The taxation of trades in assets [1]

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Abstract

When the asset market is incomplete, there typically exist taxes on trades in assets and a redistribution of revenue in the asset market that are Pareto improving.

The policy is anonymous, it economizes on complexity, and it results in *ex post* Pareto optimal allocations; it is publicly announced before markets open, thus fully and correctly anticipated by traders, it does not require that financial markets be shut down, and it does not modify the asset market structure. As such, it improves over previously proposed constrained interventions.

Key words: taxes, incomplete asset market, equilibrium, Pareto improvement.

JEL classification numbers: D52, D60, H20.

1 Introduction

Ever since Arrow (1951) and Debreu (1951) stated definitively and demonstrated the theorems of classical welfare economics, the focus has been on possible sources of failure of the Pareto optimality of competitive equilibrium allocations.

Taxation has been extensively used as an intervention policy in economies with public goods and externalities. Diamond and Mirrlees (1971) characterized tax equilibria and their optimality properties; the local structure of equilibria was chartered in Guesnerie (1977, 1979), who drew its implications for tax reform.

In Arrow (1953) or Debreu (1960), a complete market in elementary securities or in contingent commodities was shown to allow the theorems of welfare economics to encompass economies with uncertainty. The absence of a complete asset market is a well recognized reason for the Pareto suboptimality of competitive allocations.

Competitive equilibrium allocations in economies with an incomplete asset market are suboptimal in a strong sense: Pareto improvement is possible even under the restrictions implied by the incompleteness. Constrained suboptimality, defined in Diamond (1967), was formally shown in Hart (1975), and then proved robust or generic in Geanakoplos and Polemarchakis (1986).

Constrained suboptimality is a positive argument for intervention in competitive market economies. The argument for intervention in an incomplete asset market is compelling when:

1. intervention is compatible with the structural characteristics that underlie the incompleteness;

2. it is anonymous;

3. it results in ex - post Pareto optimal allocations of commodities;

4. it is anticipated by traders in markets for assets.

Compatibility restricts alternative allocations, and it is hard to assess or make precise, since standard models do not explicit the reasons for the incompleteness; it is commonly taken to mean that interventions should take the market structure as given; anonymity economizes on information and complexity; ex - post optimality guarantees against further intervention or deviations, while anticipation allows for repeated intervention.

A variety of intervention policies and corresponding notions of constrained suboptimality have been introduced in the literature, all compatible with the incompleteness of the asset market. Indeed, Citanna, Kajii and Villanacci (1998) give a general formulation of constrained suboptimality that renders it applicable across diverse scenarios or interventions. The robust constrained suboptimality results obtained to date — Geanakoplos and Polemarchakis (1986)

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for individual portfolio reallocations, Herings and Polemarchakis (1998) for rationing in asset and spot commodity markets, Citanna, Kajii and Villanacci (1998) for lump-sum taxes and transfers — fail at least one of the above -mentioned criteria for improving interventions: portfolio reallocations or lump - sum taxes and transfers are not anonymous, a fact emphasized in Kajii (1994); rationing does not yield ex - post optimal allocations of commodities. An interesting feature of lump - sum taxation is that it is anticipated and does not close down financial markets; also, it requires a minimal number of instruments for robust Pareto improvement; it fails anonymity, though Tirelli (2000a) shows that, with proportional state contingent taxes or subsidies on individual income, anonymity can be preserved; it is unappealing because it requires taxes to be announced for future events and to be partially state - contingent, requiring a lot of information to implement.

The introduction of new assets or the alteration of asset payoffs, as in Cass and Citanna (1998), Elul (1995) and, recently, Tirelli (2000b), either reduces incompleteness or, at best, ignores the reasons why assets are initially missing, and it also requires state-contingent policies.

Here, the instruments of intervention are taxes or subsidies on the purchase of assets and lump - sum redistribution of the fiscal revenue. The main result is that if the asset market is sufficiently incomplete, generically there exist Pareto improving fiscal policies.

The taxation of assets is anonymous. The resulting allocation of commodities is ex - post Pareto optimal, and there are no ex- post constraints on asset trades enforced by shutting down financial markets. Intervention is compatible with the incompleteness of the asset market. Moreover, it does not require the announcement of future or state-contingent taxes or subsidies, which could be subject to credibility constraints.

The result is easy to understand. In standard portfolio reallocation policies, individual asset holdings are directly confiscated and redistributed in order to control the state-contingent distribution of wealth. Here, a redistribution of portfolio holdings is induced through taxes or subsidies on asset prices; asset holdings are indirectly controlled by creating a bid - ask spread, which can nevertheless be negative. Anonymity is guaranteed if the number of assets exceeds the number of (types of) traders. Indeed, in this case the lump-sum, individual - specific portion of the intervention can be dispensed with.

The number of instruments employed in Pareto improving interventions here is lower than that in Citanna, Kajii and Villanacci (1998). This is due to the fact that taxes and transfers there occurred in two periods, while here they occur in only one period. Nevertheless, they imposed no essential restriction on the cardinality of the set of states of the world, the assets available or the number of individuals. Here, the number of available assets as well as the extent of the incompletenes of the asset market are required to be higher than the number of individuals.

A similar result on the welfare effects of taxation has been obtained in Bisin,

Geanakoplos, Gottardi, Minelli and Polemarchakis (2000), where the incompleteness of the market for contracts reflects the private information of individuals; indeed, taxation also yields a Pareto improvement at least in the case of adverse selection.

