CONSTRAINED INEFFICIENCY IN GEI: A GEOMETRIC ARGUMENT

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Abstract. In this paper we use global analysis to study the welfare properties of general equilibrium economies with incomplete markets (GEI). Our main result is to show that *constrained Pareto optimal* equilibria are contained in a linear submanifold of the equilibrium set. This result is explicitly derived for economies with real assets, of which real numéraire assets are a special case.

1. **Introduction**

Since Radner (1972) there has been a large body of literature studying general equilibrium economies with incomplete markets (GEI). The analysis pioneered by Arrow (1951) and Debreu (1960) on economies with uncertainty and complete markets has been extended in this new direction with contributions addressing traditional issues, such as existence and efficiency of equilibria (see Geanakoplos (1990) and Magill and Shafer (1991) for up to date surveys).

In GEI equilibria are typically not Pareto optimal. Moreover, this result persists also under weaker notions of efficiency. Although, the literature has proposed different notions of *constrained Pareto optimality* (*CPO*), economists often refer to Diamond's (1967), Stiglitz's (1982), and Geanakoplos - Polemarchakis's (1986), as the benchmark. This notion is based on the idea that, when implementing an allocation, a central planner faces the same financial constraints of the private sector. In a pure exchange economy with multiple goods, this notion requires that the planner's *attainable* set contains allocations which are *a*) resource-feasible, *b*) achievable through portfolio transfers of the existing assets.

Stiglitz (1982) was the first to provide an argument for constrained inefficiency in GEI, with multiple commodities. The intuition behind Stiglitz's result runs as follows. A portfolio change does typically determine a change in spot prices and consumption allocations. Since markets are incomplete, and consumers' marginal rates of substitution are -in general- different, a change in spot prices may induce pecuniary externalities, which are not anticipated by price taking agents. A central planner that takes into account these externalities has an advantage that can be exploited in improving the market allocation of risk.

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Geanakoplos and Polemarchakis (1986), and later Geanakoplos et al (1990), generalized Stiglitz's result, respectively, in the context of pure exchange and production GEI. Precisely, they derived conditions to prove generic constrained suboptimality of equilibria. The argument used in these classical contributions, and in other papers that followed is to show that -under certain conditions- an equilibrium can be locally Pareto improved. Indeed, they proved that an equilibrium fails to be constrained efficient when it does not satisfy the first order conditions for *CPO*. Then, they show that this result is "robust", by establishing that it holds for a generic set of economies.[1]

In this paper we propose a different approach to the analysis of the welfare properties of equilibria, based on the global analysis of the equilibrium set. Our main result is to show that *CPO* equilibria are contained in a linear submanifold of the equilibrium set.

Surprisingly, very little attention has been given to the global properties of equilibria in GEI. Balasko and Cass (1989), and later Siconolfi and Villanacci (1991), were the first to provide a global characterization of equilibria in GEI, respectively, with variable and fixed resources. Their goal however was to use these characterizations to analyze the indeterminacy of equilibria. It was only with Zhou (1997) that welfare analysis entered into the picture, with the characterization of the set of Pareto optimal (first best) equilibria.

To pursue our goal, in **section 2** of the paper, we go back to the description and definition of the notion of *CPO*, and present the underlined planner's problem. We analyze this problem and characterize its first order conditions. Then, we also discuss some relevant, generic, properties of equilibria. For expositional purposes, section 2 refers to the case of economies with real numéraire assets that was extensively considered in the literature.

In **section 3**, we use our knowledge of the planner's problem to derive the structure of the set of *CPO* equilibria. This result is derived in steps. We first show that the equilibrium set has a fiber bundle structure. The choice of the parametrization is original, and it is driven by our ultimate goal. Yet, it turns out that the structure of the bundle shares most of the properties presented for the equilibrium set of a standard Walrasian economy in Balasko (1988); namely, every fiber is a linear submanifold of the equilibrium set, and it is uniquely identified by a no-trade equilibrium. Our final step is to show that *CPO* equilibria are contained in the fibers; and precisely that each fiber contains, at most, one *CPO* equilibrium. The bundle structure, and in particular the lower dimensionality of the set of *CPO* equilibria, can be exploited to establish (generic) constrained inefficiency of equilibria. We complete our analysis for economies with real assets, of which real numéraire assets are a special case. Extensions to nominal assets, and to mixed asset structures are straightforward.

Our geometric approach to constrained suboptimality is substantially different from the one that is used in the literature, and resumed in section 2 of the paper. It does not rely on the characteristics of the parameter space, but directly on equilibria. Moreover, it establishes constrained suboptimality without having to impose any specific measure theoretical structure; just using dimensionality arguments. Obviously, the two concepts, dimensionality and measure, can be linked.

[1] See, for example, Magill and Shafer (1991), for a discussion on this approach.

2. **Constrained Pareto optimality and equilibria**

In this section we provide a notion of constrained Pareto optimality (CPO) and equilibrium. For expositional reasons, we do so in the context of a standard GEI with real numéraire assets.

2.1. **Economy and equilibria.**

Economy. The economy is of pure exchange over two periods, with uncertainty, and finitely many individuals and commodities. There are two dates indexed by 0 and 1. Uncertainty is described by a finite number $S \geq 2$ of possible states of nature in date 1. Including date 0 as one of the states, we use the indexing $s = 0, 1, .., S$, and define $N = (S + 1)$. There are a finite number $H \geq 2$ of consumer types, indexed by $h = 1, .., H$. In each state, $L \geq 2$ commodities are available for consumption. A bundle of contingent commodities for h is a vector $x^h = \left(.., x^h_{sl}, ..\right)' \in \mathbb{R}^m_+$, where $m = NL$. We allow for economies with fixed aggregate resources, $\overline{\omega} \in \mathbb{R}^m_{++}$, by letting the initial distribution of commodities across agents, $e = \left(.., e^h_{sl}, ..\right)' \in \mathbb{R}^{mH}_{++}$, be an element of the set,

$$\Omega(\overline{\omega}) = \left\{ e \in \mathbb{R}^{mH}_{++} : \sum_h e^h - \overline{\omega} = 0 \right\}$$

When $\overline{\omega}_s = \omega$ for all s, the economy has no-aggregate uncertainty. Finally, in date 0, there are also $J \geq 1$ financial assets available for trade. Assets are in zero net supply.

Each consumer, h, is initially endowed of a (column) vector $e^h = (.., e^h_{sl}, ..)' \in \mathbb{R}^m_+$ of commodities. His preferences are represented by an ordinal utility function $u^h : \mathbb{R}^m_{++} \to \mathbb{R}$. Finally, some assumptions on endowments and preferences are summarized in the following, and will be maintained throughout the paper.

Assumption 1. *(strictly positive endowments):* $e^h \in \mathbb{R}^m_{++}$

Assumption 2. *(smooth preferences):*[2] $\forall h$, u^h *is* $\mathcal{C}^{r \geq 2}$*, strictly increasing,* $(Du^h(x) \in \mathbb{R}^m_{++}, \forall x \in \mathbb{R}^m_+)$*, strictly concave,* $(bD^2u^h(x)b' < 0, \forall x \in \mathbb{R}^N_{++}, \forall b \in \mathbb{R}^m$, $b \neq 0$*, such that* $Du^h(x)\, b' = 0)$*; indifference surfaces are bounded below* $(\forall x^* \in \mathbb{R}^m_{++}$, $\left\{x \in \mathbb{R}^m_{++} : u^h(x) \geq u^h(x^*)\right\} \subset \mathbb{R}^m_{++})$.

For simplicity, in most of the analysis we will also use the following assumption.

Assumption 3. *(state-separable utilities):* $\forall h$, $u^h = \sum_s U^h_s(x^h_s)$.

With an abuse of notation, we denote the set of utilities which satisfy assumptions 2 and 3, U; finally, we let $\mathcal{U} = \times_h U$.

Markets. Commodities and assets are, respectively, traded in spot and asset markets. Spot prices are a vector $P = (.., P_{sl}, ..) \in \mathbb{R}^m_+$, and $P_{\mathbf{1}} \in \mathbb{R}^{SL}_+$ denotes its date 1 component. Commodity $l = 1$ is the *numéraire*, and its price is normalized to one: $P_{s1} = 1$, for all s.

Assets are real numéraire. Each asset j is exchanged in date 0 (before uncertainty is resolved) at a price q^j, $q = (.., q^j, ..) \in \mathbb{R}^J$. Asset j is a claim for a contingent payoff $R^j = (.., R^j_s, ..)' \in \mathbb{R}^S_+$ expressed in units of the numéraire. R is the $S \times J$

[2]We use the standard notation, $Du^h = \left(.., D_{x_{s,l}} u^h(x^h), ..\right) \in \mathbb{R}^m$, where $D_{x_{s,l}} u^h(x^h) = \partial u^h(x^h)/\partial x^h_{s,l}$. Moreover, we use $\overline{X}$ to indicate the closure of a set X.

payoff matrix. $W(q,R)$ is the $N \times J$ asset matrix, composed by $-q$ in the first row and $R_s = (..,R_s^j,..) \in \mathbb{R}_+^J$ in each of the $s = 1,..,S$ subsequent rows. Asset markets are incomplete, $J < S$.

Competitive trade and equilibria. Let us fix $(u, R, \overline{\omega})$ and denote an economy simply by an endowment distribution e in $\Omega(\overline{\omega})$. At prices (P,q), the budget set of a typical consumer h is,[3]

$$\mathcal{B}(P,q,e^h) = \left\{x : P\square\left(x - e^h\right) = W(q,R)\,\theta,\ \theta \in \mathbb{R}^J\right\}$$

where $\theta = (..,\theta^j,..)' \in \mathbb{R}^J$ denotes a portfolio of assets. The action of h is, respectively, represented by the demand functions for commodities and assets,

$$\begin{aligned} x^h(P,q,e^h) &= \left\{x : x = \arg\max\ u^h(x)\, s.t.\, x \in \mathcal{B}(P,q,e^h)\right\} \\ \theta^h\left(P,q,e^h\right) &= \left\{\theta : P\square\left(x^h(P,q,e^h) - e^h\right) = W(q,R)\,\theta\right\}, \end{aligned} \tag{2.1}$$

and spot trades are defined by $z^h(P,q,e^h) = x^h(P,q,e^h) - e^h$. First order, necessary and sufficient, conditions imply that $q = \lambda^h R$, $\lambda^h \square P = \nabla u^h(x^h)$.[4]

Finally, we define the aggregate excess demand functions for non-numéraire commodities as,

$$Z(P,q,e) = \sum_h z^h(P,q,e^h) \in \mathbb{R}^{(S+1)(L-1)}$$

Definition 1. *(Equilibria with real numéraire assets) For fixed $(u,R,\overline{\omega})$, an equilibrium for an economy e in $\Omega(\overline{\omega})$ is a triplet $\left(\overline{P},\overline{q},e\right)$ such that $Z\left(\overline{P},\overline{q},e\right) = 0$, $\sum_h \theta^h\left(\overline{P},\overline{q},e^h\right) = 0$.*

Denote by $\mathcal{E}_R$ the set of equilibria with real numéraire assets with fixed $(u, R, \overline{\omega})$. It is well known that $\mathcal{E}_R$ has a smooth manifold structure.[5] Therefore, there exists a generic set of endowments such that equilibria are locally isolated and equilibrium variables do locally behave as smooth functions of the parameters.

2.2. **Constrained Pareto Optimality.** Although the literature has proposed different notions of *Constrained Pareto Optimality* (*CPO*) in GEI, economists often refer to Diamond's (1967), Stiglitz's (1982), and Geanakoplos - Polemarchakis's (1986), as the benchmark.[6] This criterion is based on the idea that, when implementing a centralized allocation, a fictitious planner faces the same financial constraints of the private sector. In a pure exchange economy with multiple goods, this notion implies that the planner's *attainable* set contains allocations which are a) resource-feasible, b) achievable through portfolio transfers in the existing assets. Note that, since transfers occur in the first period, the attainable set of consumption allocation depends on prices. More precisely, assume that $L \geq 2$, and that centralized portfolio transfers are decided in date 0, after markets have closed. Then, the final centralized, feasible, allocation will still depend on the possible trading activity taking place on date 1 spot-markets, and thus on $P_{\mathbf{1}}$.[7] Equivalently, a resource

[3]For any two vectors $x \in \mathbb{R}^S$, $y \in \mathbb{R}^{SL}$, we define $x\square y = (....,x_s\,(y_{s1},..y_{sl},..,y_{sL})\,,...) \in \mathbb{R}^{SL}$.

[4]$\nabla u^h(x) = (..,\nabla_{sl}u^h(x),..)$ is consumer h normalized gradient at x, and $\nabla_{sl}u^h(x) = D_{x_{sl}}u^h\,(x)\,/D_{x_{s1}}u^h\,(x)$.

[5]See Geanakoplos and Polemarchakis (1986), section 6.

[6]See Magill and Shafer (1991), chapter 5, for a discussion of this subject.

