Bounded Rationality and Limited Datasets: Testable Implications, Identifiability, and Out-of-Sample Prediction*

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Abstract

Theories of bounded rationality are typically characterized over an exhaustive data set. This paper aims to operationalize some leading theories when the available data is limited, as is the case in most practical settings. How does one tell if observed choices are consistent with a theory of bounded rationality if the data is incomplete? What information can be identified about preferences? How can out-of-sample predictions be made? Our approach is contrasted with earlier attempts to examine bounded rationality theories on limited data, showing their notion of consistency is inappropriate for identifiability and out-of-sample prediction.

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1. Introduction

In the past decade, there has been a surge of interest in modeling bounded rationality. Standard rationality posits that a decision-maker maximizes a strict complete and transitive ordering; or equivalently, the alternative the decision-maker selects from each conceivable choice problem satisfies the classic property of Independence of Irrelevant Alternatives (IIA). When IIA is not satisfied, choices are said to be irrational. However, researchers interested in bounded rationality argue that irrational choices are not necessarily made in an unpredictable or unreasonable manner. Irrational choice patterns have long been observed by researchers in marketing and psychology, who believe consumer decisions arise from simplification heuristics such as elimination by criteria (e.g., Tversky 1972) and consideration sets (e.g., Wright and Barbour 1977). These features have been incorporated into structured models of bounded rationality in economics. Prominent examples include Manzini and Mariotti (2007, 2010), Cherepanov, Feddersen, and Sandroni (2010), and Masatlioglu, Nakajima and Ozbay (2011), who suggest plausible decision-making procedures that can generate many observed choice patterns. As suggested by those authors and others, understanding the procedures generating irrational choices may also be useful for finding ways to affect behavior and identifying the limits of welfare economics.

Theoretical work on bounded rationality has provided conditions that characterize choice procedures when choices from all possible problems are observed, and studies what information about underlying preferences can be identified from observing an entire choice function. However, in most realistic situations, the available data will be limited. In empirical settings, the modeler cannot control the choice problems faced by individuals. In experimental settings, generating a complete data set requires an overwhelming number of decisions by subjects (there are 26 choice problems when the alternative space contains 5 elements, 1,013 choice problems when it contains 10 elements, and 32,752 choice problems when it contains 15 elements). Given this, one should be able to use a theory to make predictions about behavior. If one does not understand the implications of a theory on partial data, then one does not know whether it can be used to forecast behavior for other choice problems that may arise (and, if so, what are all the predictions that would be consistent with the observed data).

Understanding the rational choice model in the presence of limited choice data is a classical question that is now well-understood (e.g., Samuelson 1948, Houthaker 1950, Richter 1966, Afriat 1967 and Varian 1982). Varian (1982) summarizes the three main questions that are of interest with limited data. When are observed choices consistent with the theory? What information can be identified about preferences? What out-of-sample predictions can be made? This paper studies these questions within an abstract choice setting with a finite set of alternatives, focusing on some leading theories in the literature: Manzini and Mariotti (2010)'s theory of choice by categorization; Cherepanov, Feddersen, and Sandroni (2010)'s rationalization theory; and Masatlioglu, Nakajima and Ozbay (2011)'s theory of limited attention.

These and other theories of bounded rationality pose new challenges on limited data. Because a theory has implications in all conceivable choice problems, identifying a theory's testable implications may require thinking about how the decision-maker would behave in unobserved choice problems. Out-of-sample considerations have not been an issue in the literature on rational choice with partial data; whenever the decision-maker's choices can be explained as the maximization of a transitive ordering over each observed choice set, then that ordering can be maximized over unobserved choice problems to construct predictions consistent with rationality. However, we show that out-of-sample restrictions become relevant for various theories of bounded rationality. For instance, in the theory of choice with limited attention, concluding that the consideration sets in some observed choice problems must exclude particular alternatives may have contradictory implications on the consideration sets for unobserved choice problems. This means that one cannot limit the test of consistency to simply finding a story that explains the observed data, without thinking about whether that story can be extended to unobserved choice problems. Previous attempts[1] overlook the out-of-sample restrictions implied by the underlying bounded rationality theories, and hence do not allow for out-of-sample prediction or proper identification.

A second challenge is that identifying the revealed preference is not straightforward. For rational choice theory, a choice from a set is revealed preferred to all available alternatives. However, for the models above, a choice is only revealed preferred to "considered" alternatives, which must themselves be deduced from the data. Furthermore, a notion of revealed preference that captures all available information for a bounded rationality theory when the data is complete, may fail to pick up information in nontrivial ways when there is only partial data. By contrast, for the theory of rationality, one can gather all the possible information under limited data by simply taking the transitive closure of the standard revealed preference relation.

Both of the above challenges are related to a third difficulty. If the theory involves multiple binary relations, or both binary relations and consideration sets, then there are exponentially more possible combinations of such primitives that should be checked to determine whether observed choices are consistent with the theory. It is not *a priori* clear whether tractable conditions can be found for testing consistency.

We apply our methodology to examine two categories of bounded rationality theories that do not contain each other, but nest several variations. The first is Masatlioglu, Nakajima and Ozbay (2011)'s theory of choice with limited attention, which captures a notion of awareness. It posits that when alternatives which are not paid attention to are removed, the set of alternatives that are paid attention to is unchanged. The second is Manzini and Mariotti (2010)'s theory of choice by categorization and Cherepanov, Feddersen and

[1]Those attempts include Manzini and Mariotti (2007, Corollary 1), Manzini and Mariotti (2010) and Tyson (2011).

Sandroni (2010)'s rationalization theory, which are both characterized by the axiom "weak-WARP" on complete data sets. These two theories capture the notion of simplification by criteria; they have the feature that if the decision-maker considers a particular alternative in some choice problem (e.g., because it is not dominated by another available alternative under some criterion), then he also considers that alternative within any subset in which it is contained.

We find conditions (and simple procedures for checking them) which characterize when the theories above are consistent with observed choices, identify revealed preferences, and check out-of-sample predictions. The tests we provide parallel different methods for testing standard rationality. Indeed, there are at least two equivalent ways to test whether the classical revealed preference of Samuelson (1948) is acyclic (and hence consistent with rational choice theory). One method is to use the strong axiom of revealed preference (SARP), which directly requires acyclicity. Another method is to try to enumerate the grand set of alternatives as follows: pick an alternative that is undominated within the grand set X and call it x_1, pick an alternative that is undominated in the set $X \setminus \{x_1\}$ and call it x_2, and so on and so forth. This enumeration procedure succeeds if and only if there are no cycles. In the case of Masatlioglu, Nakajima and Ozbay (2011)'s theory of choice with limited attention, where the decision-maker applies a transitive preference, we provide a test related to SARP that is based on the correct notion of preference restrictions for that theory. In the case of Manzini and Mariotti (2010)'s theory of choice by categorization and Cherepanov, Feddersen and Sandroni (2010)'s rationalization theory, we provide a method related to the enumeration procedure described above, which works despite the fact that the decision-maker may be applying a potentially cyclic preference in those theories. The methods we propose make these various theories easy to operationalize on partial data sets.

This paper is organized as follows. Section 2 formally defines the notion of a theory of choice, and presents the leading examples studied in this paper. Definitions of consistency and out-of-sample prediction are formalized in Section 3, and used in Section 4 to illustrate the problem with previous approaches. Section 5 studies Masatlioglu, Nakajima and Ozbay (2011)'s theory of choice with limited attention. Section 6 studies Manzini and Mariotti (2010)'s theory of choice by categorization and Cherepanov, Feddersen and Sandroni (2010)'s rationalization theory.

2. Theories of Choice

Let X be a (finite) set of possible alternatives. A *choice problem* is a nonempty subset of X that represents a set of feasible alternatives. The set of all *conceivable* choice problems is the set of all nonempty subsets of X, denoted by $\mathcal{P}(X)$. A *theory* $\mathcal{T}$ (e.g. rationality) offers a story that determines which option will be picked in any conceivable choice problem, as a function of primitives (e.g. preference). That is, once primitives have been fixed, a theory predicts a *choice function* $c : \mathcal{P}(X) \to X$, where $c(S)$ is an element of S for every $S \in \mathcal{P}(X)$. Here are some prominent examples from the literature.