Though taxation is anonymous, Pareto improving intervention requires information about the fundamentals of the economy, the preferences and endowments of individuals. It is then a consideration whether the information required for determining the welfare consequences of taxation can be obtained from market data, in particular from equilibrium prices. The argument in Kübler, Chiappori, Ekeland and Polemarchakis (2000) is that knowledge of the equilibrium correspondence, market data, i.e., the variation in the equilibrium prices of commodities and assets as the allocation of endowments varies, suffices to identify the profile of utilities.

2 Economies

Economies of pure exchange extend over two periods under uncertainty.

States of the world are $\mathcal{S} = \{1, \ldots, S\}$, a finite, nonempty set, and are indexed by s.

Commodities are $\mathcal{L} = \{1, \ldots, L\}$, a finite, nonempty set, and are indexed by l; they are traded in spot markets after the resolution of uncertainty. At a state of the world, s, commodities are indexed by (l, s), and a bundle of commodities is a strictly positive real vector $x_s = (\ldots, x_{l,s}, \ldots)'$; across states of the world, a bundle of commodities is $x = (\ldots, x_s, \ldots)'$.

Individuals are $\mathcal{I} = \{1, \ldots, I\}$, a finite, nonempty set, and they are indexed by i. The preferences of an individual are described by the ordinal utility function u^i, with domain the consumption set of strictly positive bundles of commodities across states of the world.

The endowment of the individual is e^i, a bundle of commodities across states of the world.

Assumption 1 *For every individual,*

1. *the utility function is smooth, differentially strictly increasing: $Du^i \gg 0$, and differentially strictly quasi-concave: if $b \neq 0$ and $Du^i b = 0$, then $b' D^2 u^i b < 0$, while, along a sequence of consumption plans, $(x_n \gg 0 : n = 1, 2, \ldots,)$, if $\lim_{n \to \infty} x_n = x \gneqq 0$, then $\lim_{n \to \infty} (\|Du^i(x_n)\|)^{-1} x_n' Du^i(x_n) = 0$, and*

2. *the endowment is strictly positive: $e^i \gg 0$.*

The boundary condition on the utility function is satisfied if the closure of the indifference surface through a consumption plan is contained in the interior of the consumption set, a stronger condition.

The preferences of an individual may, but need not admit a von Neumann-Morgenstern representation, (v^i, π^i), where v^i is the state - independent cardinal utility index, $\pi^i = (\ldots, \pi_s^i, \ldots)$ is a (subjective) probability measure on the set of states of the world, and $u^i = \mathrm{E}_{\pi^i} v^i$; alternatively, preferences may have an additively separable representation, $(\ldots, u^i, \ldots)$, where u_s^i, is a state - dependent cardinal utility index, and $u^i = \sum_{s \in \mathcal{S}} u_s^i$.

Assumption 2 *Spot markets for commodities are active:* $L \geq 2$.

Assets are $\mathcal{A} = \{1, \ldots, A\}$, a finite, nonempty set, and are indexed by a; they are exchanged prior to the resolution of uncertainty, and they are employed to transfer revenue across states of the world.

A portfolio of assets is $y = (\ldots, y_a, \ldots)'$. At a state of the world, the payoff of an asset is $r_{a,s}$, denominated in units of commodity $l = 1$, the numéraire commodity; across states of the world, the payoffs of an asset are $r_a = (\ldots, r_{a,s}, \ldots)'$; The payoffs of assets at a state of the world are $R_s = (\ldots, r_{a,s}, \ldots)$, and the matrix of asset payoffs is

$$R = (\ldots, r_a, \ldots) = (\ldots, R_s, \ldots)'.$$

The column span of the matrix of asset payoffs is $[R]$, the subspace of attainable reallocations of revenue across states of the world.

Assumption 3

1. *there are no redundant assets:* $\dim[R] = A$,

2. *the asset market is active:* $A \geq 2$, *and*

3. *the payoffs of asset* $a = 1$ *is positive:* $r_1 > 0$.

Assumption 4

1. *The economy is heterogeneous:* $I \geq 2$, *and*

2. *the asset market is sufficiently incomplete:* $\min\{A - 1, S - A\} \geq I$.

Assumption 3 and Assumtion 4, item 1, are standard, Geanakoplos and Polemarchakis (1986). Assumption 4, item 2, is strong; only one result, Pareto improvements without lump-sum transfers require it fully, while other results require a less stringent restriction; if $r_1 = (1, 0, ..., 0)'$, while $r_{a,1} = 0$, for $a > 1$, which corresponds to the exchange and consumption of commodities along with assets, the condition can be relaxed to $\min\{A - 1, S + 1 - A\} \geq I$.

An allocation of commodities is $\chi = (\ldots, x^i, \ldots)$, such that $x^i \geq 0$, for every individual; aggregate consumption is $x^a = \sum_{i \in \mathcal{I}} x^i$, while the aggregate endowment is $e^a = \sum_{i \in \mathcal{I}} e^i$. An allocation of commodities is feasible if $x^a = e^a$.

An allocation of portfolios of assets is $\psi = (\ldots, y^i, \ldots)$; the aggregate portfolio is $y^a = \sum_{i \in \mathcal{I}} y^i$. An allocation of portfolios of assets is feasible if $y^a = 0$.

The excess demand of an individual is $z^i = (..., z_s^i, ...)$, where $z_s^i = x_s^i - e_s^i$.

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The set of economies

Economies that satisfy Assumptions 1 - 4 are parametrized by structural parameters $\omega = (u, e) \in \Omega$, utilities and endowments satisfying Assumption 1. The space of utilities is endowed with the topology of C^2-uniform convergence over compact sets, while the space of endowments has the standard Euclidean topology.

