| = 2$ (Proposition 1 and Remark 3.1). Then consider the case of $|N| \geq 3$.

Suppose that $\mathcal{N}(N, V) \neq \emptyset$. Let $x \in \mathcal{N}(N, V)$. By the bilateral consistency of $\mathcal{N}$, $x_P \in \mathcal{N}(P, V_{x,P})$ for every $P \in \Pi^N$. Hence, $x_P \in \sigma(P, V_{x,P})$ for every $P \in \Pi^N$. By the converse consistency of σ, $x \in \sigma(N, V)$. Hence, $\mathcal{N}(N, V) \subseteq \sigma(N, V)$. Note in particular that $\sigma(N, V)$ is non-empty. Then we can similarly show that $\sigma(N, V) \subseteq \mathcal{N}(N, V)$. Thus, $\sigma(N, V) = \mathcal{N}(N, V)$.

Suppose that $\mathcal{N}(N, V) = \emptyset$. Let $x \in \partial V(N)$. Then by the converse consistency of $\mathcal{N}$, there exists at least one pair Q of players in N such that $x_Q \notin \mathcal{N}(Q, V_{x,Q})$. Since $(Q, V_{x,Q})$ is a two-person game, we have $\mathcal{N}(Q, V_{x,Q}) = \sigma(Q, V_{x,Q})$, so that $x_Q \notin \sigma(Q, V_{x,Q})$. By the bilateral consistency of σ, $x \notin \sigma(N, V)$. Therefore, there is no payoff profile in $\sigma(N, V)$, so that $\sigma(N, V) = \emptyset$. Thus, $\sigma(N, V) = \mathcal{N}(N, V)$. Q.E.D.

Next we show that the seven axioms used in the characterization are logically independent. In each example, the axiom in brackets is the one violated by the solution proposed.

Example 4.1 [non-emptiness]: For every $(N, V) \in \Gamma_0$, let $\sigma(N, V) = \emptyset$. Then σ vacuously satisfies all the conditions of Theorem 1 except non-emptiness for two-person games.

Example 4.2 [Pareto efficiency]: For every two-person game (P, V), define $b(P, V) = (v_i)_{i\in P}$ if $(v_i)_{i\in P} \in \text{Int}V(P)$ and $b(P, V) = \mathcal{N}(P, V)$ otherwise. For every $(N, V) \in \Gamma_0$, let

$$\sigma(N, V) = \{x \in V(N)| \quad x_P \in b(P, V_{x,P}) \quad \forall P \in \Pi^N\}.$$

Then σ satisfies all the conditions of Theorem 1 except Pareto efficiency for two-person games.

Example 4.3 [equal treatment for TU games]: For every $(N, V) \in \Gamma_0$, let

$$\sigma(N, V) = \partial V(N).$$

Then σ satisfies all the conditions of Theorem 1 except equal treatment for two-person TU games.

Example 4.4 [scale invariance]: For every $(N, V) \in \Gamma_0$, let

$$\sigma(N,V) = \{x \in \partial V(N)| \ \ x_i - v_i(x_{-\{i,j\}}) = x_j - v_j(x_{-\{i,j\}}) \ \ \forall i,j \in N\}.$$

Then σ satisfies all the conditions of Theorem 1 except scale invariance for two-person games.

Example 4.5 [local independence]: For every two-person game (P, V), define $a(P, V) = (a_i(P, V))_{i \in P}$ by

$$a_i(P,V) = \max\{x_i \in I\!R^{\{i\}} | (x_i, v_j(x_{-P})) \in V(P), \quad P = \{i,j\}\} \ \ \text{for} i,j \in P.$$

For every $(N, V) \in \Gamma_0$, let

$$\sigma(N,V) = \{x \in \partial V(N) | x_P \in [(v_i(x_{-P}))_{i \in P}, a(P, V_{x,P})] \ \ \forall P \in \Pi^N\},$$

where $[c, d] = \{(1-t)c + td | 0 \leq t \leq 1\}$ for each $c, d \in I\!R^P$. That is, for every pair of players in N, x_P is the maximal point of the feasible set $V_{x,P}(P)$ on the segment connecting $(v_i(x_{-P}))_{i \in P}$ to $a(P, V_{x,P})$ if $x \in \sigma(N, V)$. Note that σ is a modification of the Kalai- Smorodinsky bargaining solution, and it satisfies all the conditions of Theorem 1 except local independence for two-person games.