[7]A second notion, known as *weak constrained efficiency*, requires that prices must support the centralized allocation as a GEI equilibrium. This is equivalent to assume that the central planner intervenes when all markets are open. See Grossman (1977), and Grossman and Hart (1979).

feasible allocation, $x = (x_0, x_{\mathbf{1}})$, is attainable if there exist portfolio transfers θ such that x_s can be supported as a spot market equilibrium (P_s, x_s) of an economy $\widetilde{e}_s = e_s + R_s\theta$, for $s = 1, .., S$; where the following definition applies.

Definition 2. *(Spot-market equilibrium)* (P_s, x_s) *is a spot-market equilibrium in* s*, at initial endowments* $\widetilde{e}_s$*, if and only*
i) $x_s^h \in \mathbb{R}_{++}^L$ *maximizes* U_s^h *s.t.* $P_s(x_s^h - \widetilde{e}_s^h) = 0$*, for all* h
ii) $\sum_h (x_s^h - \widetilde{e}_s^h) = 0$.
A spot-market equilibrium in date 1 *is a spot-market equilibrium in* s*, for* $s = 1, .., S$*.*

Let $\widetilde{e}_{\mathbf{1}} = e_{\mathbf{1}} + \widetilde{R}\theta$, where $\widetilde{R} = (\widetilde{R}_1, .., \widetilde{R}_S)$ is a $SL \times J$ matrix whose typical column vector $\widetilde{R}_s$ is such that $\widetilde{R}_{sl}^j = R_s^j$ for $l = 1$, and 0 otherwise. A formal definition of the planner's *constrained feasible set* is,

Definition 3. *(Constrained feasible consumption allocation - CF) A consumption allocation* $x = (x_0, x_{\mathbf{1}})$ *is constrained feasible (CF) at* (x_0, e, R) *if there exists a* θ *and a* $P_{\mathbf{1}}$ *such that* $(P_{\mathbf{1}}, x_{\mathbf{1}})$ *is a spot-market equilibrium in date* 1*, at* $\widetilde{e}_{\mathbf{1}} = e_{\mathbf{1}} + \widetilde{R}\theta$*. A set of consumption allocations is said to be CF with transfers if it contains all those CF allocations which are attainable through date* 0 *transfers in the numéraire commodity.*

First, note that a competitive equilibrium is a spot market equilibrium. Let (P, q, e), with $P = (P_0, P_{\mathbf{1}})$, be a real numéraire equilibrium with allocation, (x, θ), for the economy (e, R). Then, $(P_{\mathbf{1}}, x_{\mathbf{1}})$ is a date 1 spot-market equilibrium at $\widetilde{e}_{\mathbf{1}} = e_{\mathbf{1}} + \widetilde{R}\theta$. Therefore, real numéraire equilibria are *CF*.

Moreover, since the planner's set of instruments is represented by portfolio transfers, at current market prices, these transfers and their corresponding consumption allocations are also attainable for the individual consumers.

Definition 4. *(Constrained efficient allocation - CPO) A consumption allocation is a CPO (with transfers) if it is not Pareto dominated by any other allocation that is CF (with transfers).*[8]

Before trying to explicitly write down the planner's problem capturing the definition of *CPO*, observe that the set of *CPO* is nonempty. Indeed, if R is a fixed real numéraire payoff matrix of full rank, the *CF* set is clearly nonempty and compact; hence a *CPO* is a *PO* restricted on the *CF* set.[9]

The planner problem is defined as follows. $x = (x_0, x_{\mathbf{1}})$ is *CPO* at (x_0, e, R), if and only if θ solves the following programming problem at $(e_{\mathbf{1}}, R)$:

$$\begin{array}{ll} Max_{(\theta^2,..,\theta^H)} \sum_h \delta^h v_{\mathbf{1}}^h \left(P_{\mathbf{1}}, m^h\right) & where \\ m^h = P_{\mathbf{1}} \Box e_{\mathbf{1}}^h + R\theta^h & \forall h \geq 2 \\ m^1 = P_{\mathbf{1}} \Box e_{\mathbf{1}}^1 - R \sum_{h \geq 2} \theta^h & \end{array} \tag{2.2}$$

[8] A modification to our notions of constrained feasibility and optimality would be to consider the case in which, after each policy intervention, only the asset markets close. In this case, true consumers would not be able to re-trade assets, but they would typically trade commodities on date 0 spots. Obviously, this becomes relevant in economies in which agents consume also in period 0, as for example in Magill and Shafer (1991). Then, we would say that $x = (x_0, x_{\mathbf{1}})$ is constrained feasible at (q, e, R) if and only if there exists a θ and a $P = (P_0, P_{\mathbf{1}})$ such that (P, x) is a spot-market equilibrium in both dates, at $(\widetilde{e}_0, \widetilde{e}_{\mathbf{1}}) = \left(e_0 - q\theta, e_{\mathbf{1}} + \widetilde{R}\theta\right)$. Since modifying our notion of constrained efficiency in the latter sense does not qualitatively affect the rest of the analysis, we shall hereafter refer to our definition 3.

[9] This set coincides with the set A described for the same purposes in Werner (1991).

where $v^h(\cdot)$ and δ^h, respectively, denote the indirect utility function and the welfare weight of h; while $P_{\mathbf{1}}$ is the price functional of a date 1 spot-equilibrium.[10]

It follows from the previous discussion that the central planner problem (2.2) has a solution.

Next, suppose that the spot price functional, $P_{\mathbf{1}}$, is differentiable. A *CPO* allocation satisfies the following,

$$\left(\delta^h \frac{\partial v_{\mathbf{1}}^h}{\partial m_{\mathbf{1}}^h} - \delta^1 \frac{\partial v_{\mathbf{1}}^1}{\partial m_{\mathbf{1}}^1}\right) R^j + \sum_{h'} \delta^{h'} \frac{dv_{\mathbf{1}}^{h'}}{dP_{\mathbf{1}}} \frac{\partial P_{\mathbf{1}}}{\partial \theta_j^h} = 0, \forall h \geq 2,\, \forall j. \tag{2.3}$$

where the first term on the left hand side is the aggregate income effect, and the second is the aggregate, relative price, effect.

Let $\widehat{\lambda}^h = \frac{\partial v^h}{\partial m^h}$ denote the vector of marginal utility of income for h in (2.2). Using Roy's identity,

$$\begin{aligned} \frac{dv_{\mathbf{1}}^h}{dP_{sl}} &= \frac{\partial v_{\mathbf{1}}^h}{\partial P_{sl}} + \frac{\partial v_{\mathbf{1}}^h}{\partial m^h} \frac{\partial m^h}{\partial P_{sl}} \\ &= -\widehat{\lambda}_s^h x_{sl}^h + \widehat{\lambda}_s^h e_{sl}^h = -\widehat{\lambda}_s^h z_{sl}^h \end{aligned}$$

Letting $\lambda_s^h \equiv \delta^h \widehat{\lambda}_s^h$, we can rewrite (2.3) as,

$$\left(\lambda^h - \lambda^1\right) R^j - \sum_{h',s} \sum_{l \neq 1} \lambda_s^{h'} z_{sl}^{h'} \frac{\partial P_{sl}}{\partial \theta_j^h} = 0,\ \textit{for all } h \geq 2,\ \textit{all } j$$

where $\sum_h z_{sl}^h = 0$ for all s and all $l \neq 1$.

Following Stiglitz (1982) one can test for the *CPO* of an equilibrium allocation by checking if the conditions in (2.3) hold. To ensure that the spot price functional, $P_{\mathbf{1}}$, is differentiable, we restrict the attention to the subset of economies, denoted by Ω', such that, for every $(e_0, e_{\mathbf{1}})$ in Ω', and every vector of transfers, θ, the spot-market economies $\widetilde{e}_{\mathbf{1}} = e_{\mathbf{1}} + \widetilde{R}\theta$ are regular: $P_{\mathbf{1}}(\widetilde{e}_{\mathbf{1}})$ is (at least) one time differentiable.

Then, consider an equilibrium (P, q, e), of an economy e in Ω', and allocations (x, θ). Let $\delta^h = 1/\lambda_0^h$, so that λ^h is the $S-$vector of normalized state prices of h.[11] Then, no-arbitrage implies $\left(\lambda^h - \lambda^1\right) R^j = 0$ for all h, j. The necessary conditions for the *CPO* of an equilibrium reduces to the following $J(H-1)$ equations,

$$\sum_h \left(\lambda^h \Box z_{\mathbf{1}}^h\right) D_\theta P_{\mathbf{1}} = 0 \tag{2.4}$$

where $D_\theta P_{\mathbf{1}} = (..., D_{\theta_j^h} P_{\mathbf{1}}, ...) \in \mathbb{R}^{S(L-1)\times(H-1)J}$, has typical column vector $D_{\theta_j^h} P_{\mathbf{1}} = (\partial P_{12}/\partial \theta_j^h, ..., \partial P_{SL}/\partial \theta_j^h)'$.

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[10] Note that each spot-equilibrium pairs (P_s, x_s) is a price-income equilibrium at (P_s, m_s), with $m_s = (.., m_s^h, ..)$, $m_s^h \equiv P_s \Box e_s^h + R_s \theta^h$. Where by a *price-income equilibrium*, for spot s (with aggregate resources $\overline{\omega}_s$) we mean a pair (P_s, x_s) such that x_s^h solves the consumer problem at (P_s, m_s^h) for all h, and $\sum_h x_s^h - \overline{\omega}_s = 0$.

[11] Consider the representation of a Planner problem in which δ^h is the Lagrange multipliers associated to the constraint, $v_{\mathbf{1}}^h(P, m) \geq \overline{v}_{\mathbf{1}}^h$, and take $\overline{v}_{\mathbf{1}}^h$ to be the utility level achieved at date 1, in a competitive equilibrium. Then, letting $\delta^h = 1/\lambda_0^h$ is equivalent to say that there exist welfare weights such that the original equilibrium satisfies CPO necessary conditions; this is the usual I welfare theorem. If, instead, we fix welfare weights, and we ask if an allocation that satisfies necessary conditions for CPO can be achieved at equilibrium, then we need to introduce date 0, transfers. The latter is the perspective of the II welfare theorem.

Observe that (2.4) can be interpreted by saying that a *CPO* equilibrium is such that there do not exist, feasible, portfolios redistributions that can induce indirect welfare effects; with "indirect effects" meaning effects propagating through changes in relative spot prices (i.e. "pecuniary externalities").

Remark 1. *(Assets redistribution, trade, and markets) It is now easy to see why in the notion of CPO it is important that centralized assets redistributions occur when, date* 0*, markets are closed. Precisely, consider an equilibrium* $\left(\overline{P}, \overline{q}, e\right)$ *with allocations* $(\overline{x}, \overline{\theta})$*. If we let agents trade in assets after a planner's (marginal) portfolios redistribution,* $d\theta$*, they would want to return to their original portfolios holdings,* $\overline{\theta}$*. In fact, 'expecting' prices to retain their initial value, their choice* $(\overline{x}, \overline{\theta})$ *would still be individually optimal, and satisfy market clearing at* $\left(\overline{P}, \overline{q}, e\right)$*. This is equivalent to say that, with ex-post re-trade of assets, for the economy* e*, the only CF allocation* x *at* $(\overline{x}_0, e, R)$ *is such that* $(\overline{P}_{\mathbf{1}}, \overline{x}_{\mathbf{1}})$ *is a no-trade spot-market equilibrium in date 1 at* $\widetilde{e}_{\mathbf{1}} = \overline{x}_{\mathbf{1}} = e_{\mathbf{1}} + \widetilde{R}\overline{\theta}$*. Clearly, in this case portfolio redistributions cannot have any welfare effects.*

2.2.1. ***Some sufficient conditions for (2.4) to hold***. A few, well known, sufficient conditions for (2.4) can be invoked.[12]

(1) Identical, individual, state-prices: $\lambda^h - \lambda^1 = 0$ for all h.
(2) No trade: $z^h = 0$ for all but one h.
(3) Policy interventions have no price effects: $D_\theta P_{\mathbf{1}} = 0$.

1. and 2. have obvious consequences, determining $\sum_h \lambda_s^h z_{sl}^h = 0$ for all s, all $l \neq 1$. Let us consider (3). Restricting to economies in Ω',

$$D_\theta P_{\mathbf{1}} = -(D_P Z_{\mathbf{1}}(P, m))^{-1} D_\theta Z_{\mathbf{1}}(P, m)$$

Therefore (3) occurs if and only if $D_\theta Z_{\mathbf{1}}(P, m) = 0$. Indeed, the latter holds if, at least locally (in a small neighborhood of the spot-equilibrium), preferences have identical *Gorman forms*; i.e. agents have identical marginal propensities to consume. Consider a special case of such preferences*:* identically homothetic utilities, $v^h\left(P, m^h\right) = A^h \sum_s \nu_s(P) m_s^h$, with $x_{sl}^h\left(P, m^h\right) = a_{sl}(P) m_s^h$, $a_{sl}(P) \equiv -\frac{\partial v_s(P)/\partial P_{sl}}{\nu_s(P)}$ independent of h. For all (s, l)

$$\begin{aligned} D_{\theta_j^h} Z_{sl} &= \left(\frac{\partial x_{sl}^h}{\partial m_1^h} - \frac{\partial x_{sl}^1}{\partial m_1^1}, \ldots, \frac{\partial x_{sl}^h}{\partial m_S^h} - \frac{\partial x_{sl}^1}{\partial m_S^1}\right) R^j \\ &= \left(a_{1l}(P) - a_{1l}(P), \ldots, a_{Sl}(P) - a_{Sl}(P)\right) R^j = 0 \end{aligned} \tag{2.5}$$

In the next subsection we will argue that none of the above conditions, (1) through (3), is generic.