Example 1 (Rational Choice). *The decision-maker has a strict complete and transitive preference ordering P that he uses to pick the P-maximal alternative from every choice set. Formally, the primitive P and the choice function $c : \mathcal{P}(X) \to X$ must satisfy:*

$$c(S) = \arg\max_{P} S, \text{ for all } S \in \mathcal{P}(X). \tag{1}$$

Example 2 (Shortlisting by Manzini and Mariotti (2007)). *The decision-maker has two asymmetric (possibly incomplete and/or cyclic) binary relations R_1 and R_2. From every choice problem, he first creates a shortlist of alternatives that are undominated according to R_1, and then picks the R_2-maximal alternative out of that shortlist. Formally, the primitives R_1, R_2 and the choice function $c : \mathcal{P}(X) \to X$ must satisfy:*

$$c(S) = \arg\max_{R_2}(\arg\max_{R_1} S), \text{ for all } S \in \mathcal{P}(X). \tag{2}$$

Example 3 (CHOICE BY CATEGORIZATION by Manzini and Mariotti (2010))**.** *The decision-maker follows a two-stage procedure where he first eliminates any alternatives belonging to an inferior category of options, and then maximizes an asymmetric (possibly incomplete and/or cyclic) preference relation P over the remaining alternatives. His "rationale by categorization" $\succ$ is an asymmetric and incomplete (possibly cyclic) relation on the set of potential categories $\mathcal{C}$ (the set of all subsets of X). Formally, the primitives $P, \succ$ and the choice function $c : \mathcal{P}(X) \to X$ must satisfy:*

$$c(S) = \arg\max_P \{x \in S | \nexists\, R, R' \in \mathcal{C} : x \in R, R' \succ R, R \cup R' \subseteq S\}, \textit{ for all } S \in \mathcal{P}(X). \tag{3}$$

Example 4 (RATIONALIZATION by Cherepanov, Feddersen and Sandroni (2010))**.** *The decision-maker first uses a set of rationales $R_1, \ldots, R_K$ (asymmetric and transitive relations) to eliminate those alternatives that are not optimal according to any of the rationales; he then maximizes an asymmetric (possibly incomplete and/or cyclic) preference relation P over the remaining alternatives. Formally, the primitives $R_1, \ldots, R_K, P$, and the choice function $c : \mathcal{P}(X) \to X$ must satisfy:*

$$c(S) = \arg\max_P \{x \in S | \exists i : x = \arg\max_{R_i} S\}, \textit{ for all } S \in \mathcal{P}(X). \tag{4}$$

Example 5 (LIMITED ATTENTION by Masatlioglu, Nakajima, and Ozbay (2011))**.** *For each choice problem S, the decision-maker uses a strict preference ordering P to pick the best element in his consideration set $\Gamma(S) \subseteq S$. According to the theory, the consideration set mapping $\Gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ has the property that removing ignored alternatives does not change the consideration set. Formally, the primitives P, Γ and the choice function $c : \mathcal{P}(X) \to X$*

must satisfy:[2]

$$\Gamma(S) \subseteq T \subseteq S \Rightarrow \Gamma(T) = \Gamma(S), \textit{ for all } S, T \in \mathcal{P}(X), \tag{5}$$

$$c(S) = \arg\max_P \Gamma(S), \textit{ for all } S \in \mathcal{P}(X). \tag{6}$$

3. Limited Data and Predictions

A *data set* $\mathcal{D}$ is a set of subsets of X, i.e. $\mathcal{D} \subseteq \mathcal{P}(X)$. An *observed choice function* $c_{obs} : \mathcal{D} \to X$ associates to each set S in the data the alternative in S that is selected. For any $\mathcal{D} \subseteq \mathcal{P}(X)$, we say that a choice function $\tilde{c} : \mathcal{D} \to X$ is *consistent* with a theory $\mathcal{T}$ if there exist primitives under which the choice function $c : \mathcal{P}(X) \to X$ predicted by the theory coincides with $\tilde{c}$ on $\mathcal{D}$, that is, $\tilde{c}(S) = c(S)$ for every $S \in \mathcal{D}$. Notice that a theory predicts choices for all conceivable choice problems, not just those in the data set. One objective of this paper is to provide out-of-sample predictions that are consistent with given theories of bounded rationality, such as those described in Examples 2 to 5. Given an observed choice function $c_{obs} : \mathcal{D} \to X$, a theory $\mathcal{T}$, and $S \in \mathcal{P}(X) \setminus \mathcal{D}$, the set of *predictions* consistent with $\mathcal{T}$ given c_{obs} is simply:

$$P_{\mathcal{T}}(S|c_{obs}) = \left\{ x \in S \;\middle|\; c := \begin{cases} c_{obs} & \text{on } \mathcal{D} \\ x & \text{on } \{S\} \end{cases} \text{ is consistent with } \mathcal{T} \right\}.$$

Therefore, given a theory, the question of out-of-sample prediction boils down to finding conditions on observed choices that are necessary and sufficient for consistency with that theory.

[2]Masatlioglu et al. write condition (5) as $\Gamma(S \setminus \{x\}) = \Gamma(S)$, for all $x \in S \setminus \Gamma(S)$. Condition (5) is simply an iteration of their condition. Without such iteration, their condition may easily be vacuous on limited data, which they do not study (e.g., the limited data only contains jumps of two elements or more). The iterated version (5) captures all the restrictions consistent with their motivation and original condition.

4. Inability to Forecast Using Previous Approaches

Other papers have studied bounded rationality on limited data sets from a different perspective (see Manzini and Mariotti (2007, Corollary 1), Manzini and Mariotti (2010), and Tyson (2011)). The methodology we propose is to identify conditions under which an observed choice function is consistent with a given theory, which imposes restrictions on how choice is determined for each *conceivable* problem. By contrast, those papers treat "partial theories", which impose restrictions on how choice is determined only for choice problems *in the data set*. For instance, in their Definition 4, Manzini and Mariotti (2010) introduce partial versions of the theory of choice by categorization (cf. Example 3), requiring condition (3) only for choice problems S in the data set $\mathcal{D}$ instead of all $S \in \mathcal{P}(X)$. In their Corollary 1, Manzini and Mariotti (2007) follow an analogous approach to characterize situations where observed choices are consistent with the rational shortlisting method, requiring condition (2) to hold only for choice problems S in the limited data set $\mathcal{D}$. Similarly, Tyson (2011) studies partial theories where choices over a data set $\mathcal{D}$ can be derived from the maximization of a complete and transitive preference relation applied to a consideration set mapping that is defined over $\mathcal{D}$. Varying the class of acceptable consideration set mappings generates different partial theories. Tyson defines, for instance, the partial version of the theory for limited attention, requiring conditions (5) and (6) only for $S \in \mathcal{D}$.[3]

Consider a data set $\mathcal{D}$. A *$\mathcal{D}$-theory* offers a story that determines which option will be picked in any choice problem in $\mathcal{D}$, as a function of primitives (e.g. preference, rationales, consideration set mappings, etc.). That is, once primitives have been fixed, a $\mathcal{D}$-theory predicts a *partial* choice function $c : \mathcal{D} \to X$, where $c(S)$ is an element of S for every $S \in \mathcal{D}$. Various examples of $\mathcal{D}$-theories were given in the previous paragraph, corresponding to partial versions of the (full) theories introduced Examples 2 to 5.

[3] As Tyson (2011, p. 9) points out, his meta-characterization result for the partial theories he considers does not apply to choice with limited attention, since consideration set mappings satisfying (5) do not possess the lattice structure needed for his analysis.

One could also consider a $\mathcal{D}$-theory of rational choice, requiring condition (1) only for $S \in \mathcal{D}$. But it is immediately clear that the distinction between the theory and its $\mathcal{D}$-counterpart is irrelevant for rational choice. Indeed, if maximization of a strict preference ordering on X delivers the observed choices for problems in $\mathcal{D}$, then that same ordering can be used to define choices elsewhere, while still maintaining consistency with the theory of rational choice.

More generally, the distinction between a theory and its $\mathcal{D}$-counterpart would be irrelevant if the primitives in the $\mathcal{D}$-theory could be used to define choices outside of $\mathcal{D}$ in a way that would still be consistent with the underlying theory. The problem is, however, that this generally cannot be done in the presence of bounded rationality. To illustrate this point, consider the following different cases.

Case 1 (Relating to Manzini and Mariotti (2007))**.** Take $X = \{a, b, c, d, e, f\}$, the data set $\mathcal{D} = \{ab, ac, bc, abd, ace, bcf\}$, and the observed choice function $c_{obs1} : \mathcal{D} \to X$ given by:

S	ab	ac	bc	abd	ace	bcf
$c_{obs1}(S)$	a	c	b	b	a	c

This data is consistent with the $\mathcal{D}$-theory of shortlisting, but inconsistent with the theory of shortlisting. To see this, we first find some necessary conditions on R_1 and R_2. Note that $c_{obs1}(\{a, b\}) = a$ and $c_{obs1}(\{a, b, d\}) = b$ jointly imply that aR_2b and dR_1a, with neither aR_1b nor bR_1a possible. Otherwise, either a or b could not have been chosen in the other's presence. Analogously, we infer from $c_{obs1}(\{a, c\}) = c$ and $c_{obs1}(\{a, c, e\}) = a$ that cR_2a and eR_1c, with neither aR_1c nor cR_1a possible; and we infer from $c_{obs1}(\{b, c\}) = b$ and $c_{obs1}(\{b, c, f\}) = c$ that bR_2c, and fR_1b, with neither bR_1c nor cR_1b possible. To see that the data is consistent with the $\mathcal{D}$-theory of shortlisting, it suffices to add the information that xR_2y for any $x \in \{a, b, c\}$ and any $y \in \{d, e, f\}$. Using this R_1 and R_2, condition (2) indeed delivers c_{obs1} for $S \in \mathcal{D}$. But recall that a necessary condition for rationalizing c_{obs1} is that $aR_2bR_2cR_2a$, with R_1 silent on the set $\{a,b,c\}$. Hence c_{obs1} is inconsistent with the theory of shortlisting, because there are no primitives R_1 and R_2 that simultaneously predict c_{obs1} and offer a prediction for $\{a,b,c\}$. That is, $P_{shortlist}(\{a,b,c\}|c_{obs1}) = \emptyset$.