Example 4.6 [bilateral consistency]: Let $(N, V) \in \Gamma_0$, and $i, j \in N$. Then i and j are equivalent if

$$g_i(x)(v_i(S \cup \{i\}; x_S) - v_i) = g_j(x)(v_j(S \cup \{j\}; x_S) - v_j)$$

for every $x \in \partial V(N)$, and for each $S \subseteq N \backslash \{i, j\}$, where

$$v_i(S \cup \{i\}; x_S) = \max\{y_i \in I\!R^{\{i\}} | (y_i, x_S) \in V(S \cup \{i\})\}$$

for each $i \in N$, and each $S \subseteq N \backslash \{i, j\}$. Note that $g_i(x)(v_i(x_{-\{i,j\}}) - v_i) = g_j(x)(v_i(x_{-\{i,j\}}) - v_j)$ for every $x \in \partial V(N)$ if players i and j are equivalent. Now let

$$\sigma(N,V) = \{x \in \partial V(N) | g_i(x)(x_i - v_i) = g_j(x)(x_j - v_j) \ \ \text{if } i,j \in N \text{are equivalent}\}.$$

We can verify that σ satisfies the five axioms imposed on two-person games. If $|N| = 2$, then the two players in N are equivalent, and $\sigma(N, V) = \mathcal{N}(N, V)$.

To prove that σ satisfies converse consistency, let $x \in \partial V(N)$, and $x_P \in \sigma(P, V_{x,P}) = \mathcal{N}(P, V_{x,P})$ for every $P \in \Pi^N$. Since $\mathcal{N}$ satisfies converse consistency, $x \in \mathcal{N}(N, V)$. We show that $\mathcal{N}(N,V) \subseteq \sigma(N,V)$ if $|N| \geq 3$. Let $x \in \mathcal{N}(N,V)$. Then $x \in \partial V(N)$. Suppose that players i and j in N are equivalent. Since $x \in \partial V(N)$, we have

$$g_i(x)(v_i(x_{-\{i,j\}}) - v_i) = g_j(x)(v_j(x_{-\{i,j\}}) - v_j),$$

so that

$$g_i(x)(x_i - v_i) - g_j(x)(x_j - v_j) = g_i(x)(x_i - v_i(x_{-\{i,j\}})) - g_j(x)(x_j - v_j(x_{-\{i,j\}})).$$

Since $x \in \mathcal{N}(N,V)$, we have

$$g_i(x)(x_i - v_i(x_{-\{i,j\}})) = g_j(x)(x_j - v_j(x_{-\{i,j\}})),$$

so that $g_i(x)(x_i - v_i) = g_j(x)(x_j - v_j)$. Thus $x \in \sigma(N,V)$, i.e., $\mathcal{N}(N,V) \subseteq \sigma(N,V)$. Thus, σ satisfies converse consistency.

On the other hand, σ does not satisfy bilateral consistency. Suppose that $N = \{1,2,3\}$, and V is a TU game defined by $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = 4$, $v(\{1,3\}) = 3$, $v(\{2,3\}) = 2$, and $v(\{1,2,3\}) = 6$. Then (N,V) has no pair of equivalent players, so that $\sigma(N,V) = \partial V(N)$. Let $x = (2,2,2) \in \sigma(N,V)$. However, $\sigma(\{1,2\}, V(x, -\{1,2\})) = \{(2.5, 1.5)\}$. Hence, σ does not satisfy bilateral consistency.