2.3. **Generic properties of equilibria.** Property (1) does not typically hold in an equilibrium of a GEI economy. Instead, properties (2) and (3) are non-generic for competitive economies regardless their markets are complete or incomplete. More precisely, for (1) and (2), the following result is well known, and will not be proved here.[13]

[12] By assumption, we are ruling out economies with $L = 1$. This implying no relative price effects.

[13] At the individual optimum, the consumer gradient belongs to the subspace orthogonal to the one spanned by the columns of W, i.e. to a $(N - J) > 1$ dimensional space. Thus, generically,

Lemma 1. *There exists a generic set of economies, Ω^*, such that for every e in Ω^* and every (P, q, e) in $\mathcal{E}_R$, properties (1) and (2) do not hold.*

We devote a little more discussion to show why (3) is also non-generic.

Let us consider an economy e in Ω' (with preferences and payoff structure (u, R)). Define the individual h income in spot s as a function of her portfolio decisions, and thus of $\left(P, q, e^h\right)$: $m_s^h\left(P, q, e^h\right) = P_s e_s^h + R_s \theta^h\left(P, q, e^h\right)$. Then, clearly, for every real numéraire equilibrium, $\left(P, q, e^h\right)$, with $m_s = m_s\left(P, q, e\right)$, $\left(P_s, m_s\right)$ is a (price-income) spot-market equilibrium. This does also imply the identity, $x_s^h\left(P, q, e^h\right) \equiv \widetilde{x}_s^h\left(P_s, m_s^h\left(P, q, e^h\right)\right)$, where the latest is the demand function of h at $\left(P_s, m_s^h\right)$, with $m_s^h = m_s^h\left(P, q, e^h\right)$.

Next, it is well known that the Slutzky matrix is the inverse of the Jacobian of the individual demand system. Since here we are concerned with a spot-market individual demand, $\widetilde{x}_s^h\left(P_s, m_s^h\right)$, the latter reads:

$$\left(\begin{array}{cc} D^2 U_s^h & -P_s' \\ -P_s & 0 \end{array}\right)^{-1} = \left(\begin{array}{cc} \mathsf{S} & -\kappa' \\ -\kappa & -a \end{array}\right) \tag{2.6}$$

where for spot s, S is the matrix of substitution effects, and κ' is the matrix of the individual propensity to consume, $D_{m_s^h}\widetilde{x}$. Simple computations yield,

$$D_{m_s^h}\widetilde{x}_s^h = \left(D^2 U_s^h\right)^{-1} P_s' \left(P_s (D^2 U_s^h)^{-1} P_s'\right)^{-1} \tag{2.7}$$

where U_s^h is evaluated at $x_s^h = \widetilde{x}_s^h\left(P_s, m_s^h\right)$, $D_{m_s^h}\widetilde{x}_s^h = \left(..., \partial\widetilde{x}_{sl}^h / \partial m_s^h, ...\right)' \in \mathbb{R}_+^L$, with typical element,
$\frac{\partial \widetilde{x}_{sl}^h}{\partial m_s^h} = \left(P_s (D^2 U_s^h)^{-1} P_s'\right)^{-1} \sum_{i=1}^L P_{si} \left(\frac{\partial^2 U_s^h}{\partial x_{sl}^h \partial x_{si}^h}\right)^{-1}$.

In the next lemma we will parametrize a price-income, spot-market, economy also with respect to preferences, and define individual demands correspondingly.

Lemma 2. *Let $(P_{\mathbf{1}}, m_{\mathbf{1}})$ be such that $D_{m_s^h}\widetilde{x}_s^h$ are well defined. There exists an open and dense set of utility functions, $\mathcal{U}^* \subset \mathcal{U}$, such that at their individual optimum, $\widetilde{x}_s$, agents have different marginal propensities to consume:*

$$D_{m_s^h}\widetilde{x}_s^h \neq D_{m_s^1}\widetilde{x}_s^1$$

for all $(h, s) \geq (2, 1)$.

Proof: see the Appendix.

To prove the latter lemma, we restrict the attention to a class of utilities admitting (local) quadratic perturbations. As it can be inferred by looking at (2.6), this type of perturbations have the nice property that they can be rationalized as perturbations of the Slutzky matrix,[14] which do not affect individual first order conditions, and allocations.[15]

$N - J$, state prices are distinct. This is shown, for example in Magill and Shafer (1991), Theorem 10, and it is used to argue that equilibria are typically not Pareto optimal.

Finally, no-trade can be ruled out perturbing the individual endowments on the asset span. This type of perturbations will also be illustrated later in this section of the paper.

[14] See Geanakoplos and Polemarchakis (1980).

[15] This is important also because, in our context, the initial equilibrium values are real numéraire spot prices $P_{\mathbf{1}}$, consumption allocations $x_{\mathbf{1}}$, and hence state prices.

Having in mind equation (2.5), we are now ready to define the (generic) space of economies in which, at equilibrium, property (3) does not hold, (i.e. $D_\theta P_s \neq 0$). This clearly extends to the case of $S(L-1)$ spots. Exploiting state-separability of utilities, $[D_{m^h}\widetilde{x}^h]$ denotes the $S(L-1) \times S$, block-diagonal matrix of typical element $D_{m^h_s}\widetilde{x}^h_s = \left(..., D_{m^h}\widetilde{x}^h_{sl}, ...\right)' \in \mathbb{R}^{S(L-1)\times 1}$. Therefore, $D_\theta Z_{\mathbf{1}}(P,m) = (..., D_{\theta^h_j} Z_{\mathbf{1}}, ...) \in \mathbb{R}^{S(L-1)\times J(H-1)}$, with typical element,

$$D_{\theta^h_j} Z_{\mathbf{1}} = \left([D_{m^h}\widetilde{x}^h] - [D_{m^1}\widetilde{x}^1]\right) R^j \tag{2.8}$$

By the last lemma, generically, $D_{\theta^h_j} Z_{\mathbf{1}} \neq 0$. Therefore, since a *real numéraire* equilibrium in Ω' is a spot market equilibrium, we can state the following.

Corollary 1. *For every economy $(e,u) \in \Omega' \times \mathcal{U}^*$, real numéraire equilibria satisfy,*

$$D_\theta P_{\mathbf{1}} \neq 0$$

Next, going back to the planner problem, we observe that condition (2.4) is satisfied, at an equilibrium, if and only if $\sum_h \left(\lambda^h \square z^h_{\mathbf{1}}\right) D_{\theta^h_j} P_{\mathbf{1}} = 0$ for all $(h,j) \geq (2,1)$, where $D_{\theta^h_j} P_{\mathbf{1}} = -(D_P Z_{\mathbf{1}})^{-1} D_{\theta^h_j} Z_{\mathbf{1}}$ is also evaluated at equilibrium. The following result is central to establish constrained inefficiency.

Lemma 3. *There exists a generic set of economies $\Omega^* \subset \Omega'$ such that for every $(e,u) \in \Omega^* \times \mathcal{U}^*$, real numéraire equilibria, $\left(\overline{P}, \overline{q}, e\right)$, satisfy $\sum_h (\overline{\lambda}^h \square \overline{z}^h_{\mathbf{1}}) D_{\theta^h_j} P_{\mathbf{1}} \neq 0$, for some h, j.*

Proof: see the Appendix.

The proof of the latter lemma relays on standard perturbations of the endowments. In fact, suppose that an equilibrium, $\left(\overline{P}, \overline{q}, e\right)$, is such that $\sum_h (\overline{\lambda}^h \square \overline{z}^h_{\mathbf{1}}) D_{\theta^h_j} P_{\mathbf{1}} = 0$, for all h, j. Then, perturb the endowments of some h and of 1 as to keep the value of transfers of h, $P_{\mathbf{1}} \square z^h_{\mathbf{1}}$, on the asset span, and satisfy market clearing. This will not modify any of the equilibrium variables; yet, by changing $(z^h_{\mathbf{1}}, z^1_{\mathbf{1}})$ these perturbations will change the value of $\sum_h (\overline{\lambda}^h \square z^h_{\mathbf{1}}) D_{\theta^h_j} P_{\mathbf{1}}$, whenever h is chosen such that $\overline{\lambda}^h \neq \overline{\lambda}^1$.

2.4. **Constrained inefficiency of equilibria.** Just to summarize, consider a regular equilibrium, and a planner's redistribution of portfolios. Since this occurs at date zero, when asset markets have closed, agents are prevented from re-trading assets; still portfolio changes have modified their date 1- income distribution. Thus, at the new income distribution, agents may want to modify their consumption allocations. Precisely, if their marginal propensities to consume are different, this will occur in such a way as to modify relative spot prices. Because markets are incomplete, and marginal rates of substitutions are typically different across agents, the former may produce an aggregate welfare gain. Date 0 transfers, if feasible, can be used by the planner to redistribute such gain so as to achieve a Pareto improvement. More precisely, applying lemma 3, we have:

Theorem 1. *(Constrained inefficiency of real numéraire equilibria) For every (e,u) in $\Omega^* \times \mathcal{U}^*$, real numéraire equilibria, $\left(\overline{P}, \overline{q}, e\right)$, are not CPO with transfers.*

The reason why date 0 transfers are needed is linked with the usual instrument-objective requirement. In absence of transfers, the planner can at most control $S(L-1)$ instruments, the relative price changes, attained by portfolios redistributions, to achieve a Pareto improvement over H consumers. This is possible if $H \leq S(L-1)$.[16]

2.4.1. ***Constrained inefficiency with a "large" number of types.*** When the economy is characterized by a sufficiently large number of consumer types, $H > S(L-1)$, the arguments used above, in lemma 2 leads to establish that, generically in utility space, $D_\theta P_{\mathbf{1}}$ has full row rank, $S(L-1)$, at an individual optimum.[17] This implies that $P_{\mathbf{1}}$ is a surjective function of θ (by $\widetilde{e}$), and thus that the planner can effectively control relative spot-prices through portfolios. This is a clear advantage the planner has on price-taking consumers.

Equivalently, $x = (x_0, x_{\mathbf{1}})$ is a *CPO* at (x_0, e, R), if and only if $\left(..., x_{\mathbf{1}}^h, .., \theta^h, .., P_{\mathbf{1}}\right)$ maximize $\sum_h \delta^h u_{\mathbf{1}}^h \left(x_{\mathbf{1}}^h\right)$ such that the following constraints hold,

$$\begin{array}{lll} & (.., \mu_s^h, ..) & P_{\mathbf{1}} \Box (x_{\mathbf{1}}^h - e_{\mathbf{1}}^h) - R\theta^h = 0, \; all\, h \\ (2.9) & (.., \zeta^j, ..) & \sum_h \theta_j^h = 0, \; all\, j \\ & (.., \eta_{sl}, ..) & \sum_h (x_{\mathbf{1}l}^h - e_{\mathbf{1}l}^h) = 0, \; all\ l \geq 2 \end{array}$$

the terms in parenthesis, on the left hand side, refer to the multipliers attached to the corresponding constraints.

The first order conditions of this problem, evaluated at equilibrium, hold if and only if

$$\sum_h \left(\lambda^h \Box z_{\mathbf{1}}^h\right) = 0 \tag{2.10}$$

Clearly, with "large" economies, (2.10) should replace (2.4) in our whole analysis.

3. The structure of equilibria and their welfare properties

As we have mentioned in the introduction, the GEI literature poses no attention to the analysis of the structure of *CPO* equilibria. This is an interesting problem since it allows to look at constrained inefficiency from a different angle. In particular, in this section we show that *CPO* equilibria are "exceptional"; namely,

Theorem 2. *There exist a generic set of economies in* $\Omega(\overline{\omega}) \times \mathcal{U}$*, such that CPO - equilibria are contained in a linear, lower dimensional, submanifold of the equilibrium set.*

To prove our theorem, we first derive a global parametrization of the equilibrium set. This parametrization is used to characterize equilibria as a fiber bundle. Such a characterization is simple and shares most of the properties of the one proposed for Walrasian equilibria in Balasko (1988):

- *every fiber is contained in the equilibrium set;*
- *every fiber is a linear submanifold of the equilibrium set*;
- *every fiber contains only one (and is therefore identified by a) no-trade equilibrium;*
- *each equilibrium belongs to one fiber only.*

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[16]See the remark on p. 1594 of Magill and Shafer (1991).

[17]The idea is again that one can show that it is possible to locally, and independently, control $D_{\theta^h} Z$ for as many as $S(L-1)$ consumers h, using quadratic perturbations of utilities.