Case 2 (Relating to Masatlioglu, Nakajima and Ozbay (2011)). Take $X = \{a,b,c,d,e\}$, the data set $\mathcal{D} = \{ad, de, abc, acd, bcd, bde\}$, and the observed choice function $c_{obs2} : \mathcal{D} \to X$ given by:

S	ad	de	abc	acd	bcd	bde
$c_{obs2}(S)$	d	e	c	a	b	d

This data is consistent with the $\mathcal{D}$-theory of limited attention, but inconsistent with the theory of limited attention. Consider the complete and transitive P defined by $aPcPdPbPe$, and the consideration set mapping on $\mathcal{D}$ where for each $S \in \mathcal{D}$, the consideration set $\Gamma(S)$ is comprised of $c_{obs2}(S)$ and its P-lower contour set: $\Gamma(S) = \{c_{obs2}(S)\} \cup \{x \in S | c_{obs2}(S)Px\}$ for all $S \in \mathcal{D}$. It is easy to check that (5) is satisfied for $S, T \in \mathcal{D}$, and that c_{obs2} satisfies (6) for $S \in \mathcal{D}$. However, c_{obs2} is not consistent with the theory of limited attention. This can be shown by first observing that aPc and dPb are necessary conditions on P for predicting c_{obs2}. If Γ is a consideration set mapping, then these relationships would follow immediately if $c \in \Gamma(\{a,c,d\})$ and $b \in \Gamma(\{b,d,e\})$, because a is chosen from $\{a,c,d\}$ and d is chosen from $\{b,d,e\}$. Indeed, those inclusions are necessary: if, for example, $c \notin \Gamma(\{a,c,d\})$, then property (5) of a consideration set mapping requires $\Gamma(\{a,d\}) = \Gamma(\{a,c,d\})$, requiring the choice from $\{a,d\}$ to be a rather than d. Using the fact that aPc and dPb, it is immediately clear that the theory is unable to make a consistent prediction for the choice problem $\{b,c\}$. The ranking aPc implies $a \notin \Gamma(\{a,b,c\})$, which in turn implies that $\Gamma(\{b,c\}) = \Gamma(\{a,b,c\})$ and that the choice from $\{b,c\}$ should be c. At the same time, the ranking dPb implies $d \notin \Gamma(\{b,c,d\})$, which in turn implies that $\Gamma(\{b,c\}) = \Gamma(\{b,c,d\})$ and that the choice from $\{b,c\}$

should be b, a contradiction. Hence $P_{lim-att}(\{b,c\}|c_{obs2}) = \emptyset$.

Case 3 (Relating to Manzini and Mariotti (2010))**.** Take $X = \{a,b,c,d\}$, the data set $\mathcal{D} = \{ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd\}$, and the observed choice function $c_{obs3} : \mathcal{D} \to X$ given by:

S	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd
$c_{obs3}(S)$	b	c	a	c	d	d	b	a	d	c

This data is consistent with the $\mathcal{D}$-theory of choice by categorization, but inconsistent with the theory of choice by categorization. Consider the preference P defined by xPy if $c_{obs3}(xy) = x$, the set of categories

$$\mathcal{C} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,d\}, \{c,d\}, \{b,c\}\},$$

and the rationale by categorization $\succ$ defined by $\{a,b\} \succ \{c\}$, $\{a,d\} \succ \{b\}$, $\{c,d\} \succ \{a\}$, and $\{b,c\} \succ \{d\}$. Using these primitives, it is easy to check that c_{obs3} is consistent with the $\mathcal{D}$-theory of choice by categorization. By contrast, c_{obs3} is inconsistent with the theory of choice by categorization. Indeed, for any $x \in X$, the extended choice function $\bar{c}_{obs3}$ that coincides with c on $\mathcal{D}$ and picks x out of X, is not consistent with the $\mathcal{D} \cup \{X\}$-theory of choice by categorization. One way to see this is to apply Manzini and Mariotti's (2010) characterization result (which applies when the data set contains all pairs and triples, as is the case here): a choice function $\tilde{c} : \mathcal{D} \to X$ is consistent with the $\mathcal{D}$-theory of choice by categorization if and only if $\tilde{c}$ satisfies the weak-WARP axiom. That axiom says the following: if $S, S', S'' \in \mathcal{D}$ with $x, y \in S \subset S' \subset S''$, then $\tilde{c}(S) = \tilde{c}(S'') = x$ implies that $y \neq \tilde{c}(S')$. Notice that for any $x \in X$, setting $\bar{c}_{obs3}(X) = x$ leads to a violation of weak-WARP: there is $y \in X \setminus \{x\}$ such that $x = \bar{c}_{obs3}(\{x,y\})$ where y is chosen from one of the triples containing x. In particular, $P_{categ}(X|c_{obs3}) = \emptyset$.

The previous three cases illustrate that $\mathcal{D}$-counterparts of theories are ill-suited to make consistent out-of-sample predictions. First, we showed that an observed choice function could be consistent with the $\mathcal{D}$-counterpart of a theory, and yet there is a choice problem S outside of $\mathcal{D}$ for which $P_{\mathcal{T}}(S|c_{obs}) = \emptyset$. Second, assume someone insists on using partial versions of a theory $\mathcal{T}$ to make predictions given an observed choice function $c_{obs} : \mathcal{D} \to X$ which is consistent with the $\mathcal{D}$-counterpart of $\mathcal{T}$. He then considers x to be a reasonable prediction for a problem $S \notin \mathcal{D}$ if the extended choice function (adding the selection of x from S to observed choices) is consistent with the $\mathcal{D} \cup \{S\}$-counterpart of $\mathcal{T}$. To be concrete, consider a variant of Case 2.

Case 2′ Take $X = \{a, b, c, d, e\}$, the data set $\mathcal{D} = \{ad, de, abc, acd, bcd\}$, and the observed choice function $c_{obs2'} : \mathcal{D} \to X$ given by:

S	ad	de	abc	acd	bcd
$c_{obs2'}(S)$	d	e	c	a	b

That is, this person observes all the choices from c_{obs2} except from $\{b, d, e\}$ (in which case, the data can be shown to be consistent with the theory of limited attention). Then his set of "reasonable" predictions would include the choice of d from $\{b, d, e\}$, which as we have seen, is actually inconsistent with the theory of limited attention. In other words, his notion of a "reasonable" prediction is too generous, because the test of consistency with a $\mathcal{D}$-counterpart of the theory does not refute the theory as often as it should.

Cases 1-3 above also suggest that the notion of consistency with respect to restrictions of a theory is incongruous. Given a theory $\mathcal{T}$, a person may think the choices he observes are consistent with the $\mathcal{D}$-counterpart of the theory, and yet any forecast he tries to make for another choice problem S leads to an extended choice function that is inconsistent with the $\mathcal{D} \cup \{S\}$-counterpart of the theory, under the same restricted notion of consistency. Hence, pure logic should have led him to reject the theory on $\mathcal{D}$ in the first place.

The reason behind the inadequacy of $\mathcal{D}$-theories in Cases 1-3 is that the respective theories impose joint restrictions on choices and primitives. This can be seen in conditions (2), (3), and (5). Primitives in the $\mathcal{D}$-counterpart of a theory need not be valid primitives in the theory itself. For instance, the R_1, R_2 primitives used in Case 1 to show that c_{obs1} is consistent with the $\mathcal{D}$-counterpart of the shortlisting theory are invalid primitives for the underlying theory, since they cannot pick a choice out of $\{a, b, c\}$. More generally, we showed in Case 1 that any primitives delivering c_{obs1} would be invalid for the underlying theory (similarly for Cases 2 and 3).

5. Limited Attention

The "Strong Axiom of Revealed Preference" (SARP) provides a classical test for rational choice theory. The idea is to first identify all the information that can be gleaned from observing a decision-maker pick alternatives out of some choice problems. Following Samuelson (1948), an option x is revealed preferred to an alternative y if there exists a choice problem where y is available but the decision-maker selects x instead. SARP requires the existence of a transitive preference that satisfies all the resulting restrictions.

One can also define a version of SARP, for an arbitrary set of restrictions. Suppose Σ is a set of logical statements involving a binary relation P. In the case of rational choice theory, Σ consists of the statements $c(S)Px$ for every $x \in S \setminus \{c(S)\}$ and every $S \in \mathcal{D}$. Consider the following generalized version of SARP.

SARP-Σ. *There exists an acyclic binary relation P satisfying the restrictions in Σ.*

Notice that SARP-Σ is a condition involving only a binary relation. Can such a condition be useful for the theory of Masatlioglu, Nakajima and Ozbay's (2011)? The complication is that their theory involves both a preference relation and a consideration set mapping satisfying condition (5). To answer the question above, we explore the revealed preference restrictions their theory imposes.

In their theory of choice with limited attention, a decision-maker maximizes his preference only over his consideration set. Observing that x is chosen from a choice problem S does not imply that x is better than $y \in S$, because the decision-maker may not have paid attention to y. One can only infer that x is revealed preferred to alternatives in his consideration set at S, which itself must be inferred from the choice data. If one wishes to test the theory of limited attention, then, what is all the information about preferences that can be recovered from observed choices?

Studying only full data sets (i.e., $\mathcal{D} = \mathcal{P}(X)$), Masatlioglu, Nakajima and Ozbay identify particular choice patterns that reveal information about the underlying preference and consideration sets. Specifically, observing a decision-maker who picks x in a choice problem S and an option y in the choice problem $S \setminus \{z\}$, for some $z \in S \setminus \{x\}$, is consistent with the theory of choice with limited attention only if he pays attention to z in S and he prefers x over z. For a full data set, such choice patterns contain all the information that can be gleaned (Masatlioglu, Nakajima and Ozbay 2011, Theorem 1). This need not be the case when data is limited. The following is an obvious illustration. Suppose that we only have the information that a is picked out of some choice problem S and $b \neq a$ is picked from $S' = S \setminus \{x, x'\}$, where $x, x' \in S$ and $a \in S'$. Then the above inference never applies, since S and S' differ by more than one element. Yet, one can clearly infer some information regarding the preference and consideration sets by iterating Masatlioglu, Nakajima and Ozbay's idea. Given that $a \notin \{b, x, x'\}$, it must be that at least one of x or x' is paid attention to when a is chosen from S, implying that a is revealed preferred to that element.