Example 4.7 [converse consistency]: For every $(N,V) \in \Gamma_0$. Let

$$\sigma(N,V) = \{x \in \mathcal{N}(N,V) | v_i(x_{-\{i,j\}}) = v_i(x_{-\{i,k\}}) \ \ \forall i,j,k \in N, \text{ with } j \neq i \neq k\}.$$

Note that $\sigma(N,V) = \mathcal{N}(N,V)$ if $|N| = 2$. Then σ satisfies all the conditions of Theorem 1 except converse consistency.

Remark 4.1: The solution $\mathcal{N}$ satisfies Pareto efficiency, equal treatment for TU games, scale invariance and local independence over the class of n-person smooth games, $n \geq 2$. It does not satisfy, however, non-emptiness over the same class (see Serrano [29], Example 2, and Moldovanu [19], p. 188).[9]

Remark 4.2. Theorem 1 does not extend to non-smooth problems. Although $\mathcal{N}$ satisfies consistency over this class, it fails to satisfy converse consistency (see Lensberg [16], Figure 4, p. 339).

5 The Intersection of the Core and the Nash Set

We next investigate the implications of the same axioms on the class of smooth games with non-empty cores. We find that we characterize the intersection of the core and the Nash set.

Definition. Let (N, V) be a game, S a subset of N, and x a point of $\mathbb{R}^N$. Then we say that S can improve upon x if there is $y \in V(S)$ such that $y_i > x_i$ for all $i \in S$. The *core* of (N, V) is:

$$\mathcal{C}(N, V) = \{x \in V(N) | \text{ There is no coalition that can improve upon } \quad x\}.$$

Definition. Let (N, V) be a smooth game. The *intersection of the core and the Nash set* of (N, V) is:

$$\mathcal{CN}(N, V) = \mathcal{C}(N, V) \cap \mathcal{N}(N, V).$$

Remark 5.1. The solution $\mathcal{CN}$ reduces to the Nash bargaining solution on the class of two-person smooth games $(\{i, j\}, V)$ such that $V(\{i, j\})$ is a convex set.

We can show the following in exactly the same way as for Proposition 1.

Proposition 2. Let $\Gamma_c^{\{i,j\}}$ be the class of two-person smooth games $(\{i, j\}, V)$ with non-empty cores. Then a solution on $\Gamma_c^{\{i,j\}}$ satisfies *non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance* and *local independence* if and only if it is $\mathcal{CN}$.

Similarly, we can prove the theorem below as we have done for Theorem 1.

Theorem 2. Let Γ_c be the class of smooth games with non-empty cores containing all the two-person games with the same property. A solution on Γ_c satisfies *bilateral consistency, converse consistency* and the following five axioms for two player games *non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence* if and only if it is $\mathcal{CN}$.

Moldovanu's [19] Theorem 5.2 considers the same solution on the class of NTU assignment games. His characterization implicitly assumes that the solution under investigation is single-valued for two-person games. Thus, he can use the "independence of irrelevant alternatives" as one of the axioms, instead of local independence. If one does not assume single-valuedness or local independence, all that can be shown is that the solution is a superset of the intersection of the core and the Nash set for every game in the domain: $\mathcal{CN}(N, V) \subseteq \sigma(N, V)$.

Next we provide seven examples to show that the axioms used in Theorem 2 are logically independent. As before, the axiom in brackets is the one not satisfied by the solution in the example.

Example 5.1 [non-emptiness]: Let $\sigma(N,V) = \emptyset$. It vacuously satisfies all the conditions except non-emptiness for two-person games in Γ_c.

Example 5.2 [Pareto efficiency]: For each two-person game (P,V), define $d(P,V) = (v_i)_{i \in P}$, which is the pair of the maximal payoffs the two players can attain independently. Let

$$\sigma(N,V) = \{x \in V(N) | x_P = d(P, V_{x,P}) \ \ \forall P \in \Pi^N\}.$$

Then σ satisfies all the conditions except Pareto efficiency in two-person games in Γ_c.