Quadratic perturbations of utilities are as presented in Citanna et al. (1998). For any economy $\omega \in \Omega$, a perturbed utility for an individual, i, is a function

$$u^i_{(x^{i*}, \rho, \varepsilon)}(x^i, M^i) = u^i(x^i) + (1/2)\varepsilon \rho(x^i)[(x^i - x^{i*})'M^i(x^i - x^{i*})],$$

where x^{i*} is an individual consumption plan, chosen as a function of ω, $\varepsilon > 0$ a scalar, $\rho(x^i)$ a bump function and M^i a symmetric, LS-dimensional matrix. The second derivative of this function with respect to x^i is exactly equal to $D^2 u^i(x^i) + \varepsilon M^i$ in a small open neighborhood of the equilibrium allocation x^{i*}. The vector of quadratic perturbations is $M = (..., M^i, ...)$, while $u^i(x^i, M^i)$ denotes $u^i_{(x^{i*}, \rho, \varepsilon)}(x^i, M^i)$.

A generic set of economies is an open and dense subset of Ω; a property holds generically if it holds for a generic set.

3 Fiscal policy and equilibrium

Prices of commodities at a state of the world are a row vector $p_s = (1, \ldots, p_{l,s}, \ldots) \gg 0$: commodity $l = 1$ is numéraire, and prices are strictly positive; across states of the world, prices of commodities are $p = (\ldots, p_s, \ldots)$.

At a state of the world, the value of the a bundle of commodities, x_s, at prices of commodities p_s is $p_s x_s$; across states of the world, the values of a bundle of commodities x at prices of commodities p are $p \otimes x = (\ldots, p_s x_s, \ldots)$.

Prices of assets are $q = (1, \ldots, q_a, \ldots)$. Rates of taxation or subsidy on the purchase of assets are $t = (\ldots, t_a, \ldots)$, with $t_a > -1$ for all $a \in \mathcal{A}$; the purchase prices of assets are $(1 + t) \otimes q = (\ldots, (1 + t_a)q_a, \ldots)$. The value of a portfolio of assets y at prices of assets q and rates of taxation t is $((1 + t) \otimes q)y_+ + qy_-$, where $y_{a,+} = \max\{0, y_a\}$, while $y_{a,-} = \max\{0, -y_a\}$, and $y_+ = (\ldots, y_{a,+}, \ldots)$, while $y_- = (\ldots, y_{a,-}, \ldots)$.

Aggregate fiscal revenue from the taxation of assets is $T = \sum_{i \in \mathcal{I}}(t \otimes q)y^i_+$. It is distributed across individuals according to the distribution scheme $\delta = (\ldots, \delta^i, \ldots) \gg 0$, with $\sum_{i \in \mathcal{I}} \delta^i = 1$; the revenue of an individual is $\delta^i T$; the uniform distribution scheme corresponds to $\overline{\delta} = (\ldots, (1/I), \ldots)$.

Fiscal policy is $\zeta = (t, \delta) \in \mathcal{Z}$, an open set of dimension $A + (I - 1)$. Aggregate fiscal revenue, T, is determined endogenously.

The optimization problem of an individual is

$$\max_{x,y} \quad u^i(x)$$

$$\text{s.t.} \quad ((1+t) \otimes q)y_+ - qy_- - \delta^i T \le 0,$$

$$p \otimes (x - e^i) \le Ry.$$

Definition 1 *A competitive equilibrium with fiscal policy $\zeta = (t, \delta)$ consists of a vector (χ, ψ, p, q, T) of feasible allocations and prices of commodities and assets, and a fiscal revenue such that (x^i, y^i) is a solution to the optimization problem for every individual.*

A competitive equilibrium in the standard sense is, here, a competitive equilibrium with inactive fiscal policy, $\overline{\zeta} = (0, \overline{\delta})$. At a competitive equilibrium, where fiscal policy is inactive and rates of taxation vanish, there is no fiscal revenue; the distribution scheme is immaterial and, for simplicity, it is set to be uniform.

3.1   Regularity and local existence

For every individual,

$$F^i = \begin{pmatrix} F^i_I \\ F^i_{II} \\ F^i_{III} \\ F^i_{IV} \end{pmatrix} = \begin{pmatrix} D_{x^i} u^i(d^i, x^i) - \lambda^i \otimes p \\ \lambda^i R - \mu^i q^i \\ ((1+t) \otimes q)y^i_+ - qy^i_- - \delta^i T \\ -p \otimes (x^i - e^i) + Ry^i \end{pmatrix},$$

where

$$q^i_a = \begin{cases} (1 + t_a)q_a, & \text{if} \quad y^i_a \ge 0, \\ \\ q_a & \text{if} \quad y^i_a < 0, \end{cases}$$

the Lagrange multipliers associated with the budget constraints across states of the world and the asset market are $\lambda^i = (\dots, \lambda^i_s, \dots) \gg 0$ and $\mu^i > 0$, respectively, and $\lambda^i \otimes p = (\dots, \lambda^i_s p_s, \dots)$.

Across individuals,

$$F^0 = \begin{pmatrix} F^0_V \\ F^0_{VI} \\ F^0_{VII} \end{pmatrix} = \begin{pmatrix} \tilde{x}^a - \tilde{e}^a \\ \tilde{y}^a \\ T - (t \otimes q)y^a_+ \end{pmatrix},$$

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where $\tilde{x}^a$ is the aggregate demand for commodities and $\tilde{e}^a$ the aggregate endowment of commodities other than the numéraire across states of the world, $\tilde{y}^a$ is the aggregate demand for assets other than the numéraire, and $y_+^a = \sum_{i \in \mathcal{I}} y_+^i$.