More subtly, information can be gleaned from choice problems that are not related by inclusion. For instance, suppose all we know is that a is picked out of some choice problem S and $b \neq a$ is picked from some other choice problem S', where $a, b \in S \cap S'$, but neither S nor S' is contained in the other. If the decision-maker does not pay attention to any element in $S \setminus S'$ when choosing from S, then condition (5) implies that a would remain his choice when choosing from $S \cap S'$. Similarly, if the decision-maker does not pay attention to any element in $S' \setminus S$ when choosing from S', then b would remain his choice when choosing from $S \cap S'$. Since $a \neq b$, either the decision-maker is paying attention to some element(s) of $S \setminus S'$ when choosing a from S (in which case a is revealed preferred to those), or he is paying attention to some element(s) of $S' \setminus S$ when choosing b from S' (in which case b is revealed preferred to those). That is, we can gather information about the decision-maker every time his choices exhibit a violation of the classical weak axiom of revealed preference (WARP). Such violations occur when Samuelson's revealed preference fails to be asymmetric (e.g., a is revealed preferred to b within S but b is revealed preferred to a within S').

The following definition summarizes the inferences made in the previous two paragraphs. Given an observed choice function $c_{obs} : \mathcal{D} \to X$, define $\Sigma_{LA}(c_{obs})$ to be the set of preference restrictions

$$\text{“either } c_{obs}(S) P y, \text{ for some } y \in S \setminus S', \text{ or } c_{obs}(S') P y', \text{ for some } y' \in S' \setminus S,\text{”}$$

for every $S, S' \in \mathcal{D}$ where $c_{obs}(S) \neq c_{obs}(S')$ and $c_{obs}(S), c_{obs}(S') \in S \cap S'$.

The following theorem shows that Σ_{LA} encompasses all the empirical content of the theory of choice with limited attention.

Theorem 1. *The observed choice function $c_{obs} : \mathcal{D} \to X$ is consistent with the theory of choice with limited attention if and only if SARP-$\Sigma_{LA}(c_{obs})$ is satisfied.*

This means that checking whether observed choices are consistent with the theory of limited attention can indeed be reduced to the problem of finding an acyclic relation satisfying the restrictions of Σ_{LA}. This theorem also answers the question of out-of-sample prediction.

Corollary 1. *Suppose c_{obs} is consistent with the theory of choice with limited attention. Then, an option $x \in S$ is a valid prediction for a choice problem $S \notin \mathcal{D}$ if and only if the extended choice function $\bar{c}_{obs}$, which agrees with c_{obs} on $\mathcal{D}$ and picks x from S, satisfies SARP-$\Sigma_{LA}(\bar{c}_{obs})$.*

Moreover, if one presumes the decision-maker uses the preference ordering P, which satisfies the restrictions listed in $\Sigma_{LA}(c_{obs})$, then one can also find the set of valid predictions under P for a choice problem S not in the data. Given P, x is a valid prediction for S if and only if for every set $T \in \mathcal{D}$ for which the choice from T and the choice of x from S would lead to a WARP violation, one of the following holds true under P: either xPy, for some $y \in S \setminus T$, or $c_{obs}(T)Pz$, for some $z \in T \setminus S$.

Using Theorem 1, it is easy to see why the observed choices c_{obs2} from Case 2 in Section 4 were inconsistent with the theory of limited attention. The restrictions in $\Sigma_{LA}(c_{obs2})$ are:

(i) aPc, from $c_{obs2}(\{a,c,d\}) = a$ and $c_{obs2}(\{a,d\}) = d$.

(ii) dPb, from $c_{obs2}(\{b,d,e\}) = d$ and $c_{obs2}(\{d,e\}) = e$.

(iii) Either cPa or bPd, from $c_{obs2}(\{a,b,c\}) = c$ and $c_{obs2}(\{b,c,d\}) = b$.

(iv) Either cPb or aPd, from $c_{obs2}(\{a,b,c\}) = c$ and $c_{obs2}(\{a,c,d\}) = a$.

(v) Either bPc or dPe, from $c_{obs2}(\{b,c,d\}) = b$ and $c_{obs2}(\{b,d,e\}) = d$.

It is easy to see that statements (i)-(iii) are contradictory. However, for $c_{obs2'}$ in Case $2'$ (the hypothetical situation with the same choices on all but the problem $\{b,d,e\}$, which is not observed) restrictions (ii) and (v) are dropped. Using Theorem 1 again, one can infer that $c_{obs2'}$ is consistent with the theory of choice with limited attention, as the relation P defined by aPc, bPd, and cPb is acyclic and satisfies the restrictions in $\Sigma_{LA}(c_{obs2'})$. Using Corollary 1, only b and e are valid predictions for the problem $\{b,d,e\}$. This is easy to see, since choosing either b or e would not cause any additional WARP violations, leaving the set of restrictions unchanged.

Masatlioglu, Nakajima and Ozbay (2011) note that it may be possible to generate the same choice function (over full data sets) using different combinations of preferences and consideration set mappings. Extending their Definition 3 to observed choice functions defined over limited data sets, we say that x is *revealed preferred to* y if xPy for every preference P for which there is a consideration set mapping $\Gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying (5) such that observed choices are predicted by (P, Γ) using (6). The next result shows how the more general notion of SARP allows us to identify revealed preferences between any given pair of alternatives.

Theorem 2. *Let $c_{obs} : \mathcal{D} \to X$ be an observed choice function which satisfies SARP-$\Sigma_{LA}(c_{obs})$ and consider any $x, y \in X$. Let Σ' be the set of restrictions obtained by adding the restriction yPx to $\Sigma_{LA}(c_{obs})$. Then x is revealed preferred to y if and only if SARP-Σ' is not satisfied.*

Going back to Case $2'$, one can conclude from $c_{obs2'}$ that under the theory of limited attention, the decision-maker must prefer a over both c and d, and must prefer b over d. If one were to simply apply the revealed preference as defined by Masatlioglu, Nakajima and Ozbay (2011), then it would only be possible to conclude that the decision-maker prefers a over c. This corresponds to restriction (i). However, restrictions (iii) and (iv) give more information. Using (i) and (iii), we conclude that the decision-maker prefers b over d. From (iv), either c is preferred to b or a is preferred to d. Even in the former case, by transitivity the decision-maker would conclude that a is preferred to d, because a is preferred to c and b is preferred to d.

5.1 An algorithm for checking SARP-Σ_{LA}

Theorems 1 and 2 rest on checking whether there is an acyclic relation satisfying restrictions of the form, "either there exists $x \in S \setminus S'$ such that $c_{obs}(S) \succ x$ or there exists $x' \in S' \setminus S$ such that $c_{obs}(S') \succ x'$." As we have already seen, for small data sets it may be possible to answer this question by hand. More generally, there is a conceptually simple procedure for solving the problem. Drawing an analogy between our problem, and the classical problem of scheduling tasks with "and/or" precedence constraints, we can provide an algorithm for checking SARP-Σ_{LA}.

To motivate the connection between the two problems, let's first consider a simpler problem where the restrictions only take the form "there exists $x \in S \setminus S'$ such that $c_{obs}(S)Px$." Compare this to the problem of scheduling tasks with "and/or" precedence conditions, as studied by Möhring, Skutella, and Stork (2004). There is a set of tasks to be performed one at a time. The problem is to find an ordering of those tasks which satisfies several precedence conditions of the form "task a must be scheduled before either tasks b or c." Translated to our setting, tasks becomes alternatives and the ordering over tasks becomes a preference ranking, so that the earlier statement on tasks is read as "alternative a must be preferred to either alternative b or c." This is the form of the simpler restriction above. Möhring, Skutella, and Stork give the following algorithm for solving their problem, described below for our setting:

Algorithm. Let (T, x) represent the restriction that x must be revealed preferred to some $y \in T$, and let Σ be a set of such restrictions. Recall that a *stack*, like a deck of cards, is a "last in, first out" data structure.

1. (*Initialize*). Start with an empty stack and an empty ranking of alternatives. For each $x \in X$, let $\pi(x) = \#\{(T, x) | (T, x) \in \Sigma\}$ be the number of restrictions where x must be preferred. Whenever $\pi(x) = 0$ put x on top of the stack.

2. (*Create ranking*) So long as the stack is nonempty, we repeat the following procedure. Remove the alternative from the top of the stack (call it y) and put it at the top of the current ranking. For *every* $z \in X$ and *every* restriction $(T, z) \in \Sigma$ such that $y \in T$, we remove the restriction (T, z) from Σ and decrease $\pi(z)$ by one; if $\pi(z)$ becomes zero, we add z to the top of the stack.

3. (*Conclude*) Stop when the stack is empty. If the ranking is missing any alternative in X then there was an unavoidable cycle. Otherwise, take the transitive closure of the ranking to get a preference relation satisfying all the restrictions.

