AGE MATCHING PATTERNS AND SEARCH

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Anja Sautmann

Brown University

Dept. of Economics

anja_sautmann@brown.edu

Abstract. This paper considers a marriage market with two-sided search and transferable utility in which the match payoff depends on age. It characterizes a set of payoff functions consistent with two salient marriage age patterns, (1) assortative matching by age, and (2) "differential age matching", a formalization of the age difference at marriage between men and women. Some simple conditions on slope and curvature of the payoff function in partners' ages are shown to achieve both, whereas supermodularity is not necessary for positive assortative matching, and in fact many common supermodular functions lead to negative sorting. (JEL C78, D83, J64)

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1. Introduction

Age patterns of marriage exhibit remarkable regularities. As an example, figure 1 depicts the marriage ages of women and their spouses in four different countries according to the latest Demographic and Health Surveys. While there are clearly differences, in all four countries there is strong sorting by age, and women often marry men considerably older than themselves, whereas an age difference in the opposite direction is less common and if anything usually small. Similar patterns are found in almost all countries; for example, Goldman et al. (1984) draw a similar graph using marriages from the United States in 1976-78. Consistent with these patterns men's average marriage age is higher than women's practically everywhere – the sole exception being San Marino with women on average 0.2 years older than men – and spousal age is highly correlated (United Nations (2000), Choo and Siow (2006b)). Marriage age patterns are of interest since they have implications for education, fertility and maternal health, and they affect decision making e.g. on labor supply and savings. They also play a role in the supply and demand of spouses, since population growth and the age difference at marriage determine the relative numbers of men and women in the market (see Tertilt and Neelakantan (2008), Bhat and Halli (1999) and others).

This paper develops a search model with transferable utility in which the joint payoff to a match, $f(i, j)$, depends on age i of the woman and j of the man at the time of marriage. Individuals stay in the market from age zero to at most age I and J for women and men, respectively. A matched pair leaves the market, and there is no remarriage. The ratio of unmarried men and women in the market and their age distributions are the endogenous outcome of market entry at age zero and exit from the market through marriage.

In the context of marriage the payoff function f captures individuals' preferences over their own and their spouse's age, which may be based for instance on fertility and health considerations, current wealth, or labor market value. But the model also applies more broadly to settings in which coalitions of two have to form and payoffs

Figure 1

Age at first marriage (jittered), women born before 1981 and partners, latest DHS for each country (left to right: India $n = 16802$ *(sampled from 37706 obs.), Peru* $n = 16802$*, South Africa* $n = 5410$*, Morocco* $n = 5410$ *(sampled from 8836 obs.).*

depend on time-until-match. Examples are the trade of perishable goods or time-sensitive information, or the formation of professional teams, like an executive and her assistant, or an engineer and an architect. There can be gains or losses to "aging": the value of filling a vacancy may rise over time while the skills of an unemployed worker erode, an entrepreneur and an investor may have preferences over the exact timing of capital injection and expansion, and so on.

The purpose of this paper is to characterize a set of payoff functions under which there is positive or negative assortative matching, that is the matching of similars (young-young, old-old) or opposites (young-old), as well as "differential age matching", a formalization of the difference in marriage age between men and women. Differential age matching (DAM) means that, if a woman of age b and a man of age $a < b$ marry in equilibrium, then a man b and a woman a will marry as well. DAM together with positive assortative matching (PAM) implies that a man's set of partners is "younger" than that of a woman at the same age, consistent with prevailing marriage patterns all over the world.

The theory of two-sided matching and search has long been interested in conditions for assortative matching. Usually it is assumed that the match payoff depends on a heterogenous, but fixed characteristic, for example the partners' productivity or what

Burdett and Coles (1997) call "pizzazz". Becker (1973) shows that positive assortative matching arises in a static, frictionless setting with transferable utility if the matching partners' types are complements, or equivalently if the payoff function f is supermodular ($f(x_1, y_1)+f(x_2, y_2) \geq f(x_1, y_2)+f(x_2, y_1)$ for types $x_2 > x_1$ and $y_2 > y_1$). Sattinger (1995) introduces search frictions into this model, and Shimer and Smith (2000) generalize an earlier result by Lu and McAfee (1996) for this setting and show that positive assortative matching requires supermodularity of the production function as well as log supermodularity of f_x and f_{xy}. Atakan (2006) recovers assortative matching under supermodularity alone by introducing an explicit additive search cost into the model.[1]

The present paper is closely related to Shimer and Smith's work, but instead of fixed productivity types x and y the marriage payoff depends on the partners' ages, or equivalently the time that they have spent searching in the market. As a consequence, an individual's value of search is subject to change over time, and even though all individuals are ex ante identical there will be a distribution of agents with different payoffs to marriage in the market.[2] The conditions for sorting in this setting will be shown

[1]Log supermodularity of a (positive) function f means $f(x_1, y_1)f(x_2, y_2) \geq f(x_1, y_2)f(x_2, y_1)$. There is also a rich literature on sorting with nontransferable utility. Becker showed that without search frictions both sides must have monotone preferences in their partner's type. With frictions, Smith (2006) proves that the type space can be partitioned into intervals which only match among themselves if the production function satisfies log supermodularity. Similar results for less general production functions have been proven earlier, most notably by Burdett and Coles (1997) (see Smith (2006)). Finally, Legros and Newman (2007) extend the analysis to partially transferable utility.

[2]Age-dependent match payoffs have been studied in search models of the labor market before (Ljungqvist and Sargent (1998), Coles and Masters (2000)), but in these papers only one side of the market is heterogenous (e.g. workers' productivity depends on the duration of unemployment, but all vacancies are the same) and they therefore investigate policy questions and the effects of labor market shocks, but not matching patterns *per se*.

to involve the slope and curvature of f in each partner's age. For instance, if f is decreasing in i and increasing in j at a fast enough rate, the net marriage surplus – the payoff f minus the couple's outside options – will fall in i and rise in j as well, and this guarantees PAM (as well as DAM). Remarkably, however, the conditions for sorting do not directly restrict the cross derivatives of f, and supermodularity of the payoff function is neither necessary nor sufficient for PAM. In fact, many standard supermodular production functions, most notably Cobb-Douglas functions, can be shown to lead to *negative* sorting instead. This highlights the fundamentally different role of preferences over (time-variant) age and other (fixed) characteristics for partners' matching decisions.

This paper also contributes indirectly to a body of literature on the possible causes of the positive age difference at marriage. Bergstrom and Bagnoli (1993) argue for example that men's match quality is revealed later than women's, so that high-quality men do not enter the marriage market but wait until their quality becomes common knowledge. In Siow (1998) and Giolito (2004) the marriage age difference is a consequence of women's shorter fecundity span. Coles and Francesconi (2007) propose that "fitness" decreases with age, but "success" increases, so that young individuals can trade their fitness for an old partner's success. They argue that young women match with older men more often than the reverse because their labor market opportunities are more limited.[3]

Many of these papers use specific assumptions on preferences and payoffs from marriage to explain the age difference at marriage between men and women. My focus is instead on characterizing a family of functions which guarantee differential age matching, providing a direct test of whether a particular "micro-foundation" for the marriage payoff function is consistent with DAM. Section 3 gives an example of a payoff function

[3]Choo and Siow (2006a) also study marriage payoffs by age empirically in a dynamic model without search frictions.

that is derived from a gradual loss in fecundity for women and satisfies the conditions for both PAM and DAM.

The next section describes the model and proves existence of a steady-state equilibrium. In section 3 I first define assortative matching and differential age matching. Then I illustrate the most important concepts using basic sufficient conditions on the payoff function that lead to PAM and DAM and show that a large family of payoff functions actually imply *negative* sorting, before giving a full set of general conditions. Section 4 concludes. Any omitted proofs are found in the appendix.

2. The Search Model and Marriage Market Equilibrium

The description of the model begins with the steady-state age distribution of men and women in the market. For an unmarried individual, the probability of meeting a person of the other sex at a certain age depends on the age composition of the unmarried population. This composition is the result of entry by young cohorts on the one hand, and of exit through marriage or upon reaching the final marriage age on the other. The market is in a steady state when the age distributions of singles at different ages and the relative number of men and women are constant in every period. I will start by deriving the steady-state "singles distribution", consisting of the men's and women's age distributions m and w.

Let $i \in [0, I]$ be the age of a woman, and $j \in [0, J]$ that of a man.[4] Age zero is the (normalized) age of entry into the marriage market, and I and J are the last possible marriage ages. The size of the female singles population at time t is W_t, $W_t(i)$ denotes the measure of single women at age i, and $w_t(i) = \frac{W_t(i)}{W_t}$ describes the age distribution of single women. Equivalently use M_t, $M_t(j)$ and $m_t(j)$ for single men. The singles sex ratio at t is $r_t = \frac{M_t}{W_t}$. It is assumed that equal numbers of men and women enter the

[4]As distinct from fixed types x and y used in many matching models.

marriage market at every point in time, so $W_t(0)$ and $M_t(0)$ are equal and constant for all t.[5]

Men and women meet randomly at a rate that depends on the ratio of single men to women r_t. This is modeled by assuming that the arrival rates of a potential match on each side of the market follow a Poisson process with parameters $\lambda_m(r_t)$ for the men and $\lambda_w(r_t)$ for the women. One-to-one matching requires that the number of men who meet someone in any given period equals the number of women who do. The Poisson parameters must therefore satisfy

$$r_t \equiv \frac{M_t}{W_t} = \frac{\lambda_w(r_t)}{\lambda_m(r_t)}$$

for any given r_t. Let λ_w and λ_m be positive, continuous and bounded functions (i.e. there are some minimal frictions of search). It is plausible to assume that λ_w is increasing and λ_m decreasing, so that a higher r_t raises the chance of a meeting for women and lowers that for men, since it means there are relatively more men available. As an example, $\lambda_w(r_t) = r_t^{0.5}$ and $\lambda_m(r_t) = r_t^{-0.5}$ describe symmetric meeting probabilities for the two sides of the market.[6]

Given that an individual meets someone, the model must specify the conditional probability that this person is of a particular age as a function of w_t, m_t and r_t. I will here assume proportional conditional meeting probabilities, that is, the probability that the encounter is with a person of a particular age equals the proportion of that age in the singles population. In terms of the matching function of traditional search models, this assumption implies that the matching technology has constant returns to

[5]This assumption can be relaxed e.g. to allow for a constant population growth rate or an unequal (but fixed) ratio of $W(0)$ and $M(0)$.