A function F is defined by

$$F = (\dots, F^i, \dots, F^0)';$$

elements of the domain of the function are

$$(\xi, \zeta, \omega) = (\chi, \psi, \lambda, \mu, p, q, T, t, \delta, u, e),$$

where

$$\xi = (\chi, \psi, \lambda, \mu, p, q, T)$$

are endogenous variables, with $\lambda = (\dots, \lambda^i, \dots)$, and $\mu = (\dots, \mu^i, \dots)$.

The domain of endogenous variables is Ξ, an open set of dimension $N = (ILS + IA + IS + I + S(L-1) + (A-1) + 1)$, which coincides with the dimension of the range of the function F.

For an economy $\omega \in \Omega$, a competitive equilibrium, with fiscal policy $\overline{\zeta} = (0, \overline{\delta})$, augmented with the associated Lagrange multipliers of the budget constraints of individuals, is determined as a solution to the system of equations

$$F_{(\overline{\zeta}, \omega)}(\xi) = 0.$$

Competitive equilibria, with inactive fiscal policy, exist:

$$F_{(\overline{\zeta}, \omega)}^{-1}(0) \neq \emptyset.$$

The argument in Geanakoplos and Polemarchakis (1986) applies.

Lemma 1 *There exists a generic subset of economies Ω^0, such that, for every economy $\omega \in \Omega^0$,*

1. *the function $F_{(\overline{\zeta}, \omega)}$ is transverse to 0:*

$$\dim[D_\xi F_{(\overline{\zeta}, \omega)}] = N,$$

   and, at a competitive equilibrium, with inactive fiscal policy,

2. *every individual holds a non - zero position in every asset:*

$$F_{(\overline{\zeta}, \omega)}(\xi) = 0 \quad \Rightarrow \quad y_a^i \neq 0, \quad a \in \mathcal{A}, i \in \mathcal{I},$$

   and

3. the matrix

$$\begin{pmatrix} \vdots \\ \lambda^i \otimes z^i \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \dots \lambda^i_s z^i_s \dots \\ \vdots \end{pmatrix}$$

has full row rank, I.

In this, as well as the next lemma, it is sufficient to consider perturbations only in endowments.

The domain of endogenous variables, ξ, with $y^i_a \neq 0$, for all assets and all individuals, is Ξ^0, an open set.

Lemma 2 *For every economy $\omega \in \Omega^0$, there exists an open set of fiscal policies, $\mathcal{O}_\omega \subset \mathcal{Z}$, such that*

1. *$\overline{\zeta} \in \mathcal{O}_\omega$,*

2. *if $\zeta \in \mathcal{O}_\omega$, then competitive equilibria, $\xi \in \Xi^0$, with fiscal policy ζ for the economy ω exist, they are obtained as solutions to the system of equations*

$$F_{(\zeta,\omega)}(\xi) = 0,$$

and they are locally smooth functions of the fiscal policy parameters ζ and of the quadratic perturbations M :

$$d\xi = -(D_\xi F)^{-1}(D_\zeta F d\zeta + D_M F dM).$$

The matrix in item 3 of Lemma 1 represents the relative price effects of tax reforms. For a marginal change of policy instruments, $\Delta\zeta \in \mathcal{R}^{A+I}$, and fixing $\Delta q = 0$ and $\Delta y^i = 0$, the change in individual i's indirect utility induced via a relative spot prices change is $\Delta u^i = \left(\lambda^i Z^i\right) \otimes D_\zeta p \Delta\zeta$. Then, item 3 of Lemma 1 guarantees that there is sufficient variation of utilities due to price effects. This fact is key in establishing constrained suboptimality of equilibria.

4   Pareto improving fiscal policy

Associated with an allocation of commodities, there is an allocation of utilities, $u(\chi) = (\dots, u^i(x^i), \dots)$.

An allocation of commodities, $\chi = (\dots, x^i, \dots)$, is strictly Pareto superior to another, $\chi' = (\dots, x^{i\prime}, \dots)$, if $u(\chi) \gg u(\chi')$.

Definition 2 *A feasible allocation of commodities is strictly constrained Pareto suboptimal if there exists a strictly Pareto superior, competitive equilibrium allocation with fiscal policy.*

Constrained interventions are restricted to the taxation of trades in assets and the distribution of fiscal revenue. They respect the asset structure, they yield an ex post optimal allocation of commodities; they are fully anticipated and they do not force the closure of asset markets; in the case of the uniform distribution scheme, they are anonymous.

Proposition 1 *Generically, every competitive equilibrium allocation, with inactive fiscal policy, $\overline{\zeta} = (0, \overline{\delta})$, is strictly constrained Pareto suboptimal. The fiscal policy that implements the strict Pareto improvement can be restricted to involve (i) a uniform distribution of fiscal revenue, (ii) no fiscal revenue or (iii) the taxation of trades in only one asset.*

Proposition 1 considers three different fiscal policy regimes. The first regime, (i), uses the complete array of tax (and/or subsidy) instruments, t_a, and tax revenue, T, is distributed uniformly to individuals. The second regime, (ii), coincides with the first, except that it imposes fiscal balance: $T = 0$, which eliminates the need for a distribution scheme, but reduces by 1 the available instruments. The third regime, (iii), employs only 1 tax (or subsidy) instrument, t_1, but tax revenue is distributed through individual specific lump-sum transfers, $\delta^i T$; the single tax instrument can be either an asset specific tax imposed on any asset $a = 1$, or a uniform tax rate imposed on every asset purchase.