Since the restrictions in $\Sigma_{LA}(c_{obs})$ take the more complex form "either $(S \setminus S', c_{obs}(S))$ or $(S' \setminus S, c_{obs}(S'))$," how does one apply the algorithm above? First, notice that if every pair of sets $S, S' \in \mathcal{D}$ causing a WARP violation is related by inclusion, then the restrictions in $\Sigma_{LA}(c_{obs})$ reduce to the simpler form in the algorithm, because either $S \setminus S'$ or $S' \setminus S$ is empty. Second, even when the sets causing a WARP violation are not related by inclusion, it may be trivial to simplify the form of the restriction. For a restriction in $\Sigma_{LA}(c_{obs})$ to potentially cause a cycle, it must be the case that every candidate "bottom element" in the restriction happens to be the choice from another set which is itself involved in another restriction. Suppose S, S' cause a WARP violation and are not related by inclusion, but there exists an element $x \in S \setminus S'$ which is never chosen under c_{obs}, or is chosen in some choice problem(s) but is not the "top element" in another restriction. Then the complex restriction can be replaced with $(\{x\}, c_{obs}(S))$ without affecting whether SARP-$\Sigma_{LA}(c_{obs})$ is satisfied. Third, if S and S' cause a WARP violation and are not related by inclusion, but the choice from $S \cap S'$ is observed, then the restriction associated with S, S' is redundant: it is implied from the potential restrictions associated with the pairs $S, S \cap S'$ and $S', S \cap S'$, each of which takes the simpler form.

Finally, we say that a pair of sets S, S' causing a WARP violation is *problematic* if it does not fall into one of the categories above; that is, S, S' are not related by inclusion, $S \cap S'$ is not in the data set, and every element in $S \setminus S'$ and $S' \setminus S$ is the "top element" in some other restriction in $\Sigma_{LA}(c_{obs})$. The number of problematic pairs should be small in most cases. After simplifying the restrictions according to the first three methods, any remaining complex restrictions must correspond to problematic pairs of sets. One can simply apply the algorithm for all possible decisions on whether $(S \setminus S', c_{obs}(S))$ or

$(S' \setminus S, c_{obs}(S))$ should be required for each problematic pair S, S'.

6. Rationalization and Choice by Categorization

While the theories of rationalization and choice by categorization are the result of different choice procedures, they happen to be equivalent in terms of the choice patterns they generate. Indeed, Cherepanov, Feddersen and Sandroni (2010, Corollary 2) and Manzini and Mariotti (2010, Theorem 1) each show that a choice function $c : \mathcal{P}(X) \to X$ can be explained by their respective theory if and only if it satisfies the weak-WARP axiom described in Case 3.[4] Manzini and Mariotti further show that an observed choice function defined over a data set $\mathcal{D}$ that contains all pairs and triples is consistent with the $\mathcal{D}$-*restriction* of the theory of choice by categorization if and only if it satisfies weak-WARP. As shown in Case 3, however, we cannot conclude from weak-WARP that the observed choices are consistent with the theory itself. In this section, we provide conditions characterizing when one can reach such a conclusion.

As was the case for the theory of choice with limited attention, observing that x is chosen from a choice problem S does not imply that x is better than every $y \in S$ in the theories of rationalization and choice by categorization. In the theory of rationalization, the alternative y may not have been maximal for any rationale; in the theory of choice by categorization, y may have belonged to an eliminated category. In both theories, the elements which are "paid attention to" are those surviving a process of elimination (by categorization or rationalization). In particular, an element surviving elimination in a choice problem survives elimination in any subset of that choice problem to which it belongs. This motivates the following revealed ranking, which is an adaptation of Cherepanov, Feddersen and Sandroni's (2010, Definition 1) to our setting with limited data: $x \succ y$ if y is selected (and so survives elimination) in

[4] This class also contains Manzini and Mariotti (2007)'s theory of shortlisting, which is characterized by the weak-WARP axiom and an additional axiom they call Expansion.

the choice problem $S \in \mathcal{D}$ and also belongs to a choice problem $S' \subset S$ (with $S' \in \mathcal{D}$) from which $x \neq y$ is selected. At the same time, Manzini and Mariotti (2010) use the observed choices from pairs to make inferences. Combining these approaches defines the binary relation P^*, where xP^*y if either $c_{obs}(\{x, y\}) = x$ or $x \succ y$.

Because the second-stage preference may be cyclic in the theories of rationalization and choice by categorization, we cannot apply the SARP-based approach of the previous section. This section offers a different analogy to rational choice. Consider the revealed preference of Samuelson, and notice that the following procedure offers a test for consistency with rational choice theory. Pick any element that is undominated in X, and call it x_1; pick any element that is undominated in $X \setminus \{x_1\}$ and call it x_2; and iterate this procedure until only one element remains, to be called x_n. If this procedure succeeds in creating an enumeration of X, then the revealed preference must have been acyclic and the observed choices are consistent with rationality. Even though P^* is different from the rational revealed preference (and potentially cyclic), we show that a similar enumeration procedure captures the content of these theories.

Given observed choices c_{obs}, we define the correspondence $F : \mathcal{P}(X) \to 2^X$ as follows. For each $S \in \mathcal{P}(X)$, let $F(S)$ be the set of elements $x \in S$ such that

(1) If $R \in \mathcal{D}$ and $x \in R \subseteq S$ then $c_{obs}(R)P^*x$ whenever $x \neq c_{obs}(R)$; and

(2) If $T, T' \in \mathcal{D}$ and $\{x, c_{obs}(T), c_{obs}(T')\} \subseteq S \cap T \cap T'$ then P^* is acyclic on $\{x, c_{obs}(T), c_{obs}(T')\}$.

Given F, we define a simple procedure for enumerating X.

Enumeration procedure using F. If $F(X)$ is nonempty, pick any element in $F(X)$ and call it x_1. Then, for each $k = 2, \ldots, |X|$, if $F(X \setminus \{x_1, \ldots, x_{k-1}\})$ is nonempty, pick any element in $F(X \setminus \{x_1, \ldots, x_{k-1}\})$ and call it x_k.

The enumeration procedure *succeeds* if $F(X \setminus \{x_1, \ldots, x_{k-1}\})$ is nonempty for every $k = 1, \ldots, |X|$. If the procedure succeeds, it always returns an enumeration $x_1, \ldots, x_{|X|}$ of X, no matter which element of $F(X \setminus \{x_1, \ldots, x_{k-1}\})$ is called x_k in each step. Because the set $F(X \setminus \{x_1, \ldots, x_{k-1}\})$ in each step depends on the previous choices, one might worry that the procedure will happen to work if the "right" element of $F(X \setminus \{x_1, \ldots, x_{k-1}\})$ is chosen in every step, but might fail if the "wrong" element is chosen in some step. For checkability, one would like "success" to be *history independent*: success occurs (i.e., the procedure delivers an enumeration for *any* choice of x_k in each step) if and only if the enumeration procedure delivers an enumeration for *some* choice of x_k in each step. In other words, under history independence, it suffices to use an arbitrary selection of $x_k \in F(X \setminus \{x_1, \ldots, x_{k-1}\})$ in each step to reach a conclusive verdict on the success of the enumeration procedure.

Theorem 3. *Suppose P^* is complete. Then the observed choice function $c_{obs} : \mathcal{D} \to X$ is consistent with the theories of rationalization or choice by categorization if and only if the enumeration procedure using F succeeds. Moreover, the success of the enumeration procedure using F is history independent.*

As in the previous section, the testable implications of the theory are captured in conditions pertaining to a single relation. Notice that P^* is complete as soon as $\mathcal{D}$ includes all pairs, as assumed in Manzini and Mariotti (2010) or in the classic characterization of rational choice via the property of IIA. Clearly, P^* may be complete even when not all pairwise choices are observed, provided that those missing pairs can be compared via $\succ$.

Our analysis is also helpful when observed choices lead only to an incomplete P^*. One can simply consider all possible choices over those pairs which are relevant for completing P^*, and employ the enumeration procedure on each such completion to test consistency with the data. The data is consistent with the theories of choice by categorization or rationalization if and only if there exists a completion of P^* where the enumeration procedure succeeds. However, success is already guaranteed if the incomplete P^* is acyclic. In that case, P^* admits a transitive completion and the enumeration procedure trivially succeeds (there exist elements satisfying conditions (1) and (2) in the definition of $F(S)$ for every S when the preference has no cycles). Acyclicity of P^* thus provides a sufficient condition for being consistent with the theories of rationalization or choice by categorization, independently of whether P^* is complete. If an incomplete P^* is found to be cyclic, however, success of the enumeration procedure without trying to complete P^*, as described above, is not sufficient to ensure that the observed choices are consistent with these theories.[5] Success of the enumeration procedure does remain a *necessary* condition for observed choices to be consistent with these two theories when P^* is incomplete. It is easy to check that if an observed choice function violates weak-WARP, then it will also fail the enumeration procedure. Indeed, if x and y are involved in a weak-WARP violation, then condition (2) in the definition of $F(\{x,y\})$ fails for both x and y. Therefore, success of the enumeration procedure is a stronger testable implication of these two theories on all data sets.

The enumeration procedure also permits constructing the set of possible out-of-sample predictions for any given choice problem $S \notin \mathcal{D}$.