Example 5.3 [equal treatment]: Let $\sigma(N,V) = \mathcal{C}(N,V)$, which satisfies all the conditions except equal treatment for two-person TU games in Γ_c.

Example 5.4 [scale invariance]: Let

$$\sigma(N,V) = \{x \in \mathcal{C}(N,V) | x_i - v_i(x_{-\{i,j\}}) = x_j - v_j(x_{-\{i,j\}}) \ \ \forall i,j \in N\}.$$

Then σ satisfies all the conditions except scale invariance for two-person games in Γ_c.

Example 5.5 [local independence]: Define σ as in Example 4.5, but on Γ_c (the consistent extension of the Kalai-Smorodinsky solution). Then σ satisfies all the conditions except local independence for two-person games in Γ_c.

Example 5.6 [bilateral consistency]: Let $\sigma(N,V) = \mathcal{N}(N,V)$. Then σ satisfies all the conditions except bilateral consistency.

Example 5.7 [converse consistency]: Define σ by replacing $\mathcal{N}(N,V)$ with $\mathcal{CN}(N,V)$ in the definition of $\sigma(N,V)$ in Example 4.7. Then this solution satisfies all the conditions except converse consistency.

Remark 5.2: The solution $\mathcal{CN}$ satisfies Pareto efficiency, equal treatment for TU games, scale invariance and local independence over the class of n-person smooth games with non-empty cores, $n \geq 2$. It does not satisfy, however, non- emptiness over the same class (see Moldovanu [19], p. 188).

6 Examples

This section is devoted to examine a few examples that demonstrate the usefulness of the Nash set as a solution concept.

Example 6.1. Consider the class of NTU games (N, V) such that $V(N)$ is convex and where $\forall S \subseteq N, S \neq N$, $V(S) \subseteq \{(v_i)_{i\in S}\} - \mathbb{R}^S_+$. That is, this is the class of convex pure bargaining problems. Then, $\mathcal{N}$ coincides with the Nash bargaining solution.

Example 6.2. Consider the following two person non-convex pure bargaining problem. Suppose two bargainers are negotiating over how to split two dollars and the consent of both is needed to split any pie. Suppose player i's utility function for $i = 1, 2$ is the following: $u(x_i) = x_i$ if i's share $x_i \leq 1$, while $u(x_i) = 4x_i - 3$ otherwise. Then, the reader can easily check that $\mathcal{N}$ consists of three points for this case:

$$\mathcal{N}(N, V) = \{(1, 1), (5/2, 5/8), (5/8, 5/2)\}.$$

(See Figure 2). That is, three possible splits of the pie are prescribed: equal division (the problem is symmetric) and two others where the risk loving agent receives 11/8, while the risk neutral one gets 5/8. It follows from our assumptions on NTU games that $\mathcal{N}$ is non- empty for every pure bargaining problem, convex or not.

Example 6.3. Consider the class of TU games. On this class, $\mathcal{N}$ coincides with the prekernel for TU games, and is thus always non-empty. As an illustration, consider the TU game at the end of Example 4.6. The Nash set (prekernel) of this game is the singleton $(3, 2, 1)$. Figure 3 represents the reduced problems faced by the pairs of players $(1, 2)$, $(1, 3)$ and $(2, 3)$. The payoff in the Nash set splits in half the "available surplus" to each pair determined by the threat point, which is a function of the payoff awarded to the third player.

7 Comparisons with the Literature

Our solution concept, the Nash set, is the one uncovered non-cooperatively by Serrano [29]. It is also the same solution concept as in Maschler, Owen and Peleg [18], Herrero [9] and Moldovanu [19]. The first two papers study the class of pure bargaining problems, and the third one concentrates on assignment problems. These two are subclasses of the general NTU model considered in this paper.