The fact that purchases and not sales are taxed is immaterial.

When Pareto improvement is obtained with (uniformly distributed) fiscal revenue, it suffices that $A \geq I$. If fiscal balance, $T = 0$, is imposed, then $A \geq I + 1$ is required. With individual-specific redistribution and taxation of only one asset, the required condition on the number of assets is $S - A \geq I$; this is in line with Pareto improvement obtained through lump-sum taxation, as in Citanna, Kajii and Villanacci (1998), where lump-sum before the resolution of uncertainty need not suffice for a Pareto improvement.

Assumption 4, item 2 summarizes the conditions discussed in the previous paragraph. Under (i) and (ii), these conditions correspond to the natural requirement that the number of assets exceed the number of individuals: instruments are taxes on trades in assets, while targets are the utility levels of individuals at equilibrium.

Sufficient incompleteness in the market for assets relative to the number of individuals, $S - A \geq I$, also in Assumption 4, item 2, and sufficient variability in the characteristics – and, consequently, the behavior – of individuals guarantee that marginal utility of income across individuals is maximally dispersed. In this case, taxation and redistribution not only affect non-trivially the spot price of commodities, through the state - contingent individual income profiles, but also translate into utility changes independent across individuals, hence controllable.

The proof of Proposition 1 follows the reasoning developed in Citanna et al.

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(1998, p.503), which is as follows:

$$\widetilde{\Phi}(\xi, \zeta, \omega) = \begin{pmatrix} D_\xi F & D_\zeta F \\ D_\xi u & 0 \end{pmatrix}$$

represent the derivative of the equilibrium system and of the utility vector at $\overline{\zeta} = (0, \overline{\delta})$. An equilibrium is constrained suboptimal if and only if the row rank of $\widetilde{\Phi}(\xi, \zeta, \omega)$ is full; constrained suboptimality is nothing but a violation of the first order conditions for a vector maximum or it obtains when u is a submersion on the equilibrium set. The rank condition on $\widetilde{\Phi}(\xi, \zeta, \omega)$ is indeed equivalent to showing that the system defined by

$$F_{opt}(\xi, b; \zeta, \omega) = \begin{pmatrix} b_1 D\tilde{F} + b_2 Du \\ \|b\| - 1 \end{pmatrix} = 0,$$

where (b_1, b_2) is a vector of dimension $N + I$, has no solution (ξ, b) at $F_{(\overline{\zeta}, \omega)}(\xi) = 0$, for any given ω; $D\widetilde{F}$ is nothing but an appropriately chosen submatrix of the derivative matrix DF. Hence if a planner were to choose ζ (a tax policy) to maximize the utility vector u (the social welfare) subject to $\tilde{F} = 0$, the value $\zeta = \overline{\zeta}$ would not satisfy the conditions for an optimum: $b_1 Du + b_2 D\widetilde{F} = 0$ is not solved at $\zeta = \overline{\zeta}$, and hence a tax reform (a change in taxes and redistributions) would do better. Constrained suboptimality is equivalent to the existence of a feasible direction of tax reforms in the sense of Guesnerie (1977, 1979).

Constrained suboptimality holds for a generic subset of economies in Ω. In order to show density, and using the quadratic, finite-dimensional parametrization M of utility functions, it suffices to show, according to Citanna et al. (1998, Proposition 3), that the matrix

$$D_{b,M} F_{opt} = \begin{pmatrix} D_b F_{opt} & D_M F_{opt} \end{pmatrix}$$

has full row rank.

5 Proofs

The derivative $D_{\xi,\zeta} F_\omega(\xi, \zeta, M)$ of the equilibrium system (later used to compute F_{opt}), evaluated at $(\zeta, M) = (\overline{\zeta}, 0)$, has the following structure:

$$
\begin{pmatrix} DF_I \\ DF_{II} \\ DF_{\;III} \\ \quad IV \\ DF_V \\ DF_{VI} \\ DF_{VII} \end{pmatrix}
=
\begin{pmatrix}
& (x^i) & (y^i) & (\mu^i,\lambda^i) & (\widetilde{p}) & (\widetilde{q}) \\[2ex]
& D^2 u^i & 0 & -[0\ P'] & \Lambda^i & 0 \\[2ex]
& 0 & 0 & W' & 0 & -\mu^i \begin{bmatrix} 0 \\ I_{A-1} \end{bmatrix} \\[2ex]
& -\begin{bmatrix} 0 \\ P \end{bmatrix} & W & 0 & \begin{bmatrix} 0 \\ Z^i \end{bmatrix} & \begin{bmatrix} \vdots \\ -\widetilde{y}'_i \\ 0 \\ \vdots \end{bmatrix} \\[3ex]
& \widetilde{I}_{S(L-1)} & 0 & 0 & 0 & 0 \\[2ex]
& 0 & [0\ I_{A-1}] & 0 & 0 & 0 \\[2ex]
& 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\begin{pmatrix} DF_I \\ DF_{II} \\ DF_{\;III} \\ \quad IV \\ DF_V \\ DF_{VI} \\ DF_{VII} \end{pmatrix}
=
\begin{pmatrix}
& (T) & (t_a) & (\delta^i T) \\[2ex]
& 0 & 0 & 0 \\[2ex]
& 0 & \begin{bmatrix} \vdots \\ \mu^i Q_i \\ \vdots \end{bmatrix} & 0 \\[3ex]
& \begin{bmatrix} -1/I \\ 0 \end{bmatrix} & \begin{bmatrix} \vdots \\ \left(\begin{array}{c} q \otimes y^i_+ \\ 0 \end{array} \right) \\ \vdots \end{bmatrix} & \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\[3ex]
& 0 & 0 & 0 \\[2ex]
& 0 & 0 & 0 \\[2ex]
& 1 & -q \otimes y^a_+ & 1
\end{pmatrix},
$$