[6]Note however that this cannot hold for all possible r_t, or else λ_w and λ_m are unbounded. Bounds could be introduced by assuming, say, that the maximal meeting frequency allows on average two encounters per period. We would then have $\lambda_m(r_t) = 2$ for any $r_t < 0.25$ and therefore $\lambda_w(r_t) = 2r_t$, and similarly $\lambda_m(r_t) = 2r_t^{-1}$ and $\lambda_w(r_t) = 2$ for $r_t > 4$).

scale (CRS), so that only relative numbers of men and women matter for the meeting probabilities. Other functional forms for the meeting probabilities can be adopted without changing the conclusions of the paper, provided the CRS property remains satisfied.

Upon meeting, couples decide if they want to marry. The marriage decision for a pair (i,j) is captured by a measurable marriage indicator function $\alpha : [0,I] \times [0,J] \to [0,1]$. If α lies strictly between zero and one the couple is using a mixed strategy between marrying and staying single. For given marriage behavior α, the mass of single women at age i in t can be written as

$$W_t(i) = e^{-\int_0^i \lambda_w \left(\int_0^J \alpha(x,y) m_{t-i+x}(y)\mathrm{d}y\right)\mathrm{d}x} W_{t-i}(0).$$

The exponential term is equivalent to the survival function in a hazard model, that is, it describes the probability of staying in the market past age i. The exponent of the survival function is the integral of the hazard function, here given by $\lambda_w \int_0^J \alpha(i,y) m_t(y)\mathrm{d}y$, from 0 up to i. The hazard function describes the instantaneous rate of leaving the market at age i in time t, conditional on still being single. It combines the probability of meeting ($\lambda_w m_t(y)$) and of marrying ($\alpha(i,y)$) any man between 0 and J years of age.

Normalizing by the overall population size at t, the age distribution at t is

$$w_t(i) = \frac{W_t(i)}{W_t} = \frac{e^{-\int_0^i \lambda_w \left(\int_0^J \alpha(x,y) m_{t-i+x}(y)\mathrm{d}y\right)\mathrm{d}x}}{\int_0^I e^{-\int_0^u \lambda_w \left(\int_0^J \alpha(x,y) m_{t-i+x}(y)\mathrm{d}y\right)\mathrm{d}x} \mathrm{d}u}.$$

A similar expression can be written for the men's market side. In a steady state, the age distributions are invariant, so that the time index can be suppressed and we have

$$(2.1) \qquad w(i) = \frac{e^{-\int_0^i \lambda_w(r)\left(\int_0^J \alpha(x,y) m(y)\mathrm{d}y\right)\mathrm{d}x}}{\int_0^I e^{-\int_0^u \lambda_w(r)\left(\int_0^J \alpha(x,y) m(y)\mathrm{d}y\right)\mathrm{d}x} \mathrm{d}u}$$

$$(2.2) \qquad m(j) = \frac{e^{-\int_0^j \lambda_m(r)\left(\int_0^I \alpha(x,y) w(x)\mathrm{d}x\right)\mathrm{d}y}}{\int_0^J e^{-\int_0^u \lambda_m(r)\left(\int_0^I \alpha(x,y) w(x)\mathrm{d}x\right)\mathrm{d}y} \mathrm{d}u},$$

where the ratio of men to women $r = \frac{M}{W}$ equals $\frac{w(0)}{m(0)}$ (this follows from the observation that $w(0) = 1/W$ and $m(0) = 1/M$). Clearly the age distribution functions are decreasing for all ages: after entering the market, each singles cohort is reduced gradually by those who marry. Different marriage ages for men and women are reflected in different w and m; if for instance women marry earlier than men, the age distribution of unmarried women will be more concentrated at the low ages, and there will be overall more single men than women, since each man remains in the market longer (and $m_0 < w_0$ so that $r < 1$).

The next step is to characterize marriage behavior described by α. When a man and a woman meet, they make their marriage decision by comparing the payoff to marrying with the outside value of remaining in the market and meeting other partners in the future, when they are themselves older. This outside value depends on the relative numbers of men and women at each age available for marriage, in other words, the singles distribution given by w and m, and the age of the individual i or j. In a steady-state equilibrium each individual's outside option is described by a (time-invariant) value function. To derive it I will focus on the women's side of the market, stating the equivalent expressions for the mens' side without derivations.

Denote by $f_w(i,j)$ and $f_m(i,j)$ the (expected) payoffs of marriage for a woman and a man at age i and j, respectively, and let $d(i,j)$ be the transfer paid by the bride to the groom. A couple will marry upon meeting if they both prefer marrying each other over continuing search. The woman agrees to marry if the net payoff $f_w(i,j) - d(i,j)$ exceeds the value of searching at age i, $V(i)$. Equivalently, the man compares search value $H(j)$ with $f_m(i,j) + d(i,j)$. Following the literature (e.g. Pissarides (1990)), the expected marriage transfer between ages i and j is assumed to be the generalized Nash product, given by

$$d(i,j) = \arg\max_d \left[f_w(i,j) - d - V(i)\right]^{\theta} \left[f_m(i,j) + d - H(j)\right]^{(1-\theta)}$$

where θ describes the woman's bargaining power. The woman's payoff from marrying is

$$V(i) + \theta \left(f_w(i,j) + f_m(i,j) - V(i) - H(j)\right),$$

her outside option plus a share θ of the net marriage surplus.[7] As long as this share is positive, she will agree to marry. Similarly, the groom gets his outside option plus a $(1-\theta)$ share of the net surplus. This implies that the marriage decisions of a couple (i, j) coincide and can be described by a single function α indicating a positive net marriage surplus. For the value function (but not the transfer $d(i, j)$) the source of the marriage payoff is not important, so $f_w(i, j) + f_m(i, j)$ can be summarized as $f(i, j)$. I assume that f is continuous and positive for at least some (i, j).

Define the net marriage surplus as $S(i, j) = f(i, j) - V(i) - H(j)$ and let the discount rate be ρ. The value functions are described by a system of differential equations (see appendix for the derivation) where the final conditions are given by the value of being single when leaving the marriage market, $\underline{V}$ for the woman and $\underline{H}$ for the man. $\underline{V}$ and $\underline{H}$ are assumed to be nonnegative.[8]

$$\dot{V}(i) = -\lambda_w(r)\theta \int_0^J \max\{S(i, y), 0\}\, m(y)\mathrm{d}y + \rho V(i) \tag{2.3}$$

$$V(I) = \underline{V} \tag{2.4}$$

$$\dot{H}(j) = -\lambda_m(r)(1-\theta) \int_0^I \max\{S(x, j), 0\}\, w(x)\mathrm{d}x + \rho H(j) \tag{2.5}$$

$$H(J) = \underline{H}. \tag{2.6}$$

The first term in $\dot{V}(i)$ describes the reduction in the value of search when age i passes and with it the chance of marrying someone for a surplus of $S(i, y)$. Note that the term is weakly negative, so $V(i)$ can be at most increasing by $\rho V(i)$. This second term is

[7] A possible interpretation is that either the man or the woman gets to make an ultimatum proposal for the division of the surplus, where the probability of being the proposer equals the bargaining power.

[8] Implicit in this is the normalization of the payoff of being single in the market to zero. Thus, $\underline{V}$ and $\underline{H}$ describe the value to being single, but not searching anymore. Positive values might for example indicate a cost of searching, which is not incurred after the individual leaves the market. A value of zero implies that the individual is indifferent between spending a period in or out of the market.

greater than zero only if $\rho > 0$, that is, the market participants discount the future. It describes the increase in the expected value of future matches that comes from reducing the time span until the match payoff is realized. $V(i)$ and $H(j)$ are bounded below at zero and above at $\max_{(i,j)} f(i,j)$. It should be emphasized once more that the value of search for an individual is a function of age, and therefore of time, even though the market is in a steady-state. The distribution of ages and the value functions are invariant, but an individual's search value is not.[9]

To summarize, in a steady state equilibrium the singles age distributions depend on marriage behavior, described by the indicator function α, and marriage behavior is in turn determined by the marriage payoff function and the age distributions. Men and women's expectations about meeting probabilities later in life are correct and identical to current market conditions. Formally, define:

Definition 2.1 (Steady-State Matching Equilibrium)**.** For a given marriage payoff function f and Poisson meeting rate functions λ_w and λ_m, a matching equilibrium is given by an indicator function α, value functions V and H, and singles distributions m and w such that

(1) $\alpha(i,j) = 1$ if $S(i,j) > 0$, $\alpha(i,j) = 0$ if $S(i,j) < 0$ and $\alpha(i,j) \in [0,1]$ otherwise,

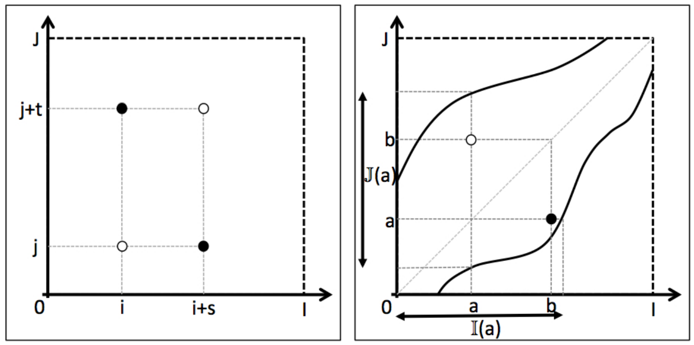
(2) m and w are age distributions for α satisfying equations (2.1) and (2.2), and

(3) $V(i)$ and $H(j)$ are the value functions for m and w given by (2.3)–(2.6).