Moreover, if one looks at the case in which there are as many assets as the number of states, this characterization reproduces Balasko's. Finally, we show that

- *each fiber contains, at most, one CPO equilibrium.*

In Walrasian economies, no-trade equilibria are Pareto optima (PO), i.e. the I Welfare Theorem applies. Therefore, no-trade Walrasian equilibria can be simply recovered using the solutions of a PO problem; the parametrization of the set of PO is indeed a global parametrization of the set of no-trade Walrasian equilibria. In extending this logic to economies with incomplete markets, we run into two obstacles. First, no-trade GEI are, typically, not PO. Second, CPO equilibria, generically, entail some trade across agents (i.e. they do not belong to the no-trade equilibrium set). We outrode the first obstacle by showing that no-trade equilibria can actually be represented as solutions of the following "modified" planner's problem: let x solve,

$$Max_{x\in\overline{\Omega}}\sum_h\sum_s \chi_s^h U_s^h(x_s^h)$$

at "welfare weights" $\chi = (1,.,\chi_s^h,..) \in \mathbb{R}^{H(S+1)}$, with $\overline{\Omega}$ denoting the closure of $\Omega(\overline{\omega})$. Differently from Pareto optima, in this problem welfare weights are state-contingent. Moreover, we will show that if one is concerned with a GEI with $(S-J)$ degrees of market incompleteness, χ has to be constrained to live in a $(H-1)+(H-1)(S-J)$- dimensional set. If markets are complete, and $J=S$, this parametrization is the one used for Walrasian no-trade equilibria. However as J decreases, falling below S, the dimensionality of the parameter space increases. In the limit -when $J=0$- the dimension of the parametrization is the one corresponding to the equilibrium set of a $(S+1)-$ spot-market economy, in which each spot is indeed an isolated Walrasian economy.

Once the set of no-trade GEI, $\mathcal{T}_{\overline{\omega},R}$, has been parametrized, the global structure of equilibria is easily derived. For every no-trade equilibrium, with prices and allocations $(\overline{P},\overline{q},\overline{x},\overline{\theta})$, one can identify the set of equilibria $(\overline{P},\overline{q},e)$ with active trade. This boils down to considering the set of economies parametrized by initial endowments, e, such that, at $(\overline{P},\overline{q},e)$, $(\overline{x},\overline{\theta})$ are feasible and satisfy market clearing. This set has dimension $n=(H-1)(m-(S-J+1))$. Therefore, for fixed aggregate resources and asset structure, $(\overline{\omega},R)$, one finds that[18]

$$\mathcal{E}_{\overline{\omega},R} \cong \mathcal{T}_{\overline{\omega},R}\times\mathbb{R}^n_{++}\cong\Omega(\overline{\omega})$$

Next, we still have to overcome the second obstacle. Even though, as we claimed, CPO equilibria do typically entail some trade, trading opportunities are limited: on top of constraining endowments to satisfy budget balance and market clearing, they have to satisfy (2.4), i.e.

$$\sum_h \left(\lambda^h\square(x_{\mathbf{1}}^h-e_{\mathbf{1}}^h)\right)D_\theta P_{\mathbf{1}}=0$$

where, $P_{\mathbf{1}},x_{\mathbf{1}}$ are no-trade equilibrium prices and allocations, and λ are the corresponding individual, marginal rates of substitutions. Thus, if we put ourselves on a fiber of the equilibrium set, identified by a no-trade equilibrium, and we move along such fiber, we might hit the unique CPO allocation associated to it. The possibility that some fibers do not contain CPO equilibria is implied by the fact

[18] $\cong$ denotes equivalence up to an homeomorphism.

that (2.4) are only necessary conditions. However, since the *CPO* problem admits solutions, *CPO* equilibria exist on some fibers.

Notice that our informal discussion is provided for a given real numéraire payoff matrix, R, of full rank J. Since each of such matrices identifies an asset span, $\mathcal{L}$ in $\mathsf{G}^{J,S}$, the same discussion would go through if we had directly parametrized the economy on $\mathsf{G}^{J,S}$ (as in Duffie and Shafer (1985)). Indeed this is exactly what we do in the rest of paper. Although at some cost, our choice turns out to be particularly useful when it comes to deal with the more general case of real-numéraire assets, or with nominal assets, for which the asset span is defined endogenously, at equilibrium. The additional cost of parametrizing with respect to $\mathcal{L}$ is that the equilibrium set retains its vector space structure only locally (on $\mathsf{G}^{J,S}$).

3.1. **Definitions.** For fixed preferences, u, and aggregate resources, $\overline{\omega}$, an economy with an "abstract" financial market structure, $\mathcal{L}$, is a pair $(e, \mathcal{L}) \in \mathbb{R}^{mH}_{++} \times \mathsf{G}^{J,S}$. $\mathsf{G}^{J,S}$ is the *Grassmanian* of $J-$planes in $\mathbb{R}^S$.

Without loss of generality,[19] assume that the first individual is financially unconstrained. The Walrasian demand of consumer 1 is a function,

$$g^1\left(p, e^1\right) = \arg \max_x \left\{u^1(x) : p\left(x - e^1\right) = 0\right\}.$$

The demand of a consumer h is,

$$f^h\left(p, \mathcal{L}, e^h\right) = \arg \max_x \left\{u^h(x) : \begin{array}{c} p\left(x - e^h\right) = 0 \\ p_{\mathbf{1}}\Box\left(x_{\mathbf{1}} - e^h_{\mathbf{1}}\right) \in \mathcal{L} \end{array}\right\} \tag{3.1}$$

for all $h \geq 2$.

We normalize prices by taking $p_{01} = 1$, and denote the set of normalized prices, $\mathbb{P} \subset \mathbb{R}^{(S+1)L-1}_{++}$. The truncated aggregate excess demand function is

$$Z(p, \mathcal{L}, e) = g^1\left(p, e^h\right) + \sum_{h=2}^{H} f^h(p, \mathcal{L}, e^h) - \overline{\omega}.$$

Then, the following sets of equilibria are defined.

$$\begin{aligned} \mathcal{S}_J &= \left\{(p, \mathcal{L}, e) \in \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega : Z(p, \mathcal{L}, e) = 0\right\}, \\ \mathcal{T}_J &= \left\{(p, \mathcal{L}, e) \in \mathcal{S}_J : g^1(p, pe^1) = e^1,\ f^h(p, \mathcal{L}, e^h) = e^h, \forall h \geq 2\right\}. \end{aligned}$$

Observe that $\mathcal{S}_J$ is the set of equilibria defined for all possible asset span $\mathcal{L}$ in $\mathsf{G}^{J,S}$. $\mathcal{T}_J$ is the subset of $\mathcal{S}_J$ for which there is no trade.

3.2. **The structure of equilibria.** To derive the structure of the equilibrium set we will appeal to Lemma 3.2.1 in Balasko (1988), limiting our proofs to the definition of the required diffeomorphisms:

Lemma 4. *(Lemma 3.2.1 in Balasko (1988)) Let $\tau : X \to Y$, and $\phi : Y \to X$, be two smooth mappings between smooth manifolds such that the composition $\tau \circ \phi : Y \to Y$ is the identity mapping. Then, the set $Z = \phi(Y)$ is a smooth submanifold of X diffeomorphic to Y.*

The definition of a parametrization, Y, is the preliminary, essential, step we will take in the next subsection.

[19] See remark 3 below.

3.2.1. ***Parametrization***. The parametrization we are going to introduce hinges upon *individual state-prices*. For every agent $h \geq 2$, a *personalized price functional*, at prices p,

$$\rho(p, \mu^h) = \left(p_0, (1+\mu_1^h)p_1, ..., (1+\mu_S^h)p_S\right) \in \mathbb{R}_+^m, \tag{3.2}$$

is a function of $\mu^h \in \mathbb{R}^S$. Let μ denote the $(H-1) \times S$ - matrix of typical row μ^h. Further, define $\mathbb{M}_J$ the set of matrices μ of rank $c = S - J$ (its closure, $\overline{\mathbb{M}}_J$, contains matrices of rank $c \leq S - J$).

The parameter set $\mathbb{M}_J$ summarizes all the 'relevant' information concerning asset markets and individual state-prices. Indeed, if markets are complete ($S = J$), then the rank of μ is zero, therefore $\mu = 0$, and $\dim \mathbb{M}_S = 0$: consumers' evaluations (marginal rates of substitutions) are all equal to p. At the other extreme, if there are no assets ($J = 0$), $\dim \mathbb{M}_0 = (H-1)S$. Thus, if $(H-1) > S$, consumer evaluations may span $\mathbb{R}^S$ (i.e. there is "maximal disagreement" among consumers).

The next two lemmas summarize all the key properties of $\mathbb{M}_J$.

Lemma 5. *$\mathbb{M}_J$ as the set of $(H-1)\times S$ matrices of rank $(S-J)$ is a submanifold of $\mathbb{R}^{(H-1)S}$ of dimension, $\dim \mathbb{M}_J = c^* = (H-1+J)(S-J)$, and codimension, $c^\circ = ((H-1)-(S-J))\,J$.*

Proof: see the Appendix.

Next, we argue that each element of $\mathbb{M}_J$ identifies an asset span, $\mathcal{L}$ in $\mathsf{G}^{J,S}$. Take $\mu \in \mathbb{M}_J$ and introduce the following block decomposition

$$\begin{pmatrix} \mu_{/J} & \mu_J \\ \overline{\mu}_{/J} & \overline{\mu}_J \end{pmatrix}$$

such that $\mu_{/J}$ is a $(S-J)-$nonsingular matrix.[20] Then, $\left(\mu_{/J} \mid \mu_J\right)$ defines a basis for $\mathcal{L}^\perp$, and thus uniquely identifies $\mathcal{L}$. Precisely, we can define a projective mapping, α_J, from $\mathbb{M}_J$ to $\mathsf{G}^{J,S}$, by taking $\alpha_J = \psi(\mu_{/J}^{-1}\mu_J)$; where ψ is an *homeomorphism*[21] of $\mathsf{G}^{J,S}$ onto $\mathbb{R}^{J(S-J)}$ that identifies the unique $(S-J)J-$coordinate system of $\mathcal{L}$. This implies that there exists a, nontrivial, $((H-1)-(S-J)) \times (S-J)$ $-$matrix Q such that $(\overline{\mu}_{/J} \mid \overline{\mu}_J) = Q(I_{S-J} \mid \psi(\mathcal{L}))$. Clearly, $Q = \overline{\mu}_{/J}\mu_{/J}^{-1}$, and $\overline{\mu}_J = \overline{\mu}_{/J}\mu_{/J}^{-1}\mu_J$, as it is shown in the proof of lemma 5.

Next, it is easily seen that different elements of $\mathbb{M}_J$ can identify the same asset span. We say that μ, μ' are in the same equivalence class $(\sim)$, $\mu \sim \mu'$, if and only if they generate the same basis for a $(S-J)$ dimensional space. Hereafter, we refer to $\mathbb{M}_J$ as the (*quotient*) *topological space*, $\mathbb{M}_J/\sim$.[22,23]

A final step is to establish the global structure of $\mathbb{M}_J$ as a fiber bundle over $\mathsf{G}^{J,S}$. This is done by regarding $\mathbb{M}_J$ as the total, topological, space, $\mathsf{G}^{J,S}$ as the base space, and α_J as the projective map. Define the *canonical vector bundle*

[20]This is always possible because μ in $\mathbb{M}_J$ is of rank $S-J$; therefore, we can permuting the rows of μ such that $(\mu_{/J} \mid \mu_J)$ is a basis for a $(S-J)$ dimensional space, and $\mu_{/J}$ is nonsingular.

[21]An *homeomorphism* is a bijective, continuous, mapping whose inverse is also continuous.

[22]This procedure uses the quotient topology on the Grassmanian to define a topology for $\mathbb{M}_J$. The next lemma helps to clarify this point.

[23]Take the closure of $\mathbb{M}_0$, $\overline{\mathbb{M}}_0$. If $\mu \in \overline{\mathbb{M}}_0$, consumer evaluations span a vector space of at most dimension S. More generally, $\overline{\mathbb{M}}_S \subset .. \subset \overline{\mathbb{M}}_J \subset .. \subset \overline{\mathbb{M}}_0$, and $\cup_{k=J}^S \mathbb{M}_k$ is an open cover of $\overline{\mathbb{M}}_J$, for all $J = 0, .., S$. The finite covering property of $\overline{\mathbb{M}}_J$ is important since it implies compactness.

over $\mathsf{G}^{J,S}$, $\upsilon = \{\mathcal{L}, y \in \mathsf{G}^{J,S} \times \mathbb{R}^S : y \in \mathcal{L}\}$, and its orthogonal complement, $\upsilon^\perp = \{\mathcal{L}, y \in \mathsf{G}^{J,S} \times \mathbb{R}^S : y \perp \mathcal{L}\}$; then the following holds,

Lemma 6. *$\mathbb{M}_J$ is a vector bundle over $\mathsf{G}^{J,S}$, $\mathbb{M}_J \cong (H-1)\upsilon^\perp$.*[24] *Its projective map, α_J, identifies a unique $\mathcal{L}$ for each μ in $\mathbb{M}_J$.*

Proof: see the Appendix.

3.2.2. ***The structure of no-trade equilibria***.