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where the first row has labels corresponding to variables of derivation; the following notation has been used: $\widetilde{I}_{S(L-1)} = [0\ I_{L-1}, ..., 0\ I_{L-1}]$ is a matrix formed by S times the $(L-1)L$ - dimensional matrix of first column of zeros and the $L-1$ - dimensional identity matrix,

$$P \quad = \quad \begin{pmatrix} \ddots & & \\ & p_s & \\ & & \ddots \end{pmatrix}_{S \times SL} , Z^i = \begin{pmatrix} \ddots & & \\ & -z_s^{i\backslash} & \\ & & \ddots \end{pmatrix}_{S \times S(L-1)} ,$$

$$\Lambda^i \quad = \quad \begin{pmatrix} -\lambda_1^i \begin{pmatrix} 0 \\ I_{L-1} \end{pmatrix} & & \\ & \ddots & \\ & & -\lambda_S^i \begin{pmatrix} 0 \\ I_{L-1} \end{pmatrix} \end{pmatrix}_{S(L-1) \times S(L-1)} ,$$

$$\mu^i Q_i \quad = \quad \begin{pmatrix} \ddots & & \\ & -\mu^i q_a I(y_{a,+}^i) & \\ & & \ddots \end{pmatrix}_{AI \times A} , W = \begin{pmatrix} -q^i \\ R \end{pmatrix} ,$$

where a backslash, $\backslash$, on a variable denotes that the first component has been deleted, and $I(y_{a,+}^i) = 1$ if $y_a^i > 0$, and is zero otherwise.

The derivative with respect to δ^i is zero everywhere at the initial point $(\overline{\zeta}, 0)$. Hence, changes in the distribution are ineffective. However, changes in the level of $\delta^i T$ of revenue distributed can be effectively used (not in conjunction with changes of T), as the corresponding derivative is nonzero. This is the reason for differentiating with respect to $\delta^i T$, rather than δ^i.

The three possible taxation methods for I objectives (the utility vector) and one budget constraint (equation (VII)) are

1. (T, t), a total of $1 + A$ instruments, requiring $1 + A \geq 1 + I$, or $A \geq I$;

2. (T, t) with $T = 0$, a total of A instruments, requiring $A \geq 1 + I$.

3. $(\delta T, t)$, a total of $I + A$ instruments, requiring $I + A \geq 1 + I$, or $A \geq 1$.

Calculations for each of the three methods correspond to the statement of Proposition 1.

Lemma 1, item 1 is straightforward. From Proposition 2 in Geanakoplos Polemarchakis (1986), rows (I) through (VI) of the matrix DF are linearly independent in a generic set Ω^0. As for (VII), $T = (t \otimes q)y_+^a$ pins down uniquely

$T = 0$ at $\overline{\zeta}$ ($t = 0$), guaranteeing upper - hemicontinuity of the equilibrium correspondence, and openness. Moreover, this equation can be perturbed: either using T (for regime (1)), while any adjustment of F^i will not have impact on this equation, again since $t = 0$; or using $\delta^1 T$ (for regime (2)).

As for Lemma 1, item 2, the property may be shown to hold by appending the equation $y^i_a = 0$ to the equilibrium system $F_{(\zeta,\omega)}(\xi) = 0$, with or without equation (VII), and applying a standard transversality argument. This also shows that equation (VII) can be perturbed (regime (3)) using t_1, for instance, since $y^i_{1,+} \neq 0$ for at least one i. Indeed, $y^i_a \neq 0$ for all i, a, when considering only the system of (I) through (VI) at $\overline{\zeta}$, and we have $\widetilde{y}^a = 0$. Similarly, for Lemma 1, item 3, it is enough to show that the I- dimensional submatrix with $s = 1, 2, ..., I$ and $l = L$ has full rank I. But it can be easily seen (by relabeling spots, if needed) that using λ^1_1 to perturb the first row, ...,λ^i_1 to perturb the ith, and so on, the system

$$\begin{pmatrix} \lambda^1_1 z^1_{L,1} & \cdots & \lambda^1_I z^1_{L,I} \\ \\ \vdots & & \vdots \\ \\ \lambda^I_1 z^I_{L,1} & \cdots & \lambda^I_I z^I_{L,I} \end{pmatrix} a = 0,$$

$$a'a - 1 = 0,$$

appended to the equilibrium system when $\zeta = \overline{\zeta}$, has no solution in a generic set of parameters, again a transversality argument. This is where we use Assumption 4, item 2, and particularly that $S - A \geq I$.