To show that such an equilibrium exists I construct a self-map $\mathcal{T}$ on the set of value functions and singles distributions whose fixed points coincide with the matching equilibria. Existence of a fixed point is then proved by applying the Eilenberg-Montgomery fixed point theorem.

[9] In a typical search model with fixed productivity types, V and H themselves, not their derivatives, are a function of the net surplus of matching. The equations here cannot easily be solved for V and H (compare with equations (6) and (8) in Shimer and Smith (2000)).

FIGURE 2.



PAM: If (i+s, j) and (i, j+t) marry, so do (i,j) and (i+s, j+t).

DAM: For any a<b, if (b, a) marry, so do (a,b).

Theorem 2.2 (Existence of a Steady-State Matching Equilibrium)**.** *Suppose marriage payoffs are given by $f(i,j)$, and λ_w and λ_m are positive, bounded and continuous functions that satisfy $r\lambda_m(r) = \lambda_w(r)$. Then there exists a steady-state matching equilibrium for this market.*

Observe that a change to who matches with whom (i.e. the α) affects the entire singles distribution and therefore each agent's value of search, leading to potential multiplicity; in other words, the equilibrium may not be unique.

3. Assortative Matching and Differential Age Matching

The discussion of equilibrium age matching patterns begins with the definitions for assortative matching and differential age matching.

Denote the matching set of a man of age j by $\mathbb{I}(j) = \{i|S(i,j) \geq 0\}$ and that of a woman at age i by $\mathbb{J}(i) = \{j|S(i,j) \geq 0\}$. For later reference define $\mathbb{M}$ to be the set of couples (i,j) in $\mathbb{R}^2$ who are willing to match with each other, i.e. $\mathbb{M} = \{(i,j)|S(i,j) \geq 0\}$. Shimer and Smith (2000) have defined assortative matching for the search context.

Definition 3.1 (Positive assortative matching)**.** Consider any $t, s > 0$. There is positive assortative matching (PAM) if for any i and j such that $i \in \mathbb{I}(j+t)$ and $i+s \in \mathbb{I}(j)$, we have $i+s \in \mathbb{I}(j+t)$ and $i \in \mathbb{I}(j)$. $\diamond$

The left panel of figure 2 illustrates PAM. Its counterpart is negative assortative matching:

Definition 3.2 (Negative assortative matching)**.** Let $t, s > 0$. There is negative assortative matching (NAM) if for any i and j such that $i \in \mathbb{I}(j)$ and $i + s \in \mathbb{I}(j + t)$, we have $i + s \in \mathbb{I}(j)$ and $i \in \mathbb{I}(j + t)$. $\diamond$

The second pattern of interest is the difference in marriage age between men and women. In a typical marriage market, a man's partner is of equal age or considerably younger than him, but rarely much older, whereas the opposite is true for women. Correspondingly, the bulk of marriages in figure 1 lie to the upper left of the 45-degree line. The following definition captures this asymmetry (see figure 2, right).

Definition 3.3 (Differential age matching)**.** Consider ages a and b such that $0 \leq a \leq b \leq I, J$. There is differential age matching (DAM) if $b \in \mathbb{I}(a)$ implies $b \in \mathbb{J}(a)$. $\diamond$

From this definition it follows immediately that with DAM a man at age a either marries only women younger than himself, or, if he does marry any woman at age $b > a$, then a woman at age a marries men of age b or older as well. Similarly, a woman of given age b either marries only men older than b, or else if $a < b$ is in her matching set, then a man at age b marries a woman at age a as well. Put differently, let i_1 and i_2 be the lowest and highest age in $\mathbb{I}(a)$, and j_1 and j_2 the lowest and highest matching age in $\mathbb{J}(a)$, for some $0 \leq a \leq I, J$. Then either $i_2 < a$ for man a or $j_2 \geq i_2 \geq a$; i.e. everyone in the man's matching set is either younger than himself or at least younger than the oldest match of a woman of the same age. Equivalently, for a woman b, either $j_1 \geq b$ or $i_1 \leq j_1 < b$, that everyone in her matching set is older than herself or at least older than the youngest match of a man of equal age.

If there is in addition positive assortative matching, then DAM means that a woman at age a considers a set of men acceptable for marriage that is "older" than the set of women that a man at the same age a considers acceptable.

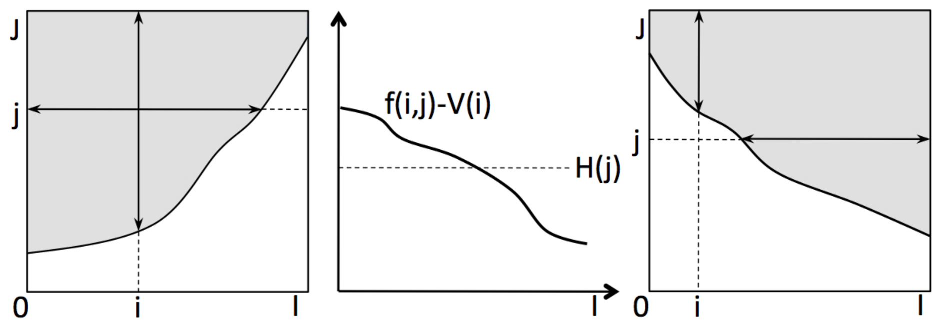
Lemma 3.4. *Suppose there is PAM. Then DAM implies that $j_1 \geq i_1$ and $j_2 \geq i_2$ for any man and woman of the same age, that is, the woman's set of partners is "older" than the man's. If all women's matching sets are nonempty intervals, the converse is also true.*

Proof: Consider a man of age a. By the argument above we only need to argue the case where $i_2 < a$. Suppose $j_2 < i_2$. Since (i_2, a) match, by DAM so do (a, i_2), and PAM then implies (a, a) marry as well. But this contradicts that i_2 and j_2 are the oldest ages that woman a and man a marry. Now take a woman at age b. Again, we only need to consider cases in which $j_1 \geq b$. Assume $i_1 > j_1$. Since (i_1, b) marry, DAM and PAM imply that (b, i_1) and thus (b, b) match, too, contradicting that i_1 and j_1 are the youngest partners of men and women at age b, respectively. For the converse, suppose women's partners are older than men's, and consider $b > a$ so that (b, a) marry. Since $j_2 \geq i_2 \geq b$, (a, j_2) also marry, and by PAM so do (a, a) and (b, j_2). But then woman a marries $j_2 \geq b$ and $a < b$, so by convexity $b \in \mathbb{J}(a)$. $\square$

The definition of differential age matching deliberately does not use average ages, but provides a distribution-free characterization similar in spirit to the definition of assortative matching by Shimer and Smith (2000). Note that DAM would be equivalent to men marrying on average younger women if the distributions m and w were identical, but of course the matching asymmetry will lead to different m and w. Nonetheless in many cases DAM will be accompanied by a positive average age difference between spouses. In general, DAM implies a positive average age gap if $w(a)m(b) \geq w(b)m(a)$ for all $a < b$.[10]

[10] To see this note that the difference in average marriage age is a multiple of $\int\int (j - i)\alpha(i, j)w(i)m(j)\mathrm{d}i\mathrm{d}j$. The integral is weakly positive on the set of ages with $\alpha(a, b) = \alpha(b, a)$. But $\alpha(a, b) > 0$ and $\alpha(b, a) = 0$ can occur only if $a < b$, so the term as a whole must be positive.

Figure 3.



There is PAM and DAM if matching sets are intervals including J for the women and 0 for the men (left; matching sets shaded grey). For the latter $S(i, j)$ must be decreasing for all i (middle). There is NAM if matching sets include J and I (right).

3.1. Interval Matching, Sorting, DAM, and Supermodularity

Before introducing the most general conditions for the matching patterns of interest, this section discusses some simpler sufficient conditions which illustrate the issues at hand.

A straightforward case in which there will be both positive sorting and differential age matching is if the matching set of each woman is an interval that includes the oldest men at age J, and each man's matching set includes the youngest (age zero) women. This is illustrated in figure 3 on the left. The figure shows the matching set $\mathbb{M}$ shaded in grey (keep in mind in this context that the graphs in figure 1 show *realized* matches between couples (i, j), which occur with a relative frequency of $\alpha(i, j)m(j)w(i)$). It is easily verified that the conditions for PAM and DAM both hold: for any (i, j) who marry, all (i', j') with $i' \leq i$ and $j' \leq j$ match as well.

Now consider a man at age j: since his outside option $H(j)$ is fixed, the shape of his matching set depends on $f(i, j) - V(i)$ as a function of i. If $f - V$ is greater than H on an interval of ages i, the matching set of j will be that interval. To get the matching pattern in figure 3 on the left it is therefore sufficient that $f - V$ is decreasing in i (fig. 3, middle), and $f - H$ is increasing in j.

Proposition 3.5 (Sufficient conditions for PAM and DAM)**.** *Suppose f is continuously differentiable and satisfies*

(DEi) $$f_i(i,j) \leq -\lambda_w \theta \max_y f(i,y) \quad \textit{for all } i,\ j, \textit{ and } \lambda_w, \textit{ and}$$

(INj) $$f_j(i,j) \geq \rho f(i,j) \quad \textit{for all } i \textit{ and } j.$$

Then there is PAM and DAM.

The right-hand side of (DEi) is a lower bound for $\dot{V}(i)$ as long as $\max_y f(i,y) \geq 0$, so that $f - V$ is decreasing on all relevant (i,j) (where $f(i,j) \geq 0$). (INj) ensures that $f_j \geq \rho H(j) \geq \dot{H}(j)$ whenever $f(i,j) \geq H(j)$, implying that $S(i,j)$ must be increasing in j if it is positive. These conditions guarantee that each man's matching set is either empty or an interval including 0, and each woman's matching set is either empty or an interval including J, thus delivering PAM and DAM.