Corollary 2. *Suppose c_{obs} is consistent with the theories of categorization and rationalization and consider $S \notin \mathcal{D}$. The set of valid predictions for S is contained in $F(S)$. If P^* is complete, then an alternative $x \in F(S)$ is a valid forecast if and only the enumeration procedure succeeds for the correspondence $\bar{F}$ associated to the extended choice function that agrees with c_{obs} on $\mathcal{D}$ and selects x from S.*

As was the case for choice with limited attention, it may be possible to generate the same choice function using different combinations of primitives (preference and rationales for Cherepanov, Feddersen and Sandroni (2010) or categories for Manzini and Mariotti (2010)). We say that x is *revealed preferred* to y if xPy for every (possibly intransitive) preference P for which there is a

[5] For an example, see Remark 1 in the appendix.

set of rationales such that observed choices are predicted using (4), under the theory of rationalization. This is equivalent to requiring that xPy for every (possibly intransitive) preference P for which there is a rationale by categorization such that observed choices are predicted using (3), under the theory of categorization. The preference P^* used to define F corresponds to *one preference* that can be used to derive observed choices when they are consistent with the theories of rationalization and choice by categories, but having xP^*y does not imply that x is revealed preferred to y. Having $x \succ y$, on the other hand, does imply that x is revealed preferred to y. With full data sets, as in Cherepanov, Feddersen and Sandroni (2010), $\succ$ coincides with the revealed preference. This is no longer the case with partial data. For instance, consider a variant of Case 3.

Case 3′ Take $X = \{a, b, c, d\}$, $\mathcal{D} = \{ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd\}$, and the observed choice function $c_{obs3'} : \mathcal{D} \to X$ given by:

S	ab	ac	ad	bc	bd	cd	abc	abd	bcd
$c_{obs3'}(S)$	b	c	a	c	d	d	b	a	c

The only difference between c_{obs3} and $c_{obs3'}$ is that the choice out of $\{a, c, d\}$ is not observed. While c_{obs3} was inconsistent with the theories of choice by rationalization and choice by categorization, $c_{obs3'}$ is consistent with those theories. In fact, it is not difficult to check that an extended choice function would violate weak-WARP if, in addition to his observed choices, the decision-maker were to pick any one of b, c or d from the choice problem X; but that it would be consistent with weak-WARP if he were to pick a from both X and $\{a, c, d\}$. Hence, any set of primitives that generates $c_{obs3'}$ must pick a out of X. Given that c is selected from $\{a, c\}$, we must conclude under these theories that c is revealed preferred to a. It is easy to check, on the other hand, that a and c are not comparable according to $\succ$.

In Corollary 2, we observed that the set of valid predictions for a choice problem $S \notin \mathcal{D}$ under the theories of rationalization or choice by categorization must be a subset of $F(S)$. Case $3'$ also illustrates that this inclusion may be strict. Indeed, it is easy to check that $d \in F(\{a, c, d\}$ in that case, and yet extending $c_{obs3'}$ by picking d out of $\{a, c, d\}$ is inconsistent with both theories (see our analysis of Case 3 in Section 4).

The next theorem shows how to use the enumeration procedure to fully identify revealed preferences between any given pair of alternatives x and y. With a complete P^*, it must be that xP^*y or yP^*x. Of course, if there is a revealed preference between x and y, then it must be consistent with P^*. For instance, if xP^*y, then it is impossible to have y revealed preferred to x. We characterize the conditions under which one can conclude that x is revealed preferred to y when xP^*y. To state the conditions parsimoniously, define the set mapping $G_{xy} : \mathcal{P}(X) \to 2^X$ derived from F as follows: $G_{xy}(S) = F(S) \backslash \{y\}$ for all S containing both x and y, and $G_{xy}(S) = F(S)$ for all other S.

Theorem 4. *Suppose c_{obs} is consistent with the theories of rationalization and choice by categorization, that P^* is complete and that xP^*y. Then x is revealed preferred to y if and only if either $x \succ y$ or the enumeration procedure using G_{xy} does not succeed. Moreover, the success of the enumeration procedure using G_{xy} is history independent.*

In Case $3'$, for instance, it is easy to see that $G_{ca}(X)$ would be empty because a is the only alternative in $F(X)$, and therefore that the enumeration procedure using G_{ca} cannot succeed. Even though $c \succ a$ does not hold, Theorem 4 allows us to conclude that c is revealed preferred to a, as was argued earlier. Moreover, Theorem 4 allows us to conclude that there is no revealed preference relationship between b and d. To see this, observe that while dP^*b, the relationship $d \succ b$ does not hold and the enumeration procedure using G_{bd} does succeed: one can check that the enumeration $x_1 = a$, $x_2 = c$, $x_3 = d$, and $x_4 = b$ would work.

Appendix

A1. Proofs for Section 5

Proof of Theorem 1. (Necessity) Suppose that $c_{obs} : \mathcal{D} \to \mathcal{P}(X)$ can be explained by the theory of choice with limited attention. That is, there is a complete and transitive relation P on X and an attention filter Γ satisfying Condition (5) such that $c_{obs}(S)$ is the P-maximal element in $\Gamma(S)$ for each $S \in \mathcal{P}(X)$. We show SARP-$\Sigma_{LA}(c_{obs})$ is satisfied. Suppose that for some $T, T' \in \mathcal{D}$, we have $c_{obs}(T), c_{obs}(T') \in T \cap T'$ and $c_{obs}(T) \neq c_{obs}(T')$. If, in contradiction to satisfying the restrictions in $\Sigma_{LA}(c_{obs})$, we have that $yPc_{obs}(T)$ for all $y \in T \backslash T'$ and $y'Pc_{obs}(T')$ for all $y' \in T' \setminus T$, then $\Gamma(T) \subsetneq T \cap T'$ and $\Gamma(T') \subsetneq T \cap T'$. Using Condition (5), $\Gamma(T) \subsetneq T \cap T'$ implies that $\Gamma(T \cap T') = \Gamma(T)$ while $\Gamma(T') \subsetneq T \cap T'$ implies $\Gamma(T \cap T') = \Gamma(T')$. But this would lead to a contradiction, since $c_{obs}(T) \neq c_{obs}(T')$ implies that $\Gamma(T) \neq \Gamma(T')$.

(Sufficiency) Following SARP-$\Sigma_{LA}(c_{obs})$, let P be an acyclic relation satisfying the restrictions in $\Sigma_{LA}(c_{obs})$. We may assume without loss of generality that P is also complete and transitive, as any acyclic relation admits a transitive completion. Define a consideration set mapping $\Gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows. For any $S \in \mathcal{D}$, define $\Gamma(S) = \{c_{obs}(S)\} \cup \{x \in S | c_{obs}(S)Px\}$. For any $S \notin \mathcal{D}$, define

$$\Gamma(S) = \begin{cases} \Gamma(T) & \text{if } S \subseteq T,\ T \in \mathcal{D}, \text{ and } \Gamma(T) \subseteq S \\ S & \text{otherwise.} \end{cases}$$

Clearly $\Gamma(S) \neq \emptyset$ for any $S \in \mathcal{P}(X)$ and the P-maximal element in $\Gamma(S)$ is $c_{obs}(S)$ for any $S \in \mathcal{D}$. To conclude the proof, we must show that Γ is well-defined and that it is an attention filter.

Suppose by contradiction that Γ is not well-defined for some S. This means that for some $S \notin \mathcal{D}$, there exist $T, T' \in \mathcal{D}$ such that $S \subseteq T \cap T'$ with $\Gamma(T) \cup \Gamma(T') \subseteq S$, but $\Gamma(T) \neq \Gamma(T')$. This implies that $c_{obs}(T) \neq c_{obs}(T')$. Consider any $y \in T \setminus T'$. Then, since $S \subseteq T'$, $y \in T \setminus S$. Moreover, since

$\Gamma(T) \subseteq S$, we know $y \in T \setminus \Gamma(T)$. By definition of $\Gamma(T)$ for $T \in \mathcal{D}$, this means $yPc_{obs}(T)$. Similarly, if $y \in T' \setminus T$, we conclude $yPc_{obs}(T')$. This contradicts the fact that P satisfies the restrictions in $\Sigma_{LA}(c_{obs})$.

To see that Γ is an attention filter, i.e. satisfies Condition (5), consider $S \in \mathcal{P}(X)$ and $x \in S \setminus \Gamma(S)$. We check that $\Gamma(S \setminus \{x\}) = \Gamma(S)$ in each of the four possible cases.

1. $S \setminus \{x\}, S \in \mathcal{D}$. Since $S \in \mathcal{D}$, and $x \notin \Gamma(S)$, we know $xPc_{obs}(S)$. Suppose that $\Gamma(S \setminus \{x\}) \neq \Gamma(S)$. Then $c_{obs}(S) \neq c_{obs}(S \setminus \{x\})$. Applying the fact that P satisfies the restrictions of $\Sigma_{LA}(c_{obs})$ to the pair of choice problems S and $S \setminus \{x\}$, we conclude that $c_{obs}(S)Px$, a contradiction.

2. $S \setminus \{x\} \in \mathcal{D}, S \notin \mathcal{D}$. Since $S \setminus \{x\} \in \mathcal{D}$, we know $\Gamma(S \setminus \{x\}) = c_{obs}(S \setminus \{x\}) \cup \{y \in S | c_{obs}(S \setminus \{x\})Py\}$. Since $S \setminus \Gamma(S) \neq \emptyset$, there exists $T \in \mathcal{D}$ with $S \subseteq T$ and $\Gamma(T) \subseteq S$. Because $T \in \mathcal{D}$, $zPc_{obs}(T)$ for all $z \in T \setminus S$. Since $\Gamma(S) = \Gamma(T)$, we know $x \in T \setminus \Gamma(T)$. Hence $xPc_{obs}(T)$. If $\Gamma(S \setminus \{x\}) \neq \Gamma(S) = \Gamma(T)$, then $c_{obs}(S \setminus \{x\}) \neq c_{obs}(T)$ contradicting the restriction of $\Sigma_{LA}(c_{obs})$ for the pair of sets T and $S \setminus \{x\}$.