In order to prove Lemma 2, for any given $\omega \in \Omega^0$, one parametrizes the utility using M, the quadratic perturbation term, and considers, for any initial $\omega \in \Omega^0$, the associated finite - dimensional parametrization (e, M) where $M = 0$. The existence of a competitive equilibrium and Lemma 1, item 1, allow the application of the Implicit Function Theorem to claim that, for any $\omega \in \Omega^0$, the system of equations $F_\omega(\xi, \zeta, M) = 0$ has a locally unique solution in an open neighborhood around $(\overline{\xi}, \overline{\zeta}, 0)$ where $\overline{\xi}$ is such that $F_{(\overline{\zeta},\omega)}(\overline{\xi}) = 0$. The Implicit Function Theorem also shows that projecting this neighborhood we get an open set O_ω around $\overline{\zeta}$, for each $\omega \in \Omega^0$. The same reasoning shows that ξ is a smooth function of ζ and M, the "independent" variables, so that the derivative can be computed as claimed.

The vector of coefficients in $F_{opt} = 0$ is

$$b = (b_1, b_2) = (\alpha, \beta, \gamma, \delta, \epsilon, \theta, b_2)'.$$

In these coefficients, $F_{opt} = 0$ writes as the system of equations

$$\alpha \qquad \alpha^i D^2 u^i - \gamma^{i\backslash} P + \delta \widetilde{I}_{S(L-1)} + b_2^i D u^i = 0, \quad \text{all } i \qquad (i)$$

$$\beta \qquad \gamma^i W + (0 \; \varepsilon) = 0, \quad \text{all } i \qquad (ii)$$

$$\gamma \qquad -\alpha^i (0 \; P') + \beta^i W' = 0, \quad \text{all } i \qquad (iii)$$

$$\psi \qquad \sum_i (\alpha^i \Lambda^i + \gamma^{i\backslash} Z^i) = 0, \qquad (iv)$$

$$\varepsilon \qquad \sum_i (\mu^i \beta^{i,a} + \gamma^{i,0} y_a^i) = 0, \quad \text{all } a > 1 \qquad (v)$$

$$\theta \qquad -\sum_i (1/I)\gamma^{i,0} + \theta = 0, \qquad (vi.a)$$
$$\sum_i [\beta^{i,a} \mu^i q_a I(y_{a,+}^i) - \gamma^{i,0} q_a y_{a,+}^i] + \theta q_a \sum_i y_{a,+}^i = 0, \quad \text{all } a \quad (vi.b)$$

$$b_2 \qquad -\gamma^{i,0} + \theta = 0, \quad \text{all } i \qquad (vi.c)$$
$$b_2' b_2 - 1 = 0, \qquad (vii)$$

where $\gamma^i = (\gamma^{i,0}, \gamma^{i\backslash})$. The last equation must be true for otherwise, in a generic set of economies, this would contradict regularity of the original incomplete markets equilibrium, Lemma 1, item 1. The first column displays the matching of variables to equations. Of equations (vi), only equations $(vi.a)$ and $(vi.b)$ should be counted using taxation regime (1); only $(vi.b)$ should be counted with regime (2); only $(vi.b)$ and $(vi.c)$ should be counted with regime (3). The number of equations, at least $N + A$, is greater than the number of variables b under Assumption 4, item 2, and the remaining variables ξ are matched by the equations $F = 0$, in number N. Therefore, that system, generically, has no solution as long as $D_{b,M} F_{opt}$ has full rank, as previously stated.

The quadratic finite-dimensional parametrization of utility used to compute $D_M F_{opt}$ allows one to perturb the Hessian of the utility function without altering its gradient at any equilibrium point. This is obtained by choosing x^{i*} to be the equilibrium consumption plan. For an economy $\omega \in \Omega^0$, equilibria are locally finite, Lemma 2, so that this construction is well-defined.

It is now possible to demonstrate the results concerning constrained suboptimality.

Lemma 3 *Constrained suboptimality is dense: for a dense subset of economies, Ω^{**} of Ω^0, $F_{opt} = 0$ has no solution.*

Proof The argument involves three methods of taxation and distribution of revenue. For each method, the proof is split in two cases, according to whether or not utility perturbations are effective — case a and case b, respectively.

Method 1: Using (T, t).

One deletes equations $(vi.c)$ and, possibly, some equations $(vi.b)$, reducing the number of equations to to I.

Case a ($\alpha^i \neq 0$, for all i): One perturbs equations (i) using M^i; equations (ii) using $\gamma_i^{\backslash}$ — this is possible by Assumption 3, item 1: $\dim[R] = A.$; equations (iii) using $\alpha^{i,s,1}$, all s, i, and equations (iv) using $\alpha^{i,s,l}$ some i, all (s, l) with $l \neq 1$.

From Lemma 1, item 2:

   i for each a, there exists $i(a)$ with $y_a^{i(a)} > 0 : I(a) \equiv \{i \in I : y_a^i > 0\} \neq$;

   ii for each a there exists $i'(a)$ with $y_a^{i'(a)} < 0 : I'(a) \equiv \{i \in I : y_a^i < 0\} \neq$;

It follows that one can use $\beta^{i,a}$ with $i \in I'(a)$ to perturb equations (v); to perturb equation $(vi.a)$ one can use θ, and choose $\beta^{i,a}$ with $i \in I(a)$ to perturb the a-th equation $(vi.b)$. Equation (vii) one perturbs by using b_2^i, for some i. The rank of $D_{b,M} F_{opt}$ is full.