Proposition 1. *$\mathcal{T}_J$, is a manifold diffeomorphic to $\mathbb{R}^{H-1}_{++} \times \mathbb{M}_J$. Moreover, as a fiber bundle over $\mathsf{G}^{J,S}$, $\mathcal{T}_J \cong \varepsilon^{H-1} \oplus (H-1)\upsilon^\perp$.*[25]

Again, the proof of this proposition uses lemma 4, and it is deferred to the Appendix. To apply this lemma, we let $Y = \mathbb{R}^{H-1}_{++} \times \mathbb{M}_J$, and $X = \mathcal{S}_J$. For the time being, we will assume that $\mathcal{S}_J$ is a smooth manifold; this is established in proposition 2 below. Moreover, we define $\tau^{\mathcal{T}} : \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega \to \mathbb{R}^{H-1}_{++} \times \mathbb{M}_J$,

$$\tau^{\mathcal{T}}(p, \mathcal{L}, e) = \left(\left(1, .., D_{01}u^1(x^1)/D_{01}u^h(x^h), ..\right), \left(\nabla_{s1}u^h(x^h)/\nabla_{s1}u^1(x^1) - 1\right)_{s\geq 1, h\geq 2}\right)$$

where, if $\tau^{\mathcal{T}}$ is restricted to $\mathcal{S}_J$, $\left(x^1, \left(x^h\right)_{h\geq 2}\right) = \left(g^1\left(p, e^1\right), \left(f^h(p, \mathcal{L}, e^h)\right)_{h\geq 2}\right)$ are equilibrium allocation;
$\phi^{\mathcal{T}} : \Delta^{H-1} \times \mathbb{M}_J \to \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega$, be such that[26]

$$\phi^{\mathcal{T}}(\delta, \mu) = \left(\nabla u^1(x^1), \alpha_J(\mu), x\right) = (p, \mathcal{L}, e),$$

where we take x to be the solution of the planner's problem,

$$Max_{x\in\overline{\Omega}} \sum_h \sum_s \chi_s^h U_s^h(x_s^h) \tag{3.3}$$

with "welfare weights" $\chi_s^1 = 1$ for all s, $\chi_0^h = \delta^h$, $\chi_s^h = \delta^h(1 + \mu_s^h)$ for all $s \geq 1$, and all $h \geq 2$; $\overline{\Omega}$ is the closure of $\Omega(\overline{\omega})$.

Remark 2. *(The structure of $\mathcal{T}_J$) When $J \in \{0, S\}$ the set of no-trade equilibria has a (globally) trivial (vector space) structure. Using $\mathcal{T}_S$ to denote the no-trade complete markets version of $\mathcal{T}_J$, it is easy to see that, $\mathcal{T}_S \cong \Delta^{H-1}$.*[27] *This is the case in which $\mu = 0$, and consumer gradients are collinear. On the other extreme, when asset markets are totally incomplete, we have $\mathcal{T}_0 \cong \times_s \Delta^{H-1}$. In all intermediate cases, $0 < J < S$, $\mathcal{T}_J \cong \mathcal{T}_S \oplus (H-1)\upsilon^\perp$; namely $\mathcal{T}_J$ retains a vector space structure only locally on $\mathsf{G}^{J,S}$.*

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[24]A fiber bundle with vector space structure on fibers is a vector bundle.
$\cong$ denotes an homeomorphism relationship.

[25]ε^n_B denotes the *trivial vector bundle* $(B \times \mathbb{R}^n, B, \alpha)$; a fiber bundle that has a global (as suppose to local) vector space structure. We drop the base space, B, from this notation, since this clearly emerges from the context.

$\oplus$ denotes the Whitney sum. This operates as a direct sum across the elements of the fibers of a vector bundle.

[26]Δ^n denotes the, strictly positive, n-dimensional simplex: $\Delta^n = \left\{y \in \mathbb{R}^{n+1}_{++} : \sum_i y_i = 1\right\}$.

[27]See Balasko (1988), section 3.3 for details on the structure of no-trade equilibria when markets are complete.

3.2.3. ***The structure of equilibria with active trade***. Next, consider equilibria in which consumers trade an initial set of resources (or endowments). The new parameter space will then be enlarged by the dimensionality of the space of relevant spot trade opportunities.

For fixed aggregate resources $\overline{\omega}$, define $\widehat{\Omega} \subset \mathbb{R}^n_{++}$ the section of $\Omega_\sigma \subset \Omega(\overline{\omega})$ of dimensionality $n = (H-1)(m-(S-J+1))$ for all $\sigma \in \Sigma$:

$$\widehat{\Omega} = \left\{ \widehat{e} \in \mathbb{R}^n : \begin{array}{l} \widehat{e} = (\widehat{e}_0, \widehat{e}_{\mathbf{1}}) \\ \widehat{e}^h_0 = (e^h_{0,l})_{l\geq 2},\ h \geq 2 \\ \widehat{e}^h_{\mathbf{1}} = \left(e^h_{\sigma(j,l)}, e^h_{/\sigma(k,l)}\right)_{j=1,..,J,\, k=1,..,S-J,\ l\geq 2},\ h\geq 2,\ \sigma \in \Sigma \end{array} \right\}$$

Proposition 2. *$\mathcal{S}_J$ is a smooth manifold diffeomorphic to $\Omega(\overline{\omega}) \times \mathbb{R}^{(S-J)J}$. Moreover, as a fiber bundle over $\mathsf{G}^{J,S}$, $\mathcal{S}_J \cong \varepsilon^{(H-1)+n} \oplus (H-1)\upsilon^{\perp}$.*

In order to apply Lemma 4, let $X = \Delta^{H-1} \times \mathbb{M}_J \times \widehat{\Omega}$, and $Y = \mathcal{S}_J$. Moreover, define $\tau^{\mathcal{S}} : \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega \to \Delta^{H-1} \times \mathbb{M}_J \times \widehat{\Omega}$, such that

$$\tau^{\mathcal{S}}(p, \mathcal{L}, e) = \left(\tau^{\mathcal{T}}(p, \mathcal{L}, e), proj_{\widehat{\Omega}}\Omega(\sigma)\right)$$

where (for given $\mathcal{L}$) σ is such that $\mathcal{L} \in W_\sigma$, and $\Omega(\sigma)$ is defined accordingly. Next, let $\phi^{\mathcal{S}} : \Delta^{H-1} \times \mathbb{M}_J \times \widehat{\Omega} \to \mathbb{P} \times \mathsf{G}^{J,S} \times \Omega$, be such that
$\phi^{\mathcal{S}}(\delta, \mu, \widehat{e}) = \left(\nabla u^1(x(\delta,\mu)), \alpha_J(\mu), e\right)$,
where $e = (e^1, (e^h)_{h\geq 2})$ is defined as follows: for given $\widehat{e} \in \widehat{\Omega}_\sigma$, with σ such that $\alpha_J(\mu) = \mathcal{L} \in W_\sigma$,[28] e is restricted to satisfy (3.4):

(3.4)

$$\begin{array}{l} e^h_{/\sigma(k,1)} = \frac{1}{p_{/\sigma(k,1)}}\left(p_{/\sigma,k}x_{/\sigma,k} - \sum_{(k,l)\neq(k,1)} p_{/\sigma(k,l)}\widehat{e}^h_{/\sigma(k,l)} + (\psi_{\sigma,k}\circ\alpha_J)p_\sigma\left(x^h_\sigma - \widehat{e}^h_\sigma\right)\right) \\ \quad k = 1,..,S-J \\ e^h_{01} = \sum_{s,l} p_{sl}x^h_{sl} - \sum_{(s,l)\neq(0,1)} p_{sl}\widehat{e}^h_{sl} \\ e^h = \left(e^h_{0,1}, \left(\widehat{e}^h_{0,l}\right)_{l\geq 2}, \left(e^h_{/\sigma(k,1)}, \left(\widehat{e}^h_{/\sigma(k,l)}\right)_{l\geq 2}\right)_k, \widehat{e}^h_\sigma\right), \textit{and for } h=1 \\ e^1_{sl} = \overline{\omega}_{sl} - \sum_{h\geq 2} e^h_{sl},\ \forall (s,l) \end{array}$$

where in the first two lines expressions are evaluated at $x^h = x^h(\delta,\mu)$, $\forall h \geq 2$. Observe that the first two restrictions in (3.4), respectively, originate from the financial constraints, and from the (Walrasian) budget constraint, $p\left(x^h - e^h\right) = 0$, for all $h \geq 2$ (both evaluated at $x = x(\delta,\mu)$). The third is just the definition of $\left(e^h\right)_{h\geq 2}$. The last restriction in (3.4) is derived from the aggregate resource constraint.

Remark 3. *(The case of a symmetric equilibrium) In this section we have considered an economy in which the first consumer is financially unconstrained. It is well known, at least since theorem* 4.2 *in Balasko and Cass (1989), that the equilibrium manifold of such an economy, is diffeomorphic to the one that treats individuals symmetrically. Just to sketch their argument, let us refer to these equilibrium set, respectively, as $\mathcal{S}_J$ and $\overline{\mathcal{S}}_J$. First, consider a $(P, \mathcal{L}, e)$ in $\overline{\mathcal{S}}_J$, with an allocation x; define $p = \nabla u^1(x)\Box P$, then $(p, \mathcal{L}, e) \in \mathcal{S}_J$. In fact, at this new price vector the*

[28]See the proof of Lemma 6 for a better understanding of how permutations of the rows of μ are used to define a local coordinate system of the asset span $\mathcal{L}$.

financial constraint of agent 1 is trivially satisfied, and $f^1\left(P,\mathcal{L},e^1\right) = g^1(p,e^1)$*. Conversely, consider* $(p,\mathcal{L},e)\in\mathcal{S}_J$*;* $\nabla u^1(x) = p$ *holds at the equilibrium allocation,* x*. Since* $p_{\mathbf{1}}\Box z^h_{\mathbf{1}}\in\mathcal{L}$ *for* $h\geq 2$*, and markets clear; then,* $x^1\in f^1\left(p,\mathcal{L},e^1\right)$*, i.e. there exist Lagrange multipliers* $\gamma^1\in\mathbb{R}^{S-J}_+$ *such that* $\mu^1 = \gamma^1(I\mid\psi_\sigma(\mathcal{L}))\pi_\sigma$*, and* $\nabla u^1(x) = \rho^1\left(p,\mu^1\right)$*.*

3.2.4. ***The fiber bundle structure of*** S_J*.* We are now going to define the fiber bundle structure of $\mathcal{S}_J$ that we shall then use to characterize *CPO* equilibria in the next section.

Definition 5. *A fiber associated to* $(\delta,\mu)\in\Delta^{H-1}\times\mathbb{M}_J$ *is a set,* $\mathcal{F}_{\delta,\mu}$*, of typical element* $(p,\mathcal{L},e)\in\mathbb{P}\times\mathsf{G}^{J,S}\times\Omega(\overline{\omega})$*:* $\mathcal{F}_{\delta,\mu}$ *is the inverse image of* $\{\delta,\mu\}\times\widehat{\Omega}$ *by* $\tau^{\mathcal{S}}$*.*

Definition 5 is better understood by looking back at proposition 2. First, observe that by propositions 1 and 2, $\mathcal{S}_J$ is homeomorphic to $\varepsilon^n\oplus\mathcal{T}_J$. This, loosely speaking, implies that we can fix no-trade equilibrium prices and allocations, and generate an n -dimensional set of equilibria with active trade, in $\mathcal{S}_J$. Precisely, recall that $\tau^{\mathcal{S}}:\mathcal{S}_J\to\Delta^{H-1}\times\mathbb{M}_J\times\widehat{\Omega}$, associates to every equilibrium, $(p,\mathcal{L},e)$ a unique $(\delta,\mu,\widehat{e})$, where $\widehat{\Omega}\subset\mathbb{R}^n$, $n\equiv(H-1)\left(m-(S-J+1)\right)$. Moreover, since each (δ,μ) identifies a no-trade equilibrium, through $\phi^{\mathcal{T}}$, $\left(\tau^{\mathcal{S}}\right)^{-1}(\delta,\mu,\widehat{e}) = (\phi^{\mathcal{T}}(\delta,\mu),\widehat{e})$. This implies that

$$\mathcal{F}_{\delta,\mu} = (\tau^{\mathcal{S}})^{-1}(\{\delta,\mu\}\times\widehat{\Omega}) \tag{3.5}$$

Notice that a fiber associated to (δ,μ) contains those equilibria in $\mathcal{S}_J$ which $i)$ are compatible with a fixed pair $(p,\mathcal{L})$ and a fixed equilibrium allocation (the corresponding no-trade equilibrium allocation), $ii)$ have different level of endowments $\widehat{e}$ in $\widehat{\Omega}$, and thus of trades z.

Our fibers have a few interesting properties, which are analogous to those established for Walrasian equilibria in Balasko (1988), apply.

- *Every fiber is a subset of* $\mathcal{S}_J$*.*

In fact, by definition, $\left(\tau^{\mathcal{S}}\right)^{-1}(\delta,\mu,\widehat{e}) = (p,\mathcal{L},e)$ is an element of $\mathcal{S}_J$.

- *Every fiber is a linear submanifold of* $\mathcal{S}_J$*, of dimension* n*.*

This also follows by definition (equation (3.5)).