A necessary condition for proposition 3.5 is that f is decreasing in i and increasing in j wherever $f(i,j) \geq 0$, so that the match payoff must be maximal at $f(0,J)$. (INj) and (DEi) do however clearly *not* require supermodularity of f (they do not preclude it either; note that a supermodular function which satisfies (DEi) at J and (INj) at $i = 0$ will automatically satisfy them at all i and j).

Next, consider a different case. Suppose (INj) holds as before, but now its counterpart for i,

(INi) $$f_i(i,j) \geq \rho f(i,j) \text{ for all } i \text{ and } j.$$

holds instead of (DEi). $S(i,j)$ is then increasing in i and j, and all matching sets are intervals including the oldest age I or J (figure 3, right). This means there is now *negative* sorting: if any (i,j) marry, then i marries any $j' > j$ and j matches with all $i' > i$ as well.

Proposition 3.6 (Sufficient Conditions for NAM)**.** *Suppose f is continuously differentiable and satisfies (INi) and (INj). Then there is* negative *assortative matching.*

The family of functions for which 3.6 applies is large. In particular, it includes many standard supermodular production functions, for example all Cobb-Douglas functions $f = Ai^a j^b$ with $\frac{a}{I}, \frac{b}{J} \geq \rho$.[11] Dividing through by $f(i,j)$, (INi) and (INj) say that there is negative assortative matching whenever f increases at a percentage rate greater than the discount rate. As $\rho \to 0$, this holds for any increasing function whose derivatives are bounded away from zero, and if f is supermodular it is sufficient that $f_i(i,0)$ and $f_j(0,j)$ are strictly positive. Last but not least, if $f(i,j) = \frac{1}{\rho}g(i,j)$, i.e. f is the present value of the flow payoff g, and g satisfies (INi) and (INj), so does f. What this means in turn is that the payoff function in a market that does *not* exhibit negative sorting – like the marriage market – cannot be a member of any of these function families. For example, the marriage payoff function cannot be increasing at a significant rate in both ages, unless individuals are very impatient.

The intuition for this result is straightforward. In a market with fixed productivity types, supermodularity of f implies that positive sorting maximizes the overall output of the economy. Since high types are best matched with other high types, low types have to marry each other as well. Even with search frictions, the market outcome approximates this matching pattern if some additional conditions are satisfied. If f instead depends on age and is increasing in both arguments, as required for prop. 3.6, payoff is maximal at the highest ages (I, J), and it is again optimal if high types, i.e. old ages, match with each other. However, this does not imply that young couples should marry: on the contrary, both sides optimally wait until they are older, and the youngest men and women do not marry at all. This leads to a matching pattern as in figure 3 and therefore negative sorting.

It is possible to generalize the above conditions further, and this is done in the next section. But first I will introduce another approach to age sorting that follows Shimer and Smith (2000). They show that there is PAM (NAM) if the matching sets $\mathbb{I}(\cdot)$ and

[11] Cobb-Douglas functions also satisfy the log supermodularity conditions in Shimer and Smith (2000).

$\mathbb{J}(\cdot)$ are nonempty intervals, $\mathbb{M}$ is closed[12], and there is "matching in the corners", that is, the couples $(0,0)$ and (I,J) marry ($(0,J)$ and $(I,0)$ for NAM).

Since $S(i,j)$ is continuous, the preimage of $\mathbb{R}_0^+$ under S, that is $\mathbb{M}$, is closed. Moreover, it is easy to guarantee that (I,J) marry each other: all that is required is $f(I,J) \geq \underline{V} + \underline{H}$, i.e. marrying at the last possible ages is preferred to being single forever. The two remaining conditions need a little more work.

The first step are conditions under which all matching sets are convex and nonempty. From here on assume that f is twice differentiable.[13] Before I showed that all matching sets are intervals if $S(i,j) = f(i,j) - V(i) - H(j)$, or equivalently, $f - V$, is monotonic in i. Now consider a man at age j and suppose $f - V$ is quasi-concave in i. Then j's matching set will be again an interval of ages i, except this time it need not include 0 or I. The following conditions (A$\cdot$) will be shown to guarantee quasi-concavity of $S(i,j)$ in i and j, respectively:

$$\text{(Ai)} \qquad f_{ii}(i,j) \leq \lambda_w \theta \min_y \{f_i(i,j) - f_i(i,y)\} + \rho f_i(i,j) \quad \text{for all } i \text{ and } \lambda_w$$

$$\text{(Aj)} \qquad f_{jj}(i,j) \leq \lambda_m (1-\theta) \min_x \{f_j(i,j) - f_j(x,j)\} + \rho f_j(i,j) \quad \text{for all } i.$$

Equation (Ai) is an assumption on the curvature of f in i. The first term of the right-hand side depends on the variation of f_i in j, and it must be (weakly) negative, so that for patient agents with $\rho \to 0$ the condition is a strengthening of concavity of f in i ((A$\cdot$) is a mnemonic for concAve).

Claim 1: Suppose that for all i on an interval (i_1, i_2), (Ai) holds for any λ_w, and $f_i(i,j) - \dot{V}(i) \leq 0$. Then $f_i(i_2,j) - \dot{V}(i_2) \leq f_i(i_1,j) - \dot{V}(i_1)$.

The claim essentially says that $f - V$ will be concave in i whenever it is decreasing, as long as f itself is "concave enough" in i. This should be understood in relative

[12] This is equivalent to their original condition of upper hemicontinuity of the matching correspondence $\mathbb{I}(j)$ or $\mathbb{J}(i)$, since the domain is compact.

[13] This can be relaxed to differentiability almost everywhere.

terms: f_{ii} may be positive if ρ and f_i are large and f_i varies only little in j. (Aj) is the equivalent of (Ai) for the men's side, and a parallel statement to Claim 1 applies.

Lemma 3.7 (Single-peaked $S(i,j)$)**.** *Suppose for a given men's age j, (Ai) is true for all i. Then j's matching set is a closed interval. Equivalently, if (Aj) holds for all j, then i's matching set is a closed interval.*

The proof uses Claim 1 to show that whenever $f(i,j) - V(i)$ is strictly decreasing at i', it will also be decreasing at all consecutive ages $i'' > i'$, implying that $S(i,j)$ is a single-peaked function and therefore quasi-concave in i (and vice versa for $f - H$).

Next, to ensure that no matching sets are empty we need

Lemma 3.8 (Nonempty matching sets)**.** *Suppose $f(i,j)$ is either nonincreasing in i or in j for all (i,j), the matching sets are convex and $(0,0)$ and (I,J) match. Then all individuals have nonempty matching sets.*

Consider the women's market side: this result stems from the observation that the surplus from an age i's *best* match forms an upper bound to the change in her outside option at that point. All lower ages in the vicinity of i will therefore also match with this best match. Put differently, if the payoff from marriage is decreasing in i for *all* j, then there must be at least one j whom the age i woman would prefer to marry today rather than tomorrow. The women's matching sets are nonempty, and the union of those sets, $\mathbb{M}$, is (path)connected. If $(0,0)$ also marry and $i = 0$ has a convex matching set, all matching sets must be nonempty. The same argument in reverse holds for the men's side.

Lemma 3.8 makes clear that the requirement of nonempty matching sets is here an actual restriction, unlike in Shimer and Smith, where symmetry ensures that all types match with themselves. It turns out that the requirement that $(0,0)$ marry is demanding as well. Suppose for instance that f is decreasing in j so that all j have nonempty matching sets. Then $(0,0)$ can be guaranteed to marry if $f(i,0) - V(i)$ is decreasing, i.e. if (DEi) holds at $j = 0$. Equivalently, if the j have nonempty matching sets then

$$\text{(DEj)} \qquad f_j(i,j) \leq -\lambda_m(1-\theta)\max_x f(x,j) \text{ for all } j \text{ and } \lambda_m.$$

must hold at $i = 0$. But these requirements are fairly strong; a necessary condition for both is for instance that $f(0,0)$ is the maximal possible marriage payoff.

Proposition 3.9. *Suppose (Ai) and (Aj) hold and (I, J) marry. If f is decreasing in i and (DEj) holds for $i = 0$, or if f is decreasing in i and (DEi) holds for $j = 0$, so that $(0,0)$ match, there is positive assortative matching.*

As in the previous set of conditions, however, supermodularity plays no direct role for positive sorting, and no log-supermodularity conditions are needed. From a technical perspective, this difference arises because the derivative of the value function in Shimer and Smith (2000) is nonlinear in the endogenous type distribution. Quasi-concavity then requires a single-crossing argument that relies on log-supermodularity. By contrast, $\dot{H}$ and $\dot{V}$ are linear in the singles distribution, making quasi-concavity easier to achieve, a property this model shares with Atakan (2006).[14]

On a more intuitive level, the difference lies in the fact that time or age can change the value of search. The productivity types x and y are *defined* by the production function $f(x, y)$, and if the types are relabeled, e.g. with a new production function $g(x, y) = f(1-x, 1-y)$, the economy remains essentially the same. Agents of different types are only connected by the fact that they are searching in the same market. By contrast, there is a natural link between different ages. Ages $i' > i$ and $j' > j$ enter the value functions of a couple (i, j) because they are part of their future. The model is fundamentally asymmetric in that old and young cannot be interchanged. Most importantly, young men and women can transform themselves into older persons simply by waiting. At early ages, both partners have all opportunities in this market still ahead of them, so it may be profitable for patient agents to wait until a later age and marry for a higher net surplus, even if the current marriage payoff given by f is

[14] I thank an anonymous referee for pointing this out to me.

high. Conversely, even "good types" with high current payoffs from f may be willing to make a "bad match" if their payoff function deteriorates quickly with age and the value of further search is low.

3.2. General Conditions for Positive and Negative Sorting

In the previous section I did not introduce the most general conditions possible for NAM and PAM. To begin with, it was noted that there is PAM if the women's matching sets are intervals including J and the men's are intervals including 0. But there will also be positive sorting (although not DAM) if the opposite holds, namely if the men's matching sets are convex and include I, and the women's matching sets are intervals including 0. Similarly, there is NAM if both the men's and the women's matching sets are intervals including age 0.