3. $S \setminus \{x\} \notin \mathcal{D}, S \in \mathcal{D}$. Since $S \in \mathcal{D}$, $\Gamma(S) = c_{obs}(S) \cup \{y \in S | c_{obs}(S)Py\}$. If $x \in S \setminus \Gamma(S)$ then $\Gamma(S) \subseteq S \setminus \{x\}$, so by construction $\Gamma(S \setminus \{x\}) = \Gamma(S)$.

4. $S \setminus \{x\}, S \notin \mathcal{D}$. Since $S \setminus \Gamma(S) \neq \emptyset$, there exists $T \in \mathcal{D}$ with $S \subseteq T$ and $\Gamma(T) \subseteq S$. Since $x \in S \setminus \Gamma(S)$, then $\Gamma(T) = \Gamma(S) \subseteq S \setminus \{x\}$ and so $\Gamma(S \setminus \{x\}) = \Gamma(T)$ by construction, and $= \Gamma(T)$, by transitivity. $\square$

Proof of Theorem 2. This is a corollary of the proof of Theorem 1. To see this, suppose that yPx for some complete and transitive P which satisfies the restrictions in $\Sigma_{LA}(c_{obs})$. The proof of sufficiency constructs a consideration set mapping Γ satisfying condition (5) for which (P, Γ) generate c_{obs}. Hence it is impossible for x to be revealed preferred to y. On the other hand, suppose that xPy for each complete and transitive preference P which satisfies the restrictions in $\Sigma_{LA}(c_{obs})$. If x is not revealed preferred to y, then there must exist a preference ordering P' with $yP'x$ and a consideration set mapping Γ' such that (P', Γ') generate c_{obs}. Since P' does not satisfy the restrictions in $\Sigma_{LA}(c_{obs})$, there exist $T, T' \in \mathcal{D}$ for which $c_{obs}(T), c_{obs}(T') \in T \cap T'$, $c_{obs}(T) \neq c_{obs}(T')$, $yP'c_{obs}(T)$ for all $y \in T \setminus T'$ and $y'P'c_{obs}(T')$ for all $y' \in T' \setminus T$. This implies $\Gamma'(T) \subseteq T \cap T'$ and $\Gamma'(T') \subseteq T \cap T'$. Using Condition (5), we conclude as before that $\Gamma'(T) = \Gamma'(T')$, which contradicts the fact that $c_{obs}(T) \neq c_{obs}(T')$. $\square$

A2. Proofs for Section 6

We begin with two preliminary lemmas.

Lemma 1. *Let $H : \mathcal{P}(X) \to 2^X$ be any set mapping satisfy the properties $H(S) \subseteq S$ for all $S \in \mathcal{P}(X)$ and $H(T) \cap S \subseteq H(S)$ whenever $S \subseteq T$ for $S, T \in \mathcal{P}(X)$. Then the success of the enumeration procedure using H is history independent.*

Proof. Trivially, success implies that there is some choice of $x_k \in H(X \setminus \{x_1, \ldots, x_{k-1}\})$ in each step k that will work. Conversely, let $(x_k)_{k=1}^{|X|}$ be an enumeration of X such that $x_k \in H(X \setminus \{x_1, \ldots, x_{k-1}\})$, for all k. We show that the enumeration procedure succeeds by establishing (the stronger property) that $H(S) \neq \emptyset$, for all $S \in \mathcal{P}(X)$. Indeed, consider any $S \in \mathcal{P}(X)$. Let x_i be the element with minimal index in S according to the enumeration $(x_k)_{k=1}^{|X|}$. It is easy to see that since $x_i \in H(X \setminus \{x_1, \ldots, x_{i-1}\})$ and $S \subseteq X \setminus \{x_1, \ldots, x_{i-1}\}$, it must be that $x_i \in H(S)$. $\square$

Lemma 2. *The success of the enumeration procedure using F is history independent. Moreover, if enumeration procedure succeeds using F, then P^* is asymmetric and $c_{obs}(S) \in F(S)$ whenever $S \in \mathcal{D}$.*

Proof. It is easy to see that F satisfies the properties that $F(S) \subseteq S$ for all $S \in \mathcal{P}(X)$ and $F(T) \cap S \subseteq F(S)$ whenever $S \subseteq T$ for $S, T \in \mathcal{P}(X)$. So

Lemma 1 implies that the success of the enumeration procedure using F is history independent.

Now, suppose the procedure using F succeeds and gives us the enumeration $x_1, \ldots, x_n$ of X, with $x_k \in F(X \setminus \{x_1, \ldots, x_{k-1}\})$, for each $k \in \{1, \ldots, n-1\}$.

We show that P^* must be asymmetric. Suppose, by contradiction, that yP^*z and zP^*y. By definition of P^*, both comparisons cannot emerge from observing the choice out of $\{y, z\}$, and hence there exists choice problems R and R' in $\mathcal{D}$ such that $y, z \in R \cap R'$, $y = c_{obs}(R)$, and $z = c_{obs}(R')$. Let i and j be the indices of y and z in the enumeration, i.e. $y = x_i$ and $z = x_k$, and assume without loss of generality that $i < j$. Consider the set $S = \{x_i, \ldots, x_n\}$. We get a contradiction with $x_i \in F(S)$, taking $T = T' = R'$ in condition (2).

Finally, suppose that $S \in \mathcal{D}$ but that $c_{obs}(S) \notin F(S)$. Note that $c_{obs}(S)$ must satisfy condition (1) in the definition of F, by asymmetry of P^*. If $c_{obs}(S)$ is not in $F(S)$ then it must violate condition (2). That is, there exist sets $T, T' \in \mathcal{D}$ such that $\{c_{obs}(S), c_{obs}(T), c_{obs}(T')\} \subseteq S \cap T \cap T'$ and P^* is cyclic on $\{c_{obs}(S), c_{obs}(T), c_{obs}(T')\} \subseteq S \cap T \cap T'$. Using the enumeration, $\{c_{obs}(S), c_{obs}(T), c_{obs}(T')\} = \{x_i, x_j, x_k\}$ where $i < j < k$. By the construction, $x_i \in F(X \setminus \{x_1, \ldots, x_{i-1}\})$. Because $x_i, x_j, x_k \in X \setminus \{x_1, \ldots, x_{i-1}\}$, this cycle would contradict the definition of x_i. $\square$

Proof of Theorem 3. We proceed with the proof for rationalization theory, with the proof for choice by categorization following *a fortiori* (both theories generate the same choice functions on $\mathcal{P}(X)$). It will be most convenient to use the equivalent, psychological filter formulation of rationalization theory given in Cherepanov, Feddersen, and Sandroni (2010); see their preliminary result. That is, the decision-maker has a psychological filter $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying

$$S \subseteq T \text{ implies } \Psi(T) \cap S \subseteq \Psi(S), \text{ for all } S, T \in \mathcal{P}(X), \tag{7}$$

and he uses an asymmetric relation P to choose from each set S the P-maximal element in $\Psi(S)$. Following Cherepanov, Feddersen and Sandroni's Proposition 2, we may also restrict attention without loss of generality to the case where P is complete and $\Psi(\{x,y\}) = \{x,y\}$ for every $x, y \in X$;

(Necessity) Let $c_{obs} : \mathcal{D} \to X$ be a choice function that can be explained by rationalization theory via (Ψ, P) (with $\Psi(\{x,y\}) = \{x,y\}$, for all $x, y \in X$). Let $c : \mathcal{P}(X) \to X$ be the choice function defined on the complete data set that is derived from (Ψ, P) (in particular, c coincides with c_{obs} on $\mathcal{D}$). By definition of the binary relation P^* (see Section 6), it must be that $P = P^*$. We have to show that the enumeration procedure using F succeeds. In fact, we can prove a stronger property, namely that $F(S) \neq \emptyset$, for all $S \in \mathcal{P}(X)$. More precisely, we can show that $c(S) \in F(S)$. If $R \in \mathcal{D}$, $c(S) \in R \subseteq S$, and $c(S) \neq c_{obs}(R) = c(R)$, then $c_{obs}(R) P^* c(S)$, as desired, since $c(S) \in \Psi(R)$. Take $Y, Z \in \mathcal{D}$ containing $\{c(S), c_{obs}(Y), c_{obs}(Z)\}$. Notice that $\Psi(\{c(S), c_{obs}(Y), c_{obs}(Z)\}) = \{c(S), c_{obs}(Y), c_{obs}(Z)\}$ because of (7). Hence P^* must be acyclic on $\{c(S), c_{obs}(Y), c_{obs}(Z)\}$, as desired, since $\Psi(\{c(S), c_{obs}(Y), c_{obs}(Z)\})$ must admit a P^*-maximal element.

(Sufficiency) Suppose that the enumeration procedure using F succeeds. The proof proceeds by constructing an attention filter $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$ such that (Ψ, P^*) deliver well-defined choices in all conceivable choice problems under rationalization theory, with these choices matching observed data.