- *Every fiber contains only one no-trade equilibrium.*

This follows from the structure of no-trade, which are diffeomorphic to $\Delta^{H-1}\times\mathbb{M}_J$ (see proposition 1).

- *Each equilibrium,* $(p,\mathcal{L},e)\in\mathcal{S}_J$*, belongs to one fiber only.*

This follows from definition 5: each fiber is identified by a unique no trade equilibrium. This does also explain the following,

- *Fibers can be "glued" together by letting* (δ,μ) *vary on* $\Delta^{H-1}\times\mathbb{M}_J$*.*

3.3. **The structure of CPO equilibria.** In this section we have worked with a different notion of the economy and equilibrium with respect to the real numéraire one used in section 2. Hence something should be said about the constrained Pareto optimality in this new setting. Indeed, we only need to adapt definition 3 as follows. For every economy $(\mathcal{L},e)$, a consumption allocation $x=(x_0,x_{\mathbf{1}})$ is constrained feasible at $(x_0,e,\mathcal{L})$ if and only if it is,
(i) attainable with transfers $z_{\mathbf{1}}$, such that $z^h_{\mathbf{1}} = (x^h_{\mathbf{1}} - e^h_{\mathbf{1}})$, and $p_{\mathbf{1}}\Box z^h_{\mathbf{1}}\in\mathcal{L}$, for all

$h \geq 2$,
(ii) supported by $p_{\mathbf{1}}$ as a, date 1, spot-market equilibrium at $\widetilde{e}_{\mathbf{1}} = e_{\mathbf{1}} + z_{\mathbf{1}}$.

Observe though, that because a change in the level of spot prices does not affect the spot-market equilibrium allocations, there are only as many as $S(L-1)$ relevant spot prices which can be controlled by the planner to achieve his objective. Hence, without loss of generality, we assume that in the planner's problem $p_{\mathbf{1}}$ is restricted to $\Delta^{S(L-1)}$ as it was for real numéraire assets in (2.2) above.

Next, to characterize *CPO* equilibria it is useful to directly appeal to our Planner's problem. A necessary condition for an equilibrium to be *CPO* is that -fiber by fiber- endowments are restricted such as to satisfy the first order conditions for a *CPO*.

For expositional simplicity, suppose that $H - 1 > S(L-1)$. Then, along a fiber identified by (δ, μ), equilibria are *CPO* if endowments are restricted such as to keep $\sum_h \mu^h \Box e^h_{\mathbf{1}}$ constant, say at $c_{\delta,\mu} \in \mathbb{R}^{S(L-1)}$. The appropriate constant, $c_{\delta,\mu}$, may vary depending of the asset structure considered. So, for example, in a GEI with real numéraire assets, condition (2.10) implies that $c_{\delta,\mu} = \sum_h \mu^h \Box x^h_{\mathbf{1}}(\delta, \mu)$. Similarly, when assets are real, $c_{\delta,\mu}(s,l) = \sum_h \mu^h_s x^h_{sl}(\delta,\mu) - R_{sl}$ for all $(s,l) \geq (1,2)$.[29]

These constancy requirements can be interpreted by saying that as e varies, pinning down equilibria of a particular fiber, it should not induce any aggregate, welfare, effect (you may read our comments on (2.4) in section 2.2).

Our discussion implies the following.

- *The set of CPO equilibria are contained in a linear submanifold of $\mathcal{S}_J$ of codimension $S(L-1)$.*

In fact, observe that for every fiber $\mathcal{F}_{\delta,\mu}$ we require that equilibria do also satisfy $\sum_h \mu^h_s e^h_{\mathbf{1}} = c_{\delta,\mu}$. The latter impose $S(L-1)$ linear restrictions on e. Moreover, by the definition of fibers, and the fact that each equilibrium belongs to a fiber only, we find that:

- *Every CPO equilibrium belongs, at most, to one fiber.*

Since the planner's problem needs not be convex, there might be points on the fibers which are not CPO equilibria even if they satisfy the first order conditions for CPO.

We can now gather our findings to prove our main result.

Proof of Theorem 2:
The set of *CPO* equilibria is contained in $\cup_{\delta,\mu}\mathcal{F}_{\delta,\mu}$. ■

Finally, one can exploit the fiber bundle structure of $\mathcal{S}_J$ to establish the, generic, constrained inefficiency of equilibria. This comes as a straightforward application of Sard's theorem,[30] implying that the set of economies with CPO-equilibria has measure zero.

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[29]For an arbitrary H, one has to explicitly account for $D_\theta P_{\mathbf{1}}$: in a GEI with real numéraire assets, one would require that -along a fiber (δ,μ)- as e varies, $\sum_h \mu^h \Box (x^h_{\mathbf{1}}(\delta,\mu) - e^h_{\mathbf{1}}) D_\theta P_{\mathbf{1}} = 0$ (i.e. condition (2.4) holds). The latter being necessary to achieve a *CPO*. Observe, however, that $D_\theta P_{\mathbf{1}}$, being a function of spot prices, depends on (δ,μ) only, and is therefore constant along each fiber (δ,μ).

[30]See, for example, Guillemin and Pollack (1974), p. 205.

3.4. **Economies and equilibria with real assets.** For completeness, we are now going to show how our construction can be used to study the efficiency properties of a GEI with real assets.

A real asset GEI is parametrized by (e, R) on $\Omega \times \mathbb{R}^{SLJ}$, with R denoting the real payoff matrix. When restricted to $\mathbb{P} \times \mathsf{G}^{J,S} \times \mathbb{R}^{SLJ}$, the set of *financial market possibilities* of this economy is,

$$\mathcal{M}_J = \{(p, \mathcal{L}, e, R) : (I \mid \psi_\sigma(\mathcal{L}))\pi_\sigma V\,(p_{\mathbf{1}} R) = 0, \sigma \in \Sigma \; s.t.\, \mathcal{L} \in W_\sigma\}$$

where $V(p_{\mathbf{1}} R)$ is the $S \times J$ asset matrix, of typical element $p_s R^j_s$. Then, the set of *pseudo equilibria* in this economy is defined by

$$\mathcal{E}_J = \{(p, \mathcal{L}, e, R) : Z(p, \mathcal{L}, e) = 0, (p, \mathcal{L}, R) \in \mathcal{M}_J)\}\,.$$

Lemma 7. *$\mathcal{M}_J$ is a manifold diffeomorphic to $\mathbb{P} \times \mathbb{R}^{SJL}$. Moreover as a fiber bundle over $\mathsf{G}^{J,S}$, $\mathcal{M}_J \cong \varepsilon^{NL-1+SJ(L-1)} \oplus J\upsilon$.*

Proof: see the Appendix.

The next proposition directly follows by Proposition 2 and Lemma 7.

Proposition 3. *For all $0 < J < S$, the set of equilibria $\mathcal{E}_J$ is diffeomorphic to $\Omega(\overline{\omega}) \times \mathbb{R}^{SJL}$. Moreover, over $\mathsf{G}^{J,S}$, it has a fiber bundle structure, $\mathcal{E}_J \cong \mathcal{S}_J \oplus \mathcal{M}_J$.*

Finally, denoting by E_s the Walrasian equilibrium manifold in the state s economy, and by E the Walrasian equilibrium manifold of an economy with a complete set of contingent markets, we have the following.

Remark 4. *Similarly to $\mathcal{S}_J$ the following consideration apply.*
1) If markets are "totally" incomplete, $J = 0$, the set of equilibria is the Cartesian product of the set of the $S+1$ spot-market equilibria, $\mathcal{E}_0 = \times_{s=0}^{S} E_s$.
2) When markets are complete, $J = S$, the set of equilibria is equivalent to the set of Walrasian equilibria, $\mathcal{E}_S = E$.
In any intermediate case, $0 < J < S$, $\mathcal{E}_J$ has the (non-trivial) structure derived in proposition 3.

Appendix

Proofs of Section 2

. *Proof of lemma 2 (sketch)*: Let $U' \subset U \subset \mathbb{R}$ be the set containing utility function of h that admit local quadratic perturbations: a utility that admits quadratic perturbations at $\overline{x}$ is a function, $u^h(x^h; \varepsilon, B^h, \overline{x}) = u^h(x^h) + \frac{\varepsilon}{2}\rho(x^h)\sum_s(x^h_s - \overline{x}_s)' B^h_s(x^h_s - \overline{x}_s)$, where for all $s > 0$, $B^h_s \in \mathbb{R}^{L\times L}$ is a symmetric matrix, $\varepsilon > 0$ is small enough such as to ensure that u^h preserves strict-concavity, $\rho : \mathbb{R}^m_+ \to [0,1]$ is a $\mathcal{C}^{r\geq 2}$ (bump[31]) function, with compact support over $\mathbb{R}^m_{++}$, taking value 1 in a neighborhood of $\overline{x}$, and 0 otherwise, for all s. Thus, at a point $\overline{x}$, we take u^h to be defined on $\mathbb{R}^m_{++} \times [0,1] \times \mathbb{B}$ where $\mathbb{B} \subset \mathbb{R}^{m\times m}$ is the set of symmetric matrices. It is easily shown that for every convergence sequence of matrices $B^h_n \to B^h$, $u^h(\cdot;\cdot, B^h_n) \to u^h(\cdot;\cdot, B^h)$ in the $\mathcal{C}^{r\geq 2}$ topology of uniform convergence.

Next, for fixed $(P_{\mathbf{1}}, m_{\mathbf{1}})$ such that $D_{m^h_s}\widetilde{x}^h_s$ are well defined, let the mapping $G^h_s : C \to C \times \mathbb{R}^L$ be such that

$$G^h_s(x^h_s, \lambda^h_s) = \begin{bmatrix} f^h_s\left(x^h_s, \lambda^h_s, P_s, m^h_s\right) \\ D_{m^h_s}\widetilde{x}^h_s - D_{m^1_s}\widetilde{x}^1_s \end{bmatrix} \tag{3.6}$$

[31]See Hirch (1976).

where $C = \mathbb{R}^L_{++} \times \mathbb{R}_+$ is the set of the endogenous variables in the individual problem, (x^h_s, λ^h_s), for all s. We are going to show that there exists an open and dense set U^*, such that for all U^h in U^*, there does not exist a $(x^h_s, \lambda^h_s) \in \mathbb{R}^{L+1}_{++}$ such that $G^h_s(x^h_s, \lambda^h_s) = 0$, all s. Observe that by adjoining the L marginal propensities equations to each spot -demand first order conditions, we have added equations without adding unknowns. Thus, to prove our result it suffices to show that we can (locally) control $D_{m^h_s}\widetilde{x}^h_s$ independently from f^h_s, and $D_{m^h_s}\widetilde{x}^1_s$. This is done by perturbing the utility function of h with respect to B^h_s, around $\overline{x}_s = f^h_s\left(x^h_s, \lambda^h_s, P_s, m^h_s\right)$ for all s (i.e. we want to make $\overline{x}$ a function of $(P_{\mathbf{1}}, m_{\mathbf{1}})$). In doing so we exploit the fact that there exists an open neighborhood of $\overline{x}$ such that the following three properties hold: $u^h(x^h; \varepsilon, B^h, \overline{x}) = u^h(x^h)$, $Du^h(x^h; \varepsilon, B^h, \overline{x}) = Du^h(x^h)$, $D^2u^h(x^h; \varepsilon, B^h, \overline{x}) = D^2u^h(x^h) + \varepsilon B^h$. So that the perturbation adopted neither changes the utility level nor its gradient, leaving unaffected the first order conditions for an individual optimum. Finally, to argue that U^* is also open, we simply observe that the property that (3.6) has independent rows is open. ■