In addition, it is possible to relax the conditions on f for decreasing and single-peaked $f - V$ and $f - H$ themselves. As a first step, define conditions

$$\text{(Xi)} \qquad f_{ii}(i,j) \geq \lambda_w \theta \max_y \{f_i(i,j) - f_i(i,y)\} + \rho f_i(i,j)$$

$$\text{(Xj)} \qquad f_{jj}(i,j) \geq \lambda_m (1-\theta) \max_x \{f_j(i,j) - f_j(x,j)\} + \rho f_j(i,j)$$

Claim 2: Suppose that for all i on an interval (i_1, i_2), (Xi) holds for any λ_w, and $f_i(i,j) - \dot{V}(i) \geq 0$. Then $f_i(i_2,j) - \dot{V}(i_2) \geq f_i(i_1,j) - \dot{V}(i_1)$.

(Xi) is the counterpart to (Ai). Now strong enough *convexity* of f ensures that $f - V$ is *convex* (but again note the relative quality of the statement, as f_{ii} may still be negative). (Xj) is again the same condition for the men's side (the equation labels stand for conveX). The next lemma uses (Xi) and (Xj) as part of conditions under which $f(i,j) - V(i)$ ($f(i,j) - H(j)$) is decreasing everywhere, so that all matching sets are intervals including age 0.

Lemma 3.10 (Decreasing $S(i,j)$)**.** *Suppose the equivalent of (DEi) holds at $i = I$, that is*

$$f_i(I,j) \leq -\lambda_w \theta \max_y [f(I,y) - \underline{V}] \textit{ for all } \lambda_w,$$

and either (DEi) or (Xi) holds for any $i \in [0, I)$. Then an age j man's matching set is a closed interval starting at 0. Similarly, if

$$f_j(i, J) \leq -\lambda_m(1-\theta)\max_x[f(x, J) - \underline{H}] \text{ for all } \lambda_m,$$

and either (DEj) or (Xj) holds on $j \in [0, J)$, then the matching set of age i is a closed interval starting at 0.

Proof: Fix j. $f - V$ is decreasing at I and at all i where (DEi) holds. Now assume by contradiction that $f(i,j) - V(i)$ is not decreasing everywhere and let i_1 be such that $f_i(i_1, j) - \dot{V}(i_1) > 0$. By continuity there must be $i > i_1$ such that $f_i(i, j) - \dot{V}(i) = 0$. Let i_2 be the smallest such i, so that $f_i(i, j) - \dot{V}(i) > 0$ for all $i \in [i_1, i_2)$. Since (Xi) holds, claim 2 applies, and we have $0 = f_i(i_2, j) - \dot{V}(i_2) \geq f_i(i_1, j) - \dot{V}(i_1)$, a contradiction. $\square$

For any subinterval on which (Xi) holds, (DEi) "anchors" $f_i - \dot{V}$ below zero at the endpoint of the interval. But by (Xi) f is convex whenever it is weakly increasing, so $f_i - \dot{V}$ cannot cross over into positive values and the payoff function must be falling everywhere.

(Xi) can be substantially less demanding than (DEi) when there is little variation in f_i across j.[15] Assume for example that $f(I, j) \leq \underline{V}$ and f_i is invariant in j. Then (Xi) only requires that f is weakly convex, and as a direct corollary from proposition 3.5 we get that any $f(i,j) = g(i) + h(j)$ with $h'(j) > 0$, $g'(i) \leq 0$ and $g''(i) \geq 0$ will exhibit PAM and DAM if agents are patient enough. Borrowing from the literature on the marriage age difference mentioned in the introduction, an example could be that the woman's aging imposes a convex cost on the marriage, perhaps as a consequence of the woman's decline in "fitness" or fecundity, but that the value of the partnership increases with the man's rising labor market success. Lemma 3.10 also relaxes the condition needed in proposition 3.9 to ensure that $(0, 0)$ marry.

[15] An equivalent to this lemma for increasing $S(i, j)$ is possible using (Ai) (see earlier versions of this paper); but for $\rho \to 0$ (Ai) implies (INi).

The next corollary quite literally pieces all the previous findings on the shape of $S(i,j)$ together into a general condition for a single-peaked surplus function $S(i,j)$. Suppose for instance the condition (Ai) from proposition 3.7 is satisfied on an interval from age 0 to i_1, and condition (DEi) holds for all $i > i_1$. Above i_1, $S(i,j)$ must be decreasing in i, and below it is concave in i whenever it is decreasing (Claim 1), so the function can be at most single-peaked. Taking this further, if (INi) holds initially $S(i,j)$ must be first increasing, then may turn downwards while (Ai) applies, and finally will be decreasing for sure under (DEi).

Corollary 3.11 (Single-peaked $S(i,j)$). *Suppose that there exist i_1 and i_2 so that (Ai) is true for all $i \in [i_1, i_2]$, (INi) holds below i_1, and the conditions of lemma 3.10 for the women's side apply for $(i_2, I]$. Then j's matching set is a closed interval. Equivalently, if (INj) holds below some j_1, (Aj) holds on $[j_1, j_2]$, and the conditions of 3.10 for the men's side apply on $(j_2, J]$, then i's matching set is a closed interval.*

This proposition subsumes the previous conditions for interval matching, since i_1 and i_2 may equal 0 and I, and j_1 and j_2 can be 0 or J.

Now the conditions for positive assortative matching can be generalized as follows.

Corollary 3.12 (Positive assortative matching). *There is PAM if*

(1) *(IN $\cdot$) holds for one market side and the conditions in lemma 3.10 (decreasing $S(i,j)$) hold for the other; or*

(2) *the conditions in lemma 3.10 hold for one market side, those in corollary 3.11 (single-peaked $S(i,j)$) hold for the other, and $f(I,J) - \underline{V} - \underline{H} \geq 0$, so that (I,J) match; or*

(3) *corollary 3.11 applies for i and j so that all matching sets are convex, $f(i,j)$ is decreasing in i (or j), and (I,J) and $(0,0)$ match.*

Both (1) and (2) imply that one market side's matching sets include age 0, and the other side's include I or J. To see this for (2), suppose the surplus function is decreasing in i only, so that the men's matching sets are convex and include 0 (lemma

3.10 holds for the women's side). If $f(I, J) \geq \underline{V} + \underline{H}$ and the women's matching sets are convex, there will be PAM, since J marries I and therefore all i, and convexity then means that the women's matching sets are intervals including J. The same argument holds with the two market sides exchanged. Part (3) uses corollary 3.11 to generalize proposition 3.9.

Parts (2) and (3) of the corollary rely on "matching in the corners" at $(0, 0)$ and (I, J). To make similar arguments for NAM, it would have to be ensured that $(0, J)$ and $(I, 0)$ marry. This is a demanding condition, and it appears that it can only be guaranteed if $\underline{V} = \underline{H} = 0$ and $f(0, J) = f(I, 0) = \max f$ (since $V(0)$ and $H(0)$ are bounded by $\max f$). This conflicts with a decreasing payoff function needed for nonempty and convex matching sets. The generalization of proposition 3.6 is therefore more limited, demonstrating again that the constraints on matching in the corners together with the requirement of nonempty matching sets are quite strong in this setting.

Corollary 3.13 (Negative assortative matching)**.** *There is NAM if*

(1) *(INi) and (INj) hold for all i and j, or*

(2) *the conditions in lemma 3.10 (decreasing $S(i, j)$) hold for all i and j.*

3.3. The Age Difference Between Couples

In addition to assortative matching, I would like to identify a set of conditions that is sufficient for DAM and compatible with the conditions for positive assortative matching. From figure 3 (left) and parts (1) and (2) of corollary 3.12 above it is clear that there is a way to immediately get the desired result: if the conditions in 3.10 (decreasing $S(i, j)$) hold for the women's side, so that men's matching sets are intervals including zero, and either (INj) holds for the men or the conditions in corollary 3.11 apply and $I \in \mathbb{I}(J)$, so that all women's matching sets are convex and include J, there will be not only PAM but also DAM.

For completeness, however, let us also consider conditions for DAM that do not require matching at the youngest or oldest ages. Take any two ages $0 \leq a < b \leq I, J$.

Differential age matching is guaranteed whenever $f(b,a) - V(b) - H(a) \geq 0$ implies $f(a,b) - V(a) - H(b) \geq 0$. That is the case if $f(a,b) - f(b,a) - (V(a) - V(b)) \geq H(b) - H(a)$, or equivalently either

$$\int_a^b f_j(b,y) - \dot{H}(y)\mathrm{d}y \geq \int_a^b f_i(x,b) - \dot{V}(x)\mathrm{d}x \qquad \text{for all } x, y < b, \text{ or}$$
$$\int_a^b f_j(a,y) - \dot{H}(y)\mathrm{d}y \geq \int_a^b f_i(x,a) - \dot{V}(x)\mathrm{d}x \qquad \text{for all } x, y > a.$$

This requires that for a man and a woman of the same age, with spouses in the same age ranges, the surplus from marriage falls faster (or rises less fast) in the female spouse's age than in the male spouse's age. If this conditions holds the surplus between an old man and a young woman is higher than between a young man and old woman at the same ages. Thus, if it is known that $f_i(\cdot, c) - \dot{V}(\cdot) \leq \delta \leq f_j(c, \cdot) - \dot{H}(\cdot)$ for some δ and all possible $i, j > c$ (or all $i, j < c$), there must be differential age matching. The next lemma guarantees just that. For given age c, consider the following conditions:

(DEi') $\qquad f_i(i,c) + \delta \leq -\lambda_w \theta \max_y f(i,y)$

(Xi') $\qquad f_{ii}(i,c) \geq \max\{0; \lambda_w \theta \max_y [f_i(i,c) + \delta - f_i(i,y)\} + \rho(f_i(i,c) + \delta)$

(INj') $\qquad f_j(c,j) + \delta \geq \rho f(i,j)$

(Aj') $\qquad f_{jj}(c,j) \leq \min\{0; \lambda_m(1-\theta) \min_x [f_j(c,j) + \delta - f_j(x,j)]\} + \rho(f_j(c,j) + \delta).$

Comparing them with (DEi), (Xi) etc. shows that they are the same, except for an additional "wedge" of $-\delta$ added to the first derivatives. However, δ can be positive or negative, so that the new equations may be more or less demanding than the old ones (and of course each of the conditions needs to hold only on a subset of (i,j)). For example, even if (DEi') and (INj) hold, f may be increasing in i or decreasing in j, respectively.