Case b ($\alpha^i = 0$, some i): To fix ideas, and without loss of generality, $i = 1$. Then, taking $l = 1$ and combining equation (i), $Du^1 = (1/b_2^1)\gamma^{1\backslash}$, with the first order conditions for $i = 1$, $Du^1 = \lambda^1$, one obtains $\gamma^{1\backslash} = b_2^1 \lambda^1$. Therefore, (i) holds only if $\delta = 0$. Similarly, from (ii) and the first order conditions, $\gamma^{i,0} = b_2^i \mu^i$, all i, and $\varepsilon = 0$; from no redundancy and (iii), $\beta^1 = 0$.
It is immediate that, for $i > 1$, $Du^i \alpha^i = 0$ (again one uses (iii) and the first order conditions), while, if $\alpha^i \neq 0$, $\alpha^i D^2 u^i \alpha^{i'} = 0$, contradicting differential strict quasi-concavity of u^i (Assumption 1). Thus, $\alpha^i = 0$ all i, and $\gamma^i = b_2^i(\mu^i, \lambda^i)$ (i.e. γ^i is colinear to (μ^i, λ^i)), $\beta^i = 0$ for all i. Substitution into the system of equations yields that (iv) becomes $\sum_i b_2^i \lambda^i Z^i = 0$. Rewriting this equation, yields

$$(b_2^1, \ldots, b_2^I) \begin{pmatrix} \lambda^1 Z^1 \\ \vdots \\ \lambda^I Z^I \end{pmatrix} = 0.$$

Out of these $S(L-1)$ equations, one extracts the I equations corresponding to the submatrix

$$\begin{pmatrix} \lambda_1^1 z_{L,1}^1 & \cdots & \lambda_I^1 z_{L,I}^1 \\ \vdots & & \vdots \\ \lambda_1^I z_{L,1}^I & \cdots & \lambda_I^I z_{L,I}^I \end{pmatrix}$$

By Lemma 1, item 3, this submatrix has full rank I and therefore (iv) implies $\nu = 0$, a contradiction to (vii), or $b_2' b_2 = 1$. Hence $\alpha^i = 0$, some i, cannot be (or there is no solution to the system of equations in this case).

Methhod 1 can be applied when $A \geq I$, a weaker requirement than $A - 1 \geq I$.

Method 2: Using (T, t) with $T = 0$.

We can delete equation $(vi.a)$, and equations $(vi.c)$. Some equations $(vi.b)$ can possibly be deleted reducing them to $I + 1$.

Case a: Equations (i) through (v) are perturbed as in Method 1; assuming without loss of generality and possibly after relabeling that the equations $(vi.b)$ left are those corresponding to $a \leq I + 1$, they are perturbed using $\beta^{i,a}$, with $i \in I(a)$ for each $a \leq I$, while the last equation $(vi.b)$ is perturbed using θ. Equation (vii) is perturbed using $b^i_,$ some i.

Case b: Exactly as in Method 1.

Method 2 requires $A - 1 \geq I$.

Method 3: Using $(\delta T, t)$.

One deletes equation $(vi.a)$ (T cannot be used as independent instrument), and all but one equation $(vi.b)$. Hence, effectively, one always use the policy variables $(\delta T, t_1)$. In using this method, it is always possible to drop all but one equation $(vi.b)$. Hence a suboptimality result can be obtained either by setting $t_a = 0$, for all $a > 1$, or by setting $t_a = t$ for all a, that is, by applying a uniform taxation of asset trades.

Case a: Equations (i) through (iv), and equation (vii) are perturbed as in Method 1. Equations (v) are perturbed using β_i, some i. Equation $(vi.b)$ is perturbed using θ, and $(vi.c)$ are perturbed with $\gamma^{i,0}$, all i.

Case b: As in previous methods. Method 3 does not require any lower bound on the asset number.

This concludes the proof. □

To illustrate the selection idea in the proof (Methods 1 and 2, (case a)), one considers the following example: there are $I = 3$ individuals and $A = 4$ assets. In the table, stars denote $y^i_a > 0$:

	$i = 1$	$i = 2$	$i = 3$
$a = 1$	$*$	$*$	
$a = 2$		$*$	$*$
$a = 3$		$*$	
$a = 4$	$*$		

This matrix, which represents the derivative of $(vi.b)$ with respect to $\beta = (\beta^1, \beta^2, \beta^3)$ up to an isomorphism, has, from (ii), is a 0 and, from (i), a $*$ in each row. For the example, the following perturbation works:

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β_3^1 in (v) and β_1^1 in (vi),

β_1^2 in (v) and β_2^2 in (vi),

β_1^3 in (v) and β_2^3 in (vi),

β_1^2 in (v) and θ in (vi).

The last part of the proof for case b) does not go through if $L = 1$: equation (iv) (corresponding to the columns of matrix DF obtained by differentiating with respect to p) would not be in the system. The system of equations in this case is essentially only in $b_2^1, \ldots, b_2^I$, and θ, with $I + 1$ unknowns, is solved by setting $\theta = \nu_i \mu^i$ (equation $(vi.c)$ in Method 3): (v) is implied by asset market clearing, and $(vi.b)$ cancels out. The choice of θ must guarantee $b_2' b_2 = 1$. Hence when $L = 1$ there is always a solution to the system, and generic constrained suboptimality cannot be shown.

Proof of Proposition 1 Method 1 corresponds to no change in δ, the revenue distribution, method 2 corresponds to no lump - sum redistribution and method 3 to taxing only one asset. Lemma 3 then established density of the constrained suboptimality property, and we are left with showing that Ω^{**} is open in Ω^0. But this is a trivial exercise, since properness of the natural projection for the system $F(\xi, \zeta, \omega) = 0$ of equilibrium equations at $\zeta = \bar{\zeta}$ is already known — Citanna et al.(1998, Lemma 1). This concludes the proof. $\square$

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