Proof of Lemma 3: Since $(e, u) \in \Omega' \times \mathcal{U}^*$, $D_\theta P_{\mathbf{1}}$ is well defined, and $D_{\theta^h_j} P_{\mathbf{1}} \neq 0$ for some $(h, j) \geq (2, 1)$, when it is evaluated at equilibrium. Fix $u \in \mathcal{U}^*$, and define Ω'' the subset of Ω' such that for every $e \in \Omega''$, there exists a regular equilibrium $\left(\overline{P}, \overline{q}, e\right)$, with allocation $\overline{x}$ and individual, normalized, state prices $\overline{\lambda}$. Then, the set
$\Omega_{hj} = \left\{e \in \Omega'' : \sum_h \left(\overline{\lambda}^h \Box \overline{z}^h_{\mathbf{1}}\right) D_{\theta^h_j} P_{\mathbf{1}} = 0\right\}$
is closed in Ω''. Hence, $\Omega'_{hj} = \Omega'' \backslash \Omega_{hj}$ is relatively open. We are now going to show that Ω'_{hj} is also dense in Ω''. Since $e \in \Omega_{hj}$ is a regular economy, then there exists an arbitrary, open, neighborhood of e, O_e, such that, $\forall \bar{e} \in O_e$, $\sum_h \left(\overline{\lambda}^h \Box \bar{z}^h_{\mathbf{1}}\right) D_{\theta^h_j} P_{\mathbf{1}} \neq 0$, and $Z\left(\overline{P}, \overline{q}, \bar{e}\right) = 0$. To see the latter, it suffices to show that there exists a marginal perturbation of endowments such as to change spot trades, z, without affecting consumption allocations and prices. Let this perturbation be $\Delta e^h_{\mathbf{1}} \in \mathbb{R}^{LS}$ (for some $h \neq 1$ such that $D_{\theta^h_j} P_{\mathbf{1}} \neq 0$) satisfy: *i)* $p_{\mathbf{1}} \Box \Delta e^h_{\mathbf{1}} = R \Delta \theta^h$ for some $\Delta \theta^h \in \mathbb{R}^J$; *ii)* $\Delta e^1_{\mathbf{1}} = -\sum_{h \geq 2} \Delta e^h_{\mathbf{1}}$. Further, let $\bar{e}^h_{\mathbf{1}} = e^h_{\mathbf{1}} + \Delta e^h_{\mathbf{1}}$, and take $\bar{z}^h_{\mathbf{1}} = \overline{x}^h_{\mathbf{1}} - \bar{e}^h_{\mathbf{1}} = z^h_{\mathbf{1}} - \Delta e^h_{\mathbf{1}}$, for all h; consumption allocations are unaffected, $\overline{x} = x$, and $Z\left(\overline{P}, \overline{q}, \bar{e}\right) = Z\left(\overline{P}, \overline{q}, e\right) = 0$. Thus, the original equilibrium allocations, x, are still demanded at $\left(\overline{P}, \bar{e}\right)$, and $\left(\overline{P}, \overline{q}\right)$ satisfies market clearing in the new economy, $\bar{e}$. Moreover, observe that also $D_{\theta^h_j} P_{\mathbf{1}}$ is not affected by the above perturbation of endowments (and the resulting changes in $m_{\mathbf{1}}$), since it only depends on x, P (see (2.7), (2.8)). Finally, $\Omega^* = \cap_{h,j} \Omega'_{hj} = \Omega'' \backslash \cap_{h,j} \Omega_{hj}$, is open and of full Lebesgue measure in $\Omega(\overline{\omega})$. ■

Proofs of Section 3

. *Proof of Lemma 5*: The fact that $\mathbb{M}_J$ is a manifold follows directly from being defined as the set of matrices of rank $(S - J)$. Take $\mu \in \mathbb{M}_J$ and introduce the following block decomposition

$$\begin{pmatrix} \mu_{/J} & \mu_J \\ \overline{\mu}_{/J} & \overline{\mu}_J \end{pmatrix}$$

such that $\mu_{/J}$ is a $(S-J)-$nonsingular matrix. Then there exists a matrix $C \subset \mathbb{R}^{S \times S}$ such that

$$MC = \begin{pmatrix} \mu_{/J} & \mu_J \\ \overline{\mu}_{/J} & \overline{\mu}_J^{'} \end{pmatrix} = \begin{pmatrix} \mu_{/J} & \mu_J \\ \overline{\mu}_{/J} & 0 \end{pmatrix}, \tag{3.7}$$

since $rank(MC) = rank(M) = S-J$ if $rank\left(\overline{\mu}_J^{'}\right) = 0$ (i.e. $\overline{\mu}_J^{'} = 0$). The latest poses a (system) of c° conditions, $c^{\circ} = ((H-1)-(S-J))\,J$. Observe that,

$$C = \begin{pmatrix} I_{S-J} & -\mu_{/J}^{-1}\mu_J \\ 0 & I_J \end{pmatrix}.$$

Hence, $\overline{\mu}_J^{'} = \overline{\mu}_J - \overline{\mu}_{/J}\mu_{/J}^{-1}\mu_J = 0$. Finally, $co\dim(\mathbb{M}_J) = c^{\circ}$, and thus $\dim(\mathbb{M}_J) = c^{*} = (H-1)\,S - c^{\circ} = (H-1+J)\,(S-J)$, exactly a number of coordinates equal to the elements of $\mu_{/J}, \mu_J, \overline{\mu}_{/J}$.[32] ■

Let A be the $(S-J)J$ matrix representing the unique local coordinate system of $\mathcal{L}$ and of its orthogonal $\mathcal{L}^{\perp}$. Further, denote by σ a permutation mapping from $\{1,..,S\}$ onto itself, by Σ the set of all such permutations, and by π_{σ} the $(S \times S)$ permutation matrix associated to σ. For every $\sigma \in \Sigma$, $W_{\sigma} = \left\{\mathcal{L} \in \mathsf{G}^{J,S} : \exists\, A \in \mathbb{R}^{(S-J)\times J} \ s.t. \ \left(\ I_{S-J} \mid \ \ A \ \right)\pi_{\sigma} \in \mathcal{L}\right\}$. $\{W_{\sigma} : \sigma \in \Sigma\}$ is an open cover of $\mathsf{G}^{J,S}$, $\mathsf{G}^{J,S} \subset \cup_{\sigma\in\Sigma} W_{\sigma}$. Finally, we define $\psi_{\sigma} : W_{\sigma} \to \mathbb{R}^{J(S-J)}$ such that $\psi_{\sigma}(\mathcal{L}) = A$. This map is a *homeomorphism* of W_{σ} onto $\mathbb{R}^{J(S-J)}$.

<u>*Proof of Lemma 6:*</u> $(\mathbb{M}_J, \mathsf{G}^{J,S}, \alpha_J)$ is a fiber bundle. Moreover, by lemma 5, for every $\mu \in \mathbb{M}_J$ there exist open neighborhoods $V_{\mu} \subset \mathbb{R}^{(H-1)S}$ and $O \subset \mathbb{R}^{c^{*}}$, and a smooth homeomorphism $h : O \to V_{\mu}$. (O, V_{μ}, h) is a local parametrization of $\mathbb{M}_J$ about μ. Moreover $h^{-1} : V_{\mu} \to O$ defines a local coordinate system for every $\mu \in V_{\mu}$. We now derive h.
Take $\mathcal{L} \in W_{\sigma}$, and let $h(\gamma, \mathcal{L}; \sigma) = \Gamma\left(\ I_{S-J} \mid \ \ \psi_{\sigma}(\mathcal{L}) \ \right)\pi_{\sigma} = \mu$, where Γ is a $(H-1) \times (S-J)$ matrix of typical row vector γ^h, and $\gamma = \left(\gamma^1, .., \gamma^H\right) \in \left(\mathbb{R}^{S-J} \setminus \{0\}\right)^{H-1}$. Thus, $\gamma^h\left(\ I_{S-J} \mid \ \ A \ \right)\pi_{\sigma} = \mu^h \in \mathcal{L}^{\perp}$, for all $h \geq 2$ (i.e. $\left(.., 1+\mu^h, ..\right) \in (H-1)\,v^{\perp}$), where $A = \psi_{\sigma}(\mathcal{L})$ is the local coordinate system of $\mathcal{L}$. Therefore, each $\mu \in Im\, h(\gamma, \mathcal{L}; \sigma)$ is an element of $\mathbb{M}_J$.[33] Since $(\gamma, A) \in \mathbb{R}^{c^{*}}$, and c^{*} is equal to the dimension of $\mathbb{M}_J$, h is a, continuous, injection.
Next, let us define the inverse mapping h^{-1}. for every $\mu \in V_{\mu}$, we can find a local coordinate system in $\mathbb{R}^{c^{*}}$ by taking $\Gamma\left(\ I_{S-J} \mid \ \ A \ \right) = \mu\pi_{\sigma'}$, $\psi_{\sigma'}^{-1}(A) = \mathcal{L} = \alpha_J(\mu)$ for some $\sigma' \in \Sigma$. More precisely, since μ in $\mathbb{M}_J$ has rank $S-J$, we can always find a permutation $\sigma' \in \Sigma$,

$$\mu\pi_{\sigma'} = \begin{pmatrix} \mu_{/J} & \mu_J \\ \overline{\mu}_{/J} & \overline{\mu}_J \end{pmatrix}$$

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[32]Notice that since $\overline{\mu}_J$ is determined when $\mu_{/J}, \mu_J, \overline{\mu}_{/J}$ are given, the latter three matrices defines a local parametrization of $\mathbb{M}_J$ of dimension $(H-1)\,S - \#\overline{\mu}_J = (H-1+J)\,(S-J)$.

[33]The fact that $\mathbb{M}_J$ does not have a vector space structure emerges clearer at this point: loosely speaking, the two set of local coordinates, (γ, A), are "interdependent".

such that $\mu_{/J}$ is a $(S-J)$ −nonsingular matrix, and $\begin{pmatrix} \mu_{/J} & \mu_J \end{pmatrix}$, induces a subspace of dimension $(S-J)$:[34] $\begin{pmatrix} \mu_{/J} & \mu_J \end{pmatrix} = \mu_{/J}\left(I \mid (\mu_{/J})^{-1}\mu_{/J}\right)$. Moreover, there exists matrix $Q \in \mathbb{R}^{r\times(S-J)}$, with $r = (H-1)-(S-J)$, such that $\begin{pmatrix} \overline{\mu}_{/J} & \overline{\mu}_J \end{pmatrix} = Q\begin{pmatrix} \mu_{/J} & \mu_J \end{pmatrix} = Q\mu_{/J}\left(I \mid (\mu_{/J})^{-1}\mu_{/J}\right)$. Clearly, $Q = \overline{\mu}_{/J}(\mu_{/J})^{-1}$, Putting things together,

$$\mu\pi_{\sigma'} = \begin{pmatrix} \mu_{/J} \\ \overline{\mu}_{/J} \end{pmatrix}\left(I_{S-J} \mid (\mu_{/J})^{-1}\mu_{/J}\right)$$

Therefore, $\mathcal{L} = \alpha_J(\mu) = \psi_{\sigma'}((\mu_{/J})^{-1}\mu_{/J})$, and $A = (\mu_{/J})^{-1}\mu_{/J}$ defines its coordinate system in $W_{\sigma'}$. The vectors γ^h form a matrix, $\Gamma^T = \left(\mu^T_{/J}, \overline{\mu}^T_{/J}\right)$. This defines h^{-1}; a continuous mapping.

We can conclude that the following diagram commutes,

$$\begin{array}{ccc} \mathbb{M}_J & \overset{h^{-1}}{\to} & \mathsf{G}^{J,S}\times\mathbb{R}^{(H-1)(S-J)} \\ \alpha_J = \rho\circ h \searrow & & \swarrow \rho(natural\, proj.) \\ & \mathsf{G}^{J,S} & \end{array}$$

Finally, being a smooth manifold, $\mathbb{M}_J$ has a local vector space structure (on fibers), and h is a diffeomorphism. Yet, $\mathbb{M}_J$ is not a trivial vector bundle. This is immediately seen once it is noticed that the latter characterization represents $\mathbb{M}_J$ as a $(H-1)$ copy of $\upsilon^{\perp}$. ■

Observe that $x \in \left(g^1\left(p, e^1\right), .., f^h\left(p, \mathcal{L}, e^h\right), ..\right)$ if and only if there exist Lagrange multipliers, $\left(\lambda^h, \gamma^h\right) \in \mathbb{R}_{++}\backslash\{0\}$, and $\gamma^h \in \mathbb{R}^{S-J}\backslash\{0\}$, such that,

$$\begin{array}{l} \left(D_0u^1\left(x^1\right), D_{\mathbf{1}}u^1\left(x^1\right)\right) = \lambda^1\left(p_0, p_{\mathbf{1}}\right) \\ p\left(x - e^1\right) = 0 \end{array} \tag{3.8}$$

$$\begin{array}{l} \left(D_0u^h(x), D_{\mathbf{1}}u^h(x)\right) = \lambda^h\left(p_0, p_{\mathbf{1}}\right) + \left(0, \gamma^h\left(I \mid \psi_\sigma(\mathcal{L})\right)\pi_\sigma\Box p_{\mathbf{1}}\right), \forall h \geq 2 \\ p\left(x - e^h\right) = 0, \forall h \geq 2 \\ \left(I \mid \psi_\sigma(\mathcal{L})\right)\pi_\sigma p_{\mathbf{1}}\Box\left(x_{\mathbf{1}} - e^h_{\mathbf{1}}\right) = 0, \forall h \geq 2 \end{array} \tag{3.9}$$

where $\mathcal{L}$ is taken to be an element of W_σ.