Lemma 3.14. *Let* $0 \leq c \leq I, J$.

(1) *Suppose there exists δ such that (DEi') holds at I and either (DEi') or (Xi') are satisfied for all other $i \in [c, I)$; and (INj') holds at J and either (INj') or (Aj') is satisfied for all $j \in [c, J)$. Then $f_j(c,j) - \dot{H}(j) \geq \delta \geq f_i(i,c) - \dot{V}(i)$ for all $i, j \geq c$.*

(2) *Alternatively, suppose (DEi') holds at $i = c$ and either (DEi') or (Xi') are satisfied for all other $i \in [0, c)$; and (INj') holds at c and either (INj') or (Aj') is satisfied for all $j \in [0, c)$. Then $f_j(c,j) - \dot{H}(j) \geq \delta \geq f_i(i,c) - \dot{V}(i)$ for all $i, j < c$.*

Now I can state:

Proposition 3.15 (Differential age matching)**.** *There is DAM if either*

(1) *the conditions in 3.10 (decreasing $S(i,j)$) hold for the women's and (INj) for the men's side,*

(2) *the conditions in 3.10 hold for the women, those in corollary 3.11 (single-peaked $S(i,j)$) hold for the men, and $I \in \mathbb{I}(J)$, or*

(3) *for any age c there is δ_c so that one set of conditions in lemma 3.14 holds.*

Importantly, all sets of conditions in proposition 3.15 are compatible with PAM. This is immediate for (1) and (2). The requirements in (3) may be combined with the conditions in corollary 3.11 to obtain convex matching sets. With δ_c greater than zero, f may be decreasing in j, so that the matching set is connected and every $\mathbb{I}(j)$ is nonempty.

3.4. An Example

To show that the requirements for positive sorting and differential age matching are satisfied under plausible preferences, I give an example of a marriage payoff function that is explicitly derived from the assumption that marriage is a precondition for (legitimate) children, and that life, or the fertile period of life, is finite and women's fecundity declines with age.

Suppose the couple receives a payoff with present value Π from having a child. The man's fertility ends with certainty at J and the woman's at I, and their outside options $\underline{V}$ and $\underline{H}$ at that date equal zero. At any age the woman has a uniform probability of losing fecundity, modeled by an exponential distribution with parameter γ. Conditional on being married and fecund, the probability of conceiving is time independent, so that the arrival rate of a child is again exponentially distributed with parameter β. Given the woman's age at marriage i, the density function of conceiving at $i+t$ is thus the probability for not having lost fecundity at that date (fecundity loss occurs at a date after $i+t$), times the conditional density of conception: $\exp(-\gamma(i+t)) \cdot \beta \exp(-\beta t)$. The expected discounted payoff from marriage at ages (i, j) of the woman and the man at marriage is then

$$f(i,j) = \Pi \beta e^{-\gamma i} \int_0^T e^{-(\gamma+\beta+\rho)t} \mathrm{d}t$$

where $T = I - i$ if $i > j$, and $T = J - j$ if $J - j \leq I - i$.

Note first that all propositions continue to hold even though $f(i,j)$ is not differentiable at $I - i = J - j$. Everywhere else, the first and second derivative with respect to j are either both zero if $J - j > I - i$, or

$$f_j(i,j) = -\Pi \beta e^{-\gamma i} e^{-(\gamma+\beta+\rho)(J-j)} \leq 0 \text{ and } f_{jj}(i,j) = (\gamma+\beta+\rho) f_j(i,j).$$

This means if $\gamma + \beta \geq \lambda_m(1-\theta)$, (Aj) holds everywhere, and the women's matching sets are convex (even though (DEj) may not be satisfied). Moreover, $f(I,J) = 0$, so I and J marry. Lastly, the first derivative with respect to i is equal to $-\gamma f(i,j)$ if $I - i > J - j$, and strictly less than that if the woman's infertile age will occur sooner than the man's, i.e. $I - i \leq J - j$. In both cases (DEi) is satisfied if $\gamma \geq \lambda_w \theta$. In other words, there will be both PAM and DAM if $\gamma + \beta$ is relatively high and the product of meeting rate and bargaining share is low. Since the incidence of infertility (γ) must be low relative to the incidence of conception (β), these conditions are more likely to be satisfied if θ, the woman's bargaining power, is small.

This example demonstrates that PAM and DAM can arise from a very simple set of assumptions about marriage preferences. More realistic payoff functions will also account for the effect of marriage on the partners' lifetime incomes, as well as the benefits to companionship, mutual insurance, or gains from specialization. The conditions in this paper provide a direct check if such preferences are consistent with the matching patterns observed in most marriage markets.

4. Conclusion

This paper studies a search model in which preferences depend on the age of both partners at the time of matching. The model is most relevant for the marriage market, but can also apply e.g. in contexts of skill loss and acquisition in the labor market.

The first result is a set of sufficient conditions on the slope and curvature of the marriage payoff function under which there is assortative matching by age. It is shown that super- or submodularity are not necessary for positive sorting by age, and that many supermodular functions in fact lead to negative sorting. The discussion highlights that the requirements of matching at $(0, 0)$ and nonempty matching sets, needed for assortative matching following the argument of Shimer and Smith (2000), can be quite restrictive in the context of age-dependent preferences.

As its second result the paper defines differential age matching and shows that DAM and PAM imply that men match with a younger set of partners than women of the same age. It then provides conditions under which the marriage surplus declines faster in a female than a male spouse's age, leading to DAM. By way of example it is shown that assortative matching and DAM can plausibly arise as a consequence of a simple marriage payoff function which models the decline in fecundity with age.

Beyond the contribution to the theory of search and matching, the results of this paper may be of use for the empirical identification of search models with age preferences. A promising direction for future research would be a search model that allows for fully general preferences over invariant characteristics as well as age.

Omitted Proofs

Proofs Section 3

Derivation of the value function: Let $p(n, \mathrm{d}t)$ denote the probability of meeting n potential marriage partners in period $\mathrm{d}t$, and $V_n(i+t)$ the expected value of such n meetings at the time of the first meeting $i+t$. We have

$$V_1(i+t) = \int_0^J \theta \max\{S(i+t, y), 0\} m(y)\mathrm{d}y + V(i+t).$$

The value of search for a woman at age i is, for any $\mathrm{d}t > 0$,

$$V(i) = p(1, \mathrm{d}t)\mathrm{E}_t(e^{-\rho t}V_1(i+t)) + \sum_{n=2}^{\infty} p(n, \mathrm{d}t)\mathrm{E}_t(e^{-\rho t}V_n(i+t)) + \left(1 - \sum_{n=1}^{\infty} p(n, \mathrm{d}t)\right) e^{-\rho \mathrm{d}t}V(i+\mathrm{d}t)$$

where ρ is the discount rate and the expectation is formed over possible meeting times $0 < t < \mathrm{d}t$. In a Poisson process with parameter λ, $\frac{1}{\mathrm{d}t}\sum_{n=2}^{\infty} p(n, \mathrm{d}t)$ goes to zero and $p(1, \mathrm{d}t) \to \lambda$ as $\mathrm{d}t \to 0$, so that

$$\dot{V}(i) = \lim_{\mathrm{d}t \to 0} \frac{V(i+\mathrm{d}t) - V(i)}{\mathrm{d}t} = -\lambda_w(r)\theta \int_0^J \max\{S(i, y), 0\}m(y)\mathrm{d}y + \rho V(i).$$

Proof of Theorem 2.2: The proof uses the following version of the Eilenberg-Montgomery fixed point theorem (Granas and Dugundji (2003), ch. 19, corollary 7.5):

Theorem. *Assume that K is an absolute retract. Then any compact acyclic map $\mathcal{T}$ of K into itself has a fixed point.*

An acyclic map is a correspondence whose values are compact, upper hemi-continuous, and acyclic. The proof constructs a map $\mathcal{T}$ described by the functions 2.1-2.2 and 2.3 and 2.5 and then verifies the conditions of the theorem.

Notation: For any functions $h(x)$ and $g(x, y)$, let $\bar{h} = \sup_x h(x)$ and $\bar{g} = \sup_{(x,y)} g(x, y)$ (on their respective domains). Similarly, let $\underline{h} = \inf_x h(x)$ and $\underline{g} = \inf_{(x,y)} g(x, y)$ respectively. These are taken to mean the max and min whenever they exist.