Applying the enumeration procedure, we find a sequence $x_1, \ldots, x_n$ of X, with $x_k \in F(X \setminus \{x_1, \ldots, x_{k-1}\})$, for each $k \in \{1, \ldots, n-1\}$. By Lemma 2, $c_{obs}(S) \in F(S)$ if $S \in \mathcal{D}$. So we may restrict attention to an enumeration with the added property that $x_k = c_{obs}(X \setminus \{x_1, \ldots, x_{k-1}\})$ for all k such that $X \setminus \{x_1, \ldots, x_{k-1}\} \in \mathcal{D}$.

For any $S \in \mathcal{P}(X)$, define $\Psi(S) =$

$$\begin{cases} \{x_k \mid k = \min\{i \text{ s.t. } x_i \in S\}\} \cup \{c_{obs}(T) \mid S \subseteq T,\ T \in \mathcal{D},\ c_{obs}(T) \in S\} & \text{if } |S| > 2, \\ S & \text{if } |S| = 2. \end{cases} \tag{8}$$

Lemma 2 tells us that P^* is asymmetric. We now complete the proof by showing that (i) $\Psi(S)$ is nonempty for each S, (ii) Ψ satisfies condition (7), (iii) for every S, $\Psi(S)$ has a P^*-maximal element, and (iv) that P^*-maximal element is equal to $c_{obs}(S)$ whenever $S \in \mathcal{D}$.

(i) Suppose $|S| > 2$, else nonemptiness of $\Psi(S)$ is trivial. Observe that $\{x_k \mid k = \min\{i \text{ s.t. } x_i \in S\}\}$ is nonempty because the x_i's are an enumeration of X.

(ii) To show that Ψ satisfies condition (7), take $R \subsetneq S$ and consider $x \in \Psi(S) \cap R$. Suppose $|R| > 2$, else the condition is trivially satisfied. If $x = x_k$, where $k = \min\{i \text{ s.t. } x_i \in S\}$, then it will also be true that $k = \min\{i \text{ s.t. } x_i \in R\}$; hence $x \in \Psi(R)$. On the other hand, if $x = c_{obs}(T)$ for some T containing S, and $x \in R$, then $x \in \Psi(R)$ because $R \subsetneq S \subseteq T$.

(iii) Suppose, by contradiction, that $\Psi(S)$ has no maximal element. If $|\Psi(S)| = 2$ this is impossible by asymmetry (see (i)), so suppose $|\Psi(S)| > 2$, and therefore that $|S| > 2$. This implies that there is a P^*-cycle within $\Psi(S)$. By completeness of P^*, there is, in particular, a P^*-cycle within $\Psi(S)$ consisting of just three elements: $xP^*yP^*zP^*x$, where $x, y, z \in \Psi(S)$. Since for each S, the set $\{x_k \mid k = \min\{i \text{ s.t. } x_i \in S\}\}$ consists of exactly one element, any other element of $\Psi(S)$ must be an observed choice. Without loss of generality let $y, z \in \{c_{obs}(T) \mid S \subseteq T,\ T \in \mathcal{D},\ c_{obs}(T) \in S\}$ and $x = x_k$, where $k = \min\{i \text{ s.t. } x_i \in S\}$. If $x = x_k$ is the element of minimal index in S, then clearly $y, z \notin \{x_1, \dots, x_{k-1}\}$. Since there exist $T', T'' \in \mathcal{D}$ containing S (hence containing $\{x_k, y, z\}$) such that $y = c_{obs}(T')$ and $z = c_{obs}(T'')$, then $x_k \notin F(X \setminus \{x_1, \dots, x_{k-1}\})$, contradicting the way x_k was constructed. Hence $\Psi(S)$ must have a P^*-maximal element.

(iv) Suppose that $S \in \mathcal{D}$. Suppose $|S| > 2$, else the result follows trivially by asymmetry of P^* (see (i)). To see that $c_{obs}(S)$ is the P^*-maximal element in $\Psi(S)$, first note that $c_{obs}(S) \in \Psi(S)$ trivially. Suppose by contradiction that for some $T \in \mathcal{D}$ containing S such that $c_{obs}(T) \in S$, we have that $c_{obs}(T) \in \Psi(S)$ and $c_{obs}(T) P^* c_{obs}(S)$. By definition of P^*, this would contradict its asymmetry. Now suppose by contradiction that $x_k P^* c_{obs}(S)$, where $k = \min\{i \text{ s.t. } x_i \in S\}$. Since x_k is the element of minimal index in S, we know that $S \subseteq X \setminus \{x_1, \ldots, x_{k-1}\}$. Observe that $S \neq X \setminus \{x_1, \ldots, x_{k-1}\}$ if $x_k \neq c_{obs}(S)$, due to the construction of x_k. But then $x_k \in S \subsetneq X \setminus \{x_1, \ldots, x_{k-1}\}$ and $x_k P^* c_{obs}(S)$ contradicts $x_k \in F(X \setminus \{x_1, \ldots, x_{k-1}\})$ (due to condition (1) of F). $\square$

Proof of Corollary 2. If $x \notin F(S)$, then either (1) there exists $R \in \mathcal{D}$ such that $x \in R \subseteq S$ and $x P^* c_{obs}(R)$, or (2) there exist $T, T' \in \mathcal{D}$ such that P^* is cyclic on $\{x, c_{obs}(T), c_{obs}(T')\} \subseteq S \cap T \cap T'$. Suppose, contrary to what we want to prove, that we can find an extension $c : \mathcal{P}(X) \to X$ of c_{obs} that satisfies weak-WARP and such that $c(S) = x$. Consider case (1) first. If $c(\{x, c_{obs}(R)\}) = x$, then we get a violation of weak-WARP, with $\{x, c_{obs}(R)\} \subseteq R \subseteq S$, given that x is picked out of the smallest and largest sets while $c_{obs}(R)$ is picked out of R. If $c(\{x, c_{obs}(R)\}) = c_{obs}(R)$, then there must exists $T, T' \in \mathcal{D}$ such that $T \subseteq T'$, $c_{obs}(T) = x$, and $c_{obs}(T') = c_{obs}(R) \in T$, given that $x P^* c_{obs}(R)$. We get a violation of weak-WARP again. Consider now case (2). To fix ideas, assume that $x P^* c_{obs}(T') P^* c_{obs}(T) P^* x$. If, $c(\{x, c_{obs}(T), c_{obs}(T')\}) = c_{obs}(T')$, then we must have $c(\{x, c_{obs}(T')\}) = c_{obs}(T')$, in order to avoid a violation of weak-WARP, given that $c(S) = x$. Given that $x P^* c_{obs}(T')$, it must be that $x \succ c_{obs}(T')$, and there exists $R, R' \in \mathcal{D}$ such that $c(R') = c_{obs}(T') \in R$ and $c_{obs}(R) = x$. Yet this leads to another violation of weak-WARP. A similar contradiction follows if c picks $c_{obs}(T)$ or x out of $\{x, c_{obs}(T), c_{obs}(T')\}$. It must thus be that the set of valid predictions for S fall within $F(S)$. The second part of the statement of Corollary 2, regarding how to apply the enumeration procedure to determine whether an option is a valid prediction when P^* is complete, immediately follows from Theorem 3 and the definition of predictions in Section 3. $\square$

Proof of Theorem 4. We begin by assuming $x \succ y$ does not hold. Then, by xP^*y, it must be that $\{x, y\} \in \mathcal{D}$ and $c_{obs}(\{x, y\}) = x$. Because c_{obs} is consistent with the theories of rationalization and choice by categorization, there is a filter Ψ such that the choice function derived from (Ψ, P^*) coincides with c_{obs} on $\mathcal{D}$. Hence x is revealed preferred to y if and only if it is impossible to find a filter Ψ' and a preference P' such that the resulting choice function coincides with c_{obs} on $\mathcal{D}$ and $yP'x$. We show it is possible to find (Ψ', P') with such properties if and only if the enumeration procedure using G_{xy} succeeds.

For one direction, assume that it is possible to find such (Ψ', P'). Let $c' : \mathcal{P}(X) \to X$ be the resulting choice function, and consider the enumeration with $x_1 = c'(X)$ and iteratively, $x_k = c'(X \setminus \{x_1, \ldots, x_{k-1}\})$, for $k = 2, \ldots, |X|$. We just need to prove that $x_k \in G_{xy}(\{x_k, \ldots, x_{|X|}\})$, for each k. Let F' be the mapping satisfying conditions (1) and (2) for c'. Because there are fewer restrictions coming from c_{obs} than from c', we know $F'(S) \subseteq F(S)$ for each $S \in \mathcal{P}(X)$. By Lemma 2, $x_k = c'(\{x_k, \ldots, x_{|X|}\}) \in F'(\{x_k, \ldots, x_{|X|}\})$. Hence $x_k \in F(\{x_k, \ldots, x_{|X|}\})$, for each k. Suppose that $x, y \in \{x_k, \ldots, x_{|X|}\}$. We must show that $x_k \neq y$. If $x_k = c'(\{x_k, \ldots, x_{|X|}\}) = y$, then in c' we have both that $c'(\{x, y\}) = x$ and that $c'(\{x_k, \ldots, x_{|X|}\}) = y$ where $x \in \{x_k, \ldots, x_{|X|}\}$. But then in c', $x \succ y$ implying, by Cherepanov, Feddersen and Sandroni (2010, Proposition 4) that it is impossible to have $yP'x$ in a rationalization of c'. Therefore, $x_k \in G_{xy}(\{x_k, \ldots, x_{|X|}\})$. Note that G_{xy} satisfies the conditions of Lemma 