<u>*Proof of proposition 1*</u>:

- Let us start with τ. When $\tau^{\mathcal{T}}$ is restricted to $\mathcal{S}_J$, x are equilibrium allocations, and $\delta^h = D_{01}u^1(x)/D_{01}u^h(x) \in \mathbb{R}_{++}$, for all $h \geq 2$. To show that $\mu \in \mathbb{M}_J$, observe that $x^h \in f^h(p, \mathcal{L}, e)$, by individual first order conditions, implies that there exists a $\gamma^h \in \mathbb{R}^{S-J}/\{0\}$ such that $\nabla_{\mathbf{1}}u^h(x) = \left(\mathbf{1}_S + \gamma^h\left(I_{S-J} \mid \psi_\sigma(\mathcal{L})\right)\Box p_{\mathbf{1}}\right.$. By definition of $\tau^{\mathcal{T}}$, $\mu^h = \gamma^h\left(I_{S-J} \mid \psi_\sigma(\mathcal{L})\right)$. Finally, assuming that $\mathcal{S}_J$ is a smooth manifold, and observing that $\tau^{\mathcal{T}}$ has smooth coordinates, we conclude that $\tau^{\mathcal{T}}$ is smooth.
- Next, consider ϕ. To show that $\phi^{\mathcal{T}}$ is well defined, and smooth, it suffices to prove that the set of solutions to (3.3) (call it $\mathcal{K}$), is -respectively- nonempty and its elements, $x(\delta, \mu)$, are smooth functions on $\mathbb{R}^{H-1}\times\mathbb{M}_J$. Nonemptiness follows from the fact that (3.3) is the maximization of a continuous function on a compact set. Because utilities are strictly concave, for every

[34]In other words, $\left(\mu^1_\sigma, .., \mu^{S-J}_\sigma, \overline{\mu}^{S-J+1}_\sigma, .., \overline{\mu}^{H-1}_\sigma\right)$ is such that the first $(S-J)$ vectors are the orthogonal basis of some $\mathcal{L} \in \mathsf{G}^{J,S}$. Moreover, since $\mu\pi_{\sigma'}$ is of rank $S-J$, there exist $c_h \in \mathbb{R}$ such that $\overline{\mu}^{h'}_\sigma = \sum_{h=1}^{S-J} c_h\mu^h_\sigma$, for all $h' = S-J+1, .., H-1$.

$(\delta, \mu) \in \mathbb{R}^{H-1} \times \mathbb{M}_J$, there is a unique solution. We establish that $x(\delta, \mu)$ is smooth in lemma 8, below.

- We then argue that the image of $\phi^{\mathcal{T}}$ is $\mathcal{T}_J$.

: $Im\, \phi^{\mathcal{T}} \subset \mathcal{T}_J$. Consider the first order (necessary and sufficient) conditions of (3.3): for all $h \geq 2$

$$\begin{array}{l} \delta^h D_0 u^h\left(x^h\right) = D_0 u^1\left(x^1\right) = \overline{p}_0 \\ D_{\mathbf{1}} u^1\left(x^1\right) = \overline{p}_{\mathbf{1}} \\ \delta^h D_{\mathbf{1}} u^h\left(x^h\right) = \rho_{\mathbf{1}}(\mu^h, \overline{p}_{\mathbf{1}}) = ((1+\mu_s^h)\overline{p}_s)_{s\geq 1} \end{array} \tag{3.10}$$

where $\overline{p} \in \mathbb{R}^m_{++}$ is the vector of Lagrange multipliers of the resource constraints; and markets clear, $x(\delta, \mu) \in \Omega(\overline{\omega})$. Notice that $p = \nabla u^1(x^1(\delta, \mu)) = \frac{1}{\overline{p}_{01}} \overline{p} \in \mathbb{P}$, and $e = x(\delta, \mu)$. We are left to check that, at $(p, \mathcal{L}, e) = \left(\nabla u^1(x^1), \alpha_J(\mu), x\right)$, agents optimize; i.e. individual first order conditions in (3.8), (3.9) hold, respectively, for $h = 1$ and all $h \geq 2$. With some re-writing, the latter are,

$$\begin{array}{l} \frac{1}{\lambda^h} D_0 u^h\left(x^h\right) = \frac{1}{\lambda^1} D_0 u^1\left(x^1\right) = p_0 \\ \frac{1}{\lambda^1} D_{\mathbf{1}} u^1\left(x^1\right) = p_{\mathbf{1}} \\ \frac{1}{\lambda^h} D_{\mathbf{1}} u^h\left(x^h\right) = \widehat{\gamma}^h\left(I \mid \psi_\sigma\left(\mathcal{L}\right)\right) \pi_\sigma \Box p_{\mathbf{1}} \end{array} \tag{3.11}$$

where $\widehat{\gamma}^h = \frac{1}{\lambda^h}\gamma^h$, for all $h \geq 2$. Clearly, $(\lambda^h)^{-1} = \delta^h$ for all $h \geq 2$, and $\lambda^1 = D_{01} u^1\left(x^1(\delta, \mu)\right)$. Second, as we argued in section 3.2.1, we use the fact that given a $\mu \in \mathbb{M}_J$, and a permutation $\sigma' \in \Sigma$, we can univocally recover a matrix $\Gamma \in \mathbb{R}^{(H-1)\times(S-J)}$ of typical element-vector $\widehat{\gamma}^h$, and a $\mathcal{L}$ in $\mathsf{G}^{J,S}$. Precisely, letting

$$\mu \pi_{\sigma'} = \begin{pmatrix} \mu_{/J} & \mu_J \\ \overline{\mu}_{/J} & \overline{\mu}_J \end{pmatrix}$$

$\mathcal{L} = \alpha_J(\mu) = \psi_{\sigma'}((\mu_{/J})^{-1}\mu_{/J})$, and $A = (\mu_{/J})^{-1}\mu_{/J}$ is the coordinate system of $\mathcal{L}$ in $W_{\sigma'}$. $\Gamma^T = \left(\mu_{/J}^T, \overline{\mu}_{/J}^T\right)$.[35]

: $Im\, \phi^{\mathcal{T}} \supset \mathcal{T}_J$. Let us show that $\phi^{\mathcal{T}} \circ \tau^{\mathcal{T}} = id_{\mathcal{T}_J}$ when $\tau^{\mathcal{T}}$ is restricted to $\mathcal{T}_J$. If $(p, \mathcal{L}, e) \in \mathcal{S}_J$, and the equilibrium allocation is $x = e$, then individual first order conditions, (3.11), hold at e, and so do (3.10) at $(\delta, \mu) = \tau(p, \mathcal{L}, e)$; because $\sum_h e^h = \overline{\omega}$, e is a solution to (3.3) at $(\delta, \mu) = \tau(p, \mathcal{L}, e)$.

- By Lemma 4, $\phi^{\mathcal{T}}$ defines the desired diffeomorphism, and when $\tau^{\mathcal{T}}$ is restricted to $\mathcal{T}_J$, $\tau^{\mathcal{T}} \circ \phi^{\mathcal{T}} = id_{\mathbb{R}^{H-1}_{++} \times \mathbb{M}_J}$.
- Finally, the fiber bundle structure of $\mathcal{T}_J$ follows immediately from the structure of $\mathbb{M}_J$. ■

Lemma 8. *$x(\delta, \mu)$ is a smooth function on $\mathbb{M}_J \times \Delta^{H-1}$*

Proof:

First, in analogy with a Pareto maximum problem, every solution of (3.3) is interior. Then, first order (necessary and sufficient) conditions of (3.3) are,

1 $\chi_s^h D_{sl} U_s^h(x^h) - D_{sl} U_s^1(x^1) = 0, h \geq 2, s \geq 0, l \geq 1$

2 $\sum_h x_{sl}^h - \overline{\omega} = 0$, $s \geq 0, l \geq 1$,

We denote this system as $F(x; \chi, \overline{\omega}) = 0$. Since $\mathcal{K} = ImF^{-1}(0)$, 0 is a regular

[35] For the full argument, see the definition of the mapping $h(\gamma, \mathcal{L}; \sigma)$ in the proof of Lemma *6.*

value of F if its Jacobian, D_xF, is of full rank. Since u^h are $\mathcal{C}^{r\geq 2}$, F and $x(\delta,\mu)$ are $\mathcal{C}^{r\geq 1}$, by the Implicit Function Theorem. Computing D_xF:

$$\begin{array}{ccccc}
-D^2U^1 & \ddots & & & 0 \\
 & & \ddots & & \\
\vdots & & \chi_s^h D_s^2 U_s^h & & \\
 & & & \ddots & \\
-D^2U^1 & 0 & & & \ddots \\
I_m & I_m & I_m & & I_m
\end{array} \tag{3.12}$$

Since column operations do not affect the rank of (3.12), subtract the first column block from the h^{th}, for $h = 2,..,H$; then, move the resulting H^{th} row block (a block–row vector of typical element I_m in the first H blocks) to the top row block. The following matrix representation is obtained,

$$\begin{pmatrix} I_m & 0 \\ * & O \end{pmatrix} \tag{3.13}$$

$$O = \begin{pmatrix} G^2 & D^2U^1 & \cdots & D^2U^1 \\ D^2U^1 & \ddots & & \vdots \\ \vdots & & \ddots & D^2U^1 \\ D^2U^1 & \cdots & D^2U^1 & G^H \end{pmatrix}$$

$$G^h = \begin{pmatrix} \chi_1^2 D_1^2 U_1^h + D_1^2 U_1^1 & & 0 \\ & \ddots & \\ 0 & & \chi_S^h D_S^2 U_S^h + D_S^2 U_S^1 \end{pmatrix}$$

and $G_s^h \in \mathbb{R}^{L\times L}$. We are going to show that O is of full rank, because otherwise negative definiteness of D^2u^h (in assumption 2) would be contradicted. For $r^1 \in \mathbb{R}^{(H-1)m}$, r^1O has typical (L-vector) element, $\chi_s^h r_{s,h}^1 D_s^2 U_s^h + \left(\sum_{h=2}^H r_{s,h}^1\right) D_s^2 U_s^1$. Post multiplying the latter by $r_{s,h}^{1^T}$, and summing over $h \geq 2$, yields

$$\sum_{h=2}^H \chi_s^h \left(r_{s,h}^1 \left(D_s^2 U_s^h\right) r_{s,h}^{1^T} \right) + \left(\sum_{h=2}^H r_{s,h}^1\right) D_s^2 U_s^1 \left(\sum_{h=2}^H r_{s,h}^1\right)^T \tag{3.14}$$

By assumption 2), the two terms in the latter expression are negative, and so is their sum. Hence, (3.14) is equal to zero if and only if $r_{s,h}^1 = 0$, for all s and all $h \geq 2$. Finally, observe that $\mathbb{M}_J \times \Delta^{H-1} \subset \mathbb{R}^{(H-1)(S+1)}$. ■

Proof of Proposition 2:

Observe that the endowments restrictions in (3.4) imply that $Im\phi^{\mathcal{S}} \subset \mathcal{S}_J$. $Im\phi^{\mathcal{S}} \supset \mathcal{S}_J$, follows from observing that $\phi^{\mathcal{S}} \circ \tau^{\mathcal{S}} = id_{\mathcal{S}}$. $\phi^{\mathcal{S}}, \tau^{\mathcal{S}}$ are smooth functions, since their coordinates are smooth. Therefore, $\mathcal{S}_J = \phi^{\mathcal{S}}(\Delta^{H-1} \times \mathbb{M}_J \times \widehat{\Omega})$, is a smooth manifold. ■

Proof of Lemma 7:[36] We proceed by showing that $\mathcal{M}_J$ and $\varepsilon^{NL-1+SJ(L-1)} \oplus J\upsilon$ are homeomorphic. Then, since the latter is a vector bundle, it has a vector space structure on fibers (over $\mathsf{G}^{J,S}$); therefore for $\mathcal{M}_J$ to be a manifold of dimension $(NL-1)+SJL = (NL-1)+(S-J)J+J^2+SJ(L-1)$, we need to show that such an homeomorphic is in fact a diffeomorphism (i.e. it is smooth and has a smooth inverse). To define the desired homeomorphism, let $r = (r(1), r(-1)) \in \mathbb{R}^{SLJ}$ represent the whole vector of real payoffs in R, where $r(1) = \left(r_{11}^1, ..., r_{S1}^J\right) \in \mathbb{R}^{SJ}$ refers to good $l = 1$ and $r(-1) \in \mathbb{R}^{SJ(L-1)}$ to all the remaining goods. Define $\phi^{\mathcal{M}}(p, \mathcal{L}, R) = \left(p, \mathcal{L}, \left(V^1, .., V^J\right), r(-1)\right)$ such that $V^j \in \mathcal{L}$ for all j. That is $\left(\mathcal{L}, \left(V^1, .., V^J\right)\right)$ is an element of the J-copy of the canonical vector bundle υ, $J\upsilon$, whose dimension, over $\mathsf{G}^{J,S}$, is J^2.
Next, let $\left(\mathcal{L}, \left(V^1, .., V^J\right)\right) \in J\upsilon$, $\tau^{\mathcal{M}}\left(p, \mathcal{L}, \left(V^1, .., V^J\right), r(-1)\right) = (p, \mathcal{L}, R)$ is defined such that R is formed by $r = (r(1), r(-1))$ with
$r_s^j(1) = R_{s,1}^j = \frac{1}{p_{s,1}}\left(V_s^j - \sum_{l>1} p_{sl} r_{sl}^j(-1)\right)$ for all $(s,j) \geq (1,1)$.
Thus, $\phi^{\mathcal{M}}$ is an homeomorphism between $\mathcal{M}_J$ and $\varepsilon^{NL-1+SJ(L-1)} \oplus J\upsilon$, with its inverse, ${\phi^{\mathcal{M}}}^{-1} = \tau^{\mathcal{M}}$. Finally, to show that $\phi^{\mathcal{M}}$ is also a diffeomorphism it suffices to show 0 is a regular value of $(I \mid \psi_\sigma(\mathcal{L}))\pi_\sigma V(p_{\mathbf{1}}R)$, or that its Jacobian with respect to R is of full row rank $(S-J)J$ (as in Fact 7, Duffie and Shafer (1985)).■

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[36] This proof is substantially the same of Theorem 6 in Zhou (1997).

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