Let K be the product $\mathcal{V} \times \mathcal{H} \times \mathcal{W} \times \mathcal{M}$ with function tuples $k = (V, H, w, m)$ as elements. $\mathcal{V}$ is the set of continuous functions on $[0, I]$, bounded below by zero and above by $\overline{f}$. Equivalently, $\mathcal{H}$ are the sets of continuous functions on $[0, J]$ bounded by zero and $\overline{f}$. $\mathcal{W}$ ($\mathcal{M}$) are the continuous functions on $[0, I]$ ($[0, J]$), bounded by

$$\begin{aligned}
\underline{w} &= \frac{1}{I} e^{-\bar{\lambda}_w I} \text{ and } \overline{\overline{w}} = \frac{\bar{\lambda}_w}{1 - e^{-\bar{\lambda}_w I}} \\
\underline{m} &= \frac{1}{J} e^{\bar{\lambda}_m J} \text{ and } \overline{\overline{m}} = \frac{\bar{\lambda}_m}{1 - e^{-\bar{\lambda}_m J}}
\end{aligned}$$

The mapping $\mathcal{T}$ is defined as follows. Let $\mathcal{T}[(V, H, w, m)] = (\hat{V}, \hat{H}, \hat{w}, \hat{m})$. $\hat{w}(i)$ and $\hat{m}(j)$ equal the right-hand sides of (2.1) and (2.2); $\hat{V}(i)$ and $\hat{H}(j)$ are given by

$$\hat{V}(i) = \max\left\{0, \lambda_w(r)\theta \int_i^I \int_0^J \max\{0, f(x, y) - V(x) - H(y)\} m(y) \mathrm{d}y - \rho V(x) \mathrm{d}x + \underline{V}\right\}$$

$$\hat{H}(j) = \max\left\{0, \lambda_m(r)(1-\theta) \int_j^J \int_0^I \max\{0, f(x, y) - V(x) - H(y)\} w(x) \mathrm{d}x - \rho H(y) \mathrm{d}y + \underline{H}\right\}$$

For any given k, denote the set of (i, j) where $f(i, j) - V(i) - H(j) = 0$ by A_k and let $\bar{A}_k = [0, I] \times [0, J] \backslash A_k$. For short, also define $A_k(i) = A_k \cap ([0, i] \times [0, J])$ and $\bar{A}_k(i) = \bar{A}_k \cap ([0, i] \times [0, J])$. Finally, define $\alpha(i, j)$ to be 0 whenever $S(i, j) < 0$ and 1 if $S(i, j) > 0$, and let $\alpha(A_k) = c$ be a constant between 0 and 1. In other words, α describes a particular mixed strategy in which all couples who are indifferent between marrying and not marrying randomize in the same way. Note that $\mathcal{T}(k)|_{\mathcal{V}} = \hat{V}$ and $\mathcal{T}(k)|_{\mathcal{H}} = \hat{H}$ are singletons for any k. $\mathcal{T}(k)|_{\mathcal{W}}$ and $\mathcal{T}(k)|_{\mathcal{M}}$, on the other hand, may be sets of functions $\hat{w}$ and $\hat{m}$ if A_k is a set with nonzero measure: in this case, some couples may choose mixed strategies. These sets are described by varying c continuously between 0 and 1.

Step 1: Endow K with the norm implied by the metric $d(k_1, k_2) = \max_{(f \in \{V, H, w, m\})} [\sup_x |f_1(x) - f_2(x)|]$. Each of the function spaces constituting K is closed and convex, so that K inherits those properties. A convex subset of a normed linear space is an absolute retract.

Step 2: $\mathcal{T}$ is compact if its image is contained in a compact subset of K. It is sufficient to prove that the image of $\mathcal{T}$ on each subspace of K is a bounded, equicontinuous set, and has therefore a compact closure by the Arzelà-Ascoli theorem[16]. The product C of these closures is compact (Tychonoff), and $\mathcal{T}(K) \subset C$. We therefore consider $\hat{V}(i)$ and $\hat{w}(i)$ separately (with the equivalent proofs for $\hat{H}(j)$ and $\hat{m}(j)$ omitted). It is easily verified that they lie in $\mathcal{V}$ and $\mathcal{W}$ respectively, so we are left with proving equicontinuity. For any (V, H, w, m) we have

$$\begin{aligned}
\left|\hat{V}(i_1) - \hat{V}(i_2)\right| &= \left| \int_{i_1}^{i_2} \lambda_w(r)\theta \int_0^J \max\{0, f(x,y) - V(x) - H(y)\} m(y)\mathrm{d}y - \rho V(x)\mathrm{d}x \right| \\
&\leq \bar{\lambda}_w \theta \bar{f} \left| i_1 - i_2 \right|
\end{aligned}$$

Therefore $|i_1 - i_2| < \frac{1}{\bar{\lambda}_w \theta \bar{f}}\epsilon$ implies $\left|\hat{V}(i_1) - \hat{V}(i_2)\right| < \epsilon$, for all $\hat{V} \in \mathcal{T}(K) \cap \mathcal{V}$. Similarly,

$$\begin{aligned}
|\hat{w}(i_1) - \hat{w}(i_2)| &= \hat{w}(i_1) \left| 1 - \exp\left(\left(\int_{i_1}^{i_2} \lambda_w(r) \int_0^J \alpha(x,y) m(y) \mathrm{d}(y) \mathrm{d}x \right) \right) \right| \\
&\leq \overline{\overline{w}} \left| 1 - exp(-\bar{\lambda}_w \left| i_1 - i_2 \right|) \right|
\end{aligned}$$

This implies that $|\hat{w}(i_1) - \hat{w}(i_2)| < \epsilon$ whenever

$$|i_1 - i_2| < \frac{1}{\underline{\lambda}_w} ln\left(1 - \frac{\epsilon}{\overline{\overline{w}}}\right),$$

for all $\hat{w} \in \mathcal{T}(K) \cap \mathcal{W}$.

Step 3: Consider any $\hat{w}$ in $\mathcal{T}(k)|_{\mathcal{W}}$. There must be an α and c such that

$$(A.1) \qquad \hat{w}(i) = \frac{e^{-\lambda_w\left(\frac{w(0)}{m(0)}\right)\left(\int_{\bar{A}_k(i)} \alpha(x,y)m(y)\mathrm{d}(x,y) + \int_{A_k(i)} \alpha(x,y)\ m(y)\mathrm{d}(x,y)\right)}}{\int_0^I e^{-\lambda_w\left(\frac{w(0)}{m(0)}\right)\left(\int_{\bar{A}_k(u)} \alpha(x,y)m(y)\mathrm{d}(x,y) + \int_{A_k(u)} \alpha(x,y)\ m(y)\mathrm{d}(x,y)\right)}\mathrm{d}u}.$$

Observe that α is uniquely defined to be either 1 or 0 on $\bar{A}_k$, and that $\alpha(x,y) = c$ on A_k is unique as well (consider $\hat{w}(0)$). $\hat{w}$ is continuous in c. Now let $g(i,j,t)$ be defined as follows:

$$g(i,j,t) = \begin{cases} (1-t)c + t & \text{if } (i,j) \in A_k \\ \alpha(i,j) & \text{else} \end{cases}$$

[16]More precisely, by Arzelà-Ascoli a set of functions on a metric space is relatively compact if it is bounded and uniformly equicontinuous, and a relatively compact set has a compact closure. An equicontinuous set of functions is uniformly equicontinuous if its domain is compact (see Ok (2007) p. 262-64).

and define a function $F : \mathcal{T}(k)|_{\mathcal{W}} \times [0,1] \to \mathcal{T}(k)|_{\mathcal{W}}$ where $F(\hat{w},t)(i)$ is given by (A.1), except that $\alpha(x,y)$ is replaced by $g(x,y,t)$. Then $F(\hat{w},0) = \hat{w}$ and $F(\hat{w},1)$ is identical for the entire set $\mathcal{T}(k)|_{\mathcal{W}}$. This means $\mathcal{T}(k)|_{\mathcal{W}}$ is contractible and therefore acyclic, and the same argument can be made for $\mathcal{T}(k)|_{\mathcal{M}}$. Together this implies that $\mathcal{T}$ has acyclic values.

Step 4: $\mathcal{T}$ is compact-valued: consider a convergent sequence $\hat{k}^n \in \mathcal{T}(k)$ with limit $\hat{k}$. There exists a sequence c^n such that $\alpha(A_k) = c^n$ for each $\hat{k}^n$ and $\alpha(A_k) = c$ for $\hat{k}$, given w, m. Moreover, $c^n \to c$. But c must converge to a point in $[0,1]$, and this implies $\hat{k} \in \mathcal{T}(k)$.

Step 5: Last I have to show that $\mathcal{T}$ is upper hemicontinuous. Consider a sequence $k^n \to k$ in K and let $\hat{k}^n \in \mathcal{T}(k^n)$. Since the closure of $\mathcal{T}(K)$ is compact, $\hat{k}^n$ has a convergent subsequence; w.l.o.g. let it be $\hat{k}^n$. I need to show that this limit point lies in $\mathcal{T}(k)$. First, let k^n such that $d(k^n,k) = \delta$ and denote $\hat{V}^n = \mathcal{T}(k^n)|_{\mathcal{V}}$ and $\hat{V} = \mathcal{T}(k)|_{\mathcal{V}}$. Now

$$
\begin{aligned}
\left|\hat{V}^n(i) - \hat{V}(i)\right| \leq\ & \left|\lambda_w(r^n) - \lambda_w(r)\right| I\theta\bar{f} \\
& + I\bar{\lambda}_w\theta\bar{f}\int_0^J \left|m^n(y) - m(y)\right| \mathrm{d}y \\
& + \int_i^I \bar{\lambda}_w\theta\int_0^J m_2(y)\left|V(x) + H(y) - V^n(x) - H^n(y)\right| \mathrm{d}y\mathrm{d}x \\
& + \rho\int_i^I \left|V^n(x) - V(x)\right| \mathrm{d}x \\
\leq\ & \left|\lambda_w\left(\frac{w^n(0)}{m^n(0)}\right) - \lambda_w\left(\frac{w(0)}{m(0)}\right)\right| I\theta\bar{f} + \left[IJ\bar{\lambda}_w\theta\bar{f} + 2I\bar{\lambda}_w\theta + \rho I\right]\delta
\end{aligned}
$$

Since λ_w is continuous, for any ϵ there exists a small enough δ such that $\left|\hat{V}_1(i) - \hat{V}_2(i)\right| < \epsilon$. Thus, $\hat{V}^n \to \hat{V}$ and by a similar argument, $\hat{H}^n \to \hat{H}$.