Conley and Wilkie's [3] single-valued solution is not continuous. Specifically, consider a sequence of non-convex problems that converges (in the Hausdorff topology, or in the one specified below) to a convex problem. Along the sequence this solution prescribes the Kalai and Smorodinsky solution, but "jumps" to Nash's in the limit. Instead, the Nash set is upper hemicontinuous with the following notion of convergence: consider a sequence of games (N, V_k) with representations g_k^S for every $S \subseteq N$. We say that such a sequence converges to the game (N, V) with representations g^S for every $S \subseteq N$ if

1.
$$\forall S \subseteq N, \quad \lim_k \sup_{x_S \in I\!R^S} \{|g_k^S(x_S) - g^S(x_S)|\} = 0$$
and

2.
$$\lim_k \sup_{x \in I\!R^N} \{|\nabla g_k(x) - \nabla g(x)|\} = 0.$$

One can easily construct examples to show that the Nash set fails to satisfy lower hemicontinuity, though. This seems to challenge Herrero's [9], but her property A5 has a misleading label, since the sequences are allowed to be chosen arbitrarily both in the domain and the range of the correspondence.

Kaneko's [13] solution picks only the global maxima of the Nash product. Therefore, it violates Nash's symmetry axiom for non- convex problems, such as in Example 6.2. Besides, the Nash solution and its extensions, as opposed to other prominent bargaining solutions, should be guided by local considerations around the solution point. Specifically, the Nash set selects those points in the Pareto frontier where the utility elasticity is 1 between every pair of players. This property is met at local maxima and minima of the Nash product. More importantly, we are not recommending local or global extrema of a particular function: the axioms are our starting point. We have utilized axioms that are central in important characterizations of the Nash solution, investigated their implications over a larger domain, and discovered that they lead to the constrained critical points of the Nash product.

Finally, we touch on the existence question. Under our weak general assumptions, it is well known that the Nash set is non-empty in TU games and pure bargaining problems. For symmetric NTU games, the Nash set is also non- empty. The same can be said for NTU assignment games

(Moldovanu [19]). It is an open and interesting question to identify other classes where an existence result obtains.

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(

Footnotes)

1. In fact, previous attempts to extend the prekernel to NTU games (Kalai [11], Billera and McLean [2]) were not satisfactory. Both papers try to extend the notions of a "player's surplus" and a "coalition excess" to NTU games. In doing so, they come up with solution concepts that even fail to be scale invariant, a minimal requirement that solutions must satisfy. In addition, their solutions lack any clear economic content.

2. We adopt the name given to this solution by Maschler, Owen and Peleg [18]. Like Herrero [9], they were only concerned with non-convex pure bargaining problems.

3. In this sense, our work resembles Aumann's [1] and Hart's [8] axiomatizations of the Shapley NTU value and the Harsanyi value, respectively. These theorems combine the axioms of Nash [22] and those of Shapley [31].

4. Shimomura [32] characterizes the core by axioms imposed also on bilateral problems only.

5. Local independence is introduced and studied by Nagahisa [20] for the characterization and implementation of the Walrasian rule in exchange economies. See also Dutta, Sen and Vohra [5], Nagahisa and Suh [21], Saijo, Tatamitani and Yamato [28], and the related conditions of localness (Lensberg [15]) and independence of locally irrelevant alternatives (Herrero [9]). The basic idea can be traced back to Inada [10], who proposes this condition as a modification of Arrow's IIA axiom in economic environments.

6. We use an extension of Lensberg's axiom to NTU games, first utilized by Peleg [25] in his characterization of the core. Peleg [25] is also the source of our converse consistency axiom. See Thomson [33] for a comprehensive survey of the vast literature related to consistency.

7. For each pair of vectors (x, y), we write $x \geq y$ if $x_i \geq y_i \quad \forall i$.

8. For each pair of vectors (p, p'), "p is proportional to p'" means that there exists a non-zero real number t such that $p = tp'$.

9. A solution satisfies local independence over a class of N-player games, $|N| > 2$ if it does so for every two person reduced game as defined in section 2. The definitions of all the other axioms are straightforward for this case.