In the same manner, let $\hat{w}^n = k^n|_{\mathcal{W}}$ and $\hat{w} \in \mathcal{T}(k)|_{\mathcal{W}}$. Calling the numerator of (2.1) $\phi(i)$ and the denominator Φ for short, we have

$$\begin{aligned} |\hat{w}^n(i) - \hat{w}(i)| &= \left| \frac{\phi^n(i)}{\Phi^n \Phi} (\Phi - \Phi^n) + \frac{1}{\Phi} (\phi^n(i) - \phi(i)) \right| \\ &\leq \overline{\overline{w}}^2 |\Phi - \Phi^n| + \overline{\overline{w}} |\phi^n(i) - \phi(i)| \\ &\leq \left(\overline{\overline{w}}^2 I + \overline{\overline{w}} \right) \max_i |\phi^n(i) - \phi(i)| \end{aligned}$$

But $\max_i |\phi^n(i) - \phi(i)|$ is at most

$$\max_i \left| 1 - \exp\left(\int_0^i \left(\lambda_w(r) \int_0^J \alpha(x,y) m(y) \mathrm{d}y - \lambda_w(r^n) \int_0^J \alpha^n(x,y) m^n(y) \mathrm{d}y \right) \mathrm{d}x \right) \right|$$

The expression in the exponent is bounded by

$$|\lambda_w(r^n) - \lambda_w(r)| + \bar{\lambda}_w IJ \, |m(y) - m^n(y)| + \int_0^I \int_0^J I_{\alpha(x,y) \neq \alpha^n(x,y)} m(y) \mathrm{d}y \mathrm{d}x.$$

Thus, $\hat{w}$ is a limit point of $\hat{w}^n$, provided that the size of the set of (i,j) at which the signs of $S^n(i,j)$ and $S(i,j)$ differ shrinks to zero. But note that for any $(i,j) \in \bar{A}_k$, since $S(i,j)^n \to S(i,j) \neq 0$, there must be an n such that $sign(S(i,j)^n) = sign(S(i,j))$. If A_k has measure zero, we are done. Otherwise, for any $(i,j) \in A_k$ there must be a convergent sequence $\alpha(i,j)^n \to \alpha$, or else $\hat{w}^n$ could not converge. But then there must be some $\hat{w} \in \mathcal{T}(k)|_{\mathcal{W}}$ such that $c = \alpha$ on A_k, and this $\hat{w}$ is the limit point of $\hat{w}^n$. The same argument holds for $\hat{m}^n$. $\square$

Proof of Claim 1 (p. 17): Using the value function equation (2.3),

$$\begin{aligned}
\dot V(i_2)-\dot V(i_1) &= \lambda_w\theta\int_0^J[\alpha(i_1,y)-\alpha(i_2,y)][f(i_1,y)-V(i_1)-H(y)]m(y)\mathrm{d}y \\
&\quad+\lambda_w\theta\int_0^J\alpha(i_2,y)[V(i_2)-V(i_1)-(f(i_2,y)-f(i_1,y))]m(y)\mathrm{d}y+\rho(V(i_2)-V(i_1)) \\
&\geq \lambda_w\theta\int_0^J\alpha(i_2,y)\left[\int_{i_1}^{i_2}\Big(\dot V(x)-f_i(x,y)\Big)\,\mathrm{d}x\right]m(y)\mathrm{d}y+\rho\int_{i_1}^{i_2}\dot V(x)\mathrm{d}x \\
&\geq \lambda_w\theta\int_0^J\alpha(i_2,y)\left[\int_{i_1}^{i_2}(f_i(x,j)-f_i(x,y))\,\mathrm{d}x\right]m(y)\mathrm{d}y+\rho\int_{i_1}^{i_2}f_i(x,j)\mathrm{d}x \\
&\geq \int_{i_1}^{i_2}f_{ii}(x,j)\mathrm{d}x=f_i(i_2,j)-f_i(i_1,j).
\end{aligned}$$

The first inequality holds because the terms $[\alpha(i_1,y)-\alpha(i_2,y)]$ and $[f(i_1,y)-V(i_1)-H(y)]$, if not zero, must have the same sign under optimal matching behavior. The second inequality follows since for all $x\in[i_1,i_2]$, $\dot V(x)\geq f_i(x,j)$.The last step comes from (Ai) (note that $\min_y\{f_i(i,j)-f_i(i,y)\}\leq 0$). But this means that $f_i(i_2,j)-\dot V(i_2)\leq f_i(i_1,j)-\dot V(i_1)$. □

Proof of Claim 2 (p. 20):

$$\begin{aligned}
\dot V(i_2)-\dot V(i_1) &= \lambda_w\theta\int_0^J[\alpha(i_1,y)-\alpha(i_2,y)]\,[f(i_2,y)-V(i_2)-H(y)]\,m(y)\mathrm{d}y \\
&\quad-\lambda_w\theta\int_0^J\alpha(i_1,y)\,[f(i_2,y)-f(i_1,y)-(V(i_2)-V(i_1))]\,m(y)\mathrm{d}y+\rho(V(i_2)-V(i_1)) \\
&\leq \lambda_w\theta\int_0^J\alpha(i_1,y)\left[\int_{i_1}^{i_2}(\dot V(x)-f_i(x,y))\mathrm{d}x\right]m(y)\mathrm{d}y+\rho\int_{i_1}^{i_2}\dot V(x)\mathrm{d}x \\
&\leq \int_{i_1}^{i_2}f_{ii}(x,j)\mathrm{d}x=f_i(i_2,j)-f_i(i_1,j),
\end{aligned}$$

and therefore $f_i(i_1,j)-\dot V(i_1)\leq f_i(i_2,j)-\dot V(i_2)$. □

Proof of Lemma 3.8: Define a function $S^*(i)=\max_j f(i,j)-V(i)-H(j)$ for $i\in[0,I]$ and constant at $S^*(0)$ for all $i<0$, with the arg max denoted by $j^*(i)$. Since f, V and H are continuous, so is S^*. Assume that for some i'', $\mathbb{J}(i'')$ is nonempty. Now suppose there exists an $i<i''$ such that $S^*(i)<0$. There must be an $i'\in(i,i'')$ such that $S^*(x) \leq 0$ for all $x \in [i, i']$ and $S^*(i') = 0$. But then

$$\begin{aligned} V(i') - V(i) &= -\lambda_w \theta \int_i^{i'} \int_0^J \alpha(x,y) \left[f(x,y) - V(x) - H(y) \right] m(y) \mathrm{d}y \mathrm{d}x + \rho \int_i^{i'} V(x) \mathrm{d}x \\ &\geq -\lambda_w \theta \int_i^{i'} \max\{0, S^*(x)\} \mathrm{d}x + \rho \int_i^{i'} V(x) \mathrm{d}x \geq 0. \end{aligned}$$

Since f is decreasing in i, this means that $f(i, j^*(i')) - V(i) - H(j^*(i')) \geq f(i', j^*(i')) - V(i') - H(j^*(i')) = S(^*(i')) = 0$; in other words, $j^*(i')$ must be in the matching set of i (as well as all x). The connecting path for $\mathbb{M}$ is given by $S^*(i) \times i$. □

Proof of Proposition 3.9: Suppose $f_i(i_1, j) - \dot{V}(i_1) < 0$, but there exists an $i > i_1$ such that $f_i(i, j) - \dot{V}(i) > 0$. There must be a point between i_1 and i with $f_i(i, j) - \dot{V}(i) = 0$; let i_2 be the smallest of them. By claim 1, $f_i(i_2, j) - \dot{V}(i_2) \leq f_i(i_1, j) - \dot{V}(i_1) < 0$. □

Proof of Lemma 3.14: Hold j fixed at a. (DEi') directly bounds $f_i(i,a) - \dot{V}(i) \leq \delta$. Suppose by contradiction that there is an i_1 such that $f_i(i_1, a) - \dot{V}(i_1) > \delta$. Let i_2 be the smallest $i > i_1$ such that $f_i(i, a) - \dot{V}(i) = \delta$. Then $\dot{V}(x) \leq f_i(x, a) - \delta$ for all $x \in [i_1, i_2]$. By an analog argument to that for claim 2,

$$\begin{aligned} \dot{V}(i_2) - \dot{V}(i_1) &\leq \lambda_w \theta \int_0^J \alpha(i_1, y) \left[\int_{i_1}^{i_2} (\dot{V}(x) - f_i(x,y)) \mathrm{d}x \right] m(y) \mathrm{d}y + \rho \int_{i_1}^{i_2} \dot{V}(x) \mathrm{d}x \\ &\leq \lambda_w \theta \int_0^J \alpha(i_1, y) \left[\int_{i_1}^{i_2} (f_i(x,a) - \delta - f_i(x,y)) \mathrm{d}x \right] m(y) \mathrm{d}y + \rho \int_{i_1}^{i_2} (f_i(x,a) - \delta) \mathrm{d}x \\ &\leq \int_{i_1}^{i_2} f_{ii}(x,a) \mathrm{d}x = f_i(i_2, a) - f_i(i_1, a) \end{aligned}$$

and therefore $f_i(i_1, a) - \dot{V}(i_1) \leq f_i(i_2, a) - \dot{V}(i_2) = \delta$; contradiction. Similarly, let $i = a$. (INj') bounds $f_j - \dot{H}$ above δ. Suppose that there is an j_1 such that $f_j(a, j_1) - \dot{H}(j_1) < \delta$. Let j_2 be the smallest age above j_1 at which $f_j(a, j) - \dot{H}(j) = \delta$. Then by a similar argument

$$\begin{aligned} \dot{H}(j_2) - \dot{H}(j_1) &\geq \lambda_m (1-\theta) \int_0^I \alpha(x, j_2) \left[\int_{j_1}^{j_2} (f_j(a,y) - \delta - f_j(x,y)) \mathrm{d}y \right] w(x) \mathrm{d}x + \rho \int_{j_1}^{j_2} (f_j(a,y) - \delta) \mathrm{d}y \\ &\geq \int_{j_1}^{j_2} f_{jj}(x,a) \mathrm{d}x = f_j(j_2, a) - f_j(j_1, a) \end{aligned}$$

and $f_j(j_1, a) - \dot{H}(j_1) \geq f_j(j_2, a) - \dot{H}(j_2) = \delta$, contradiction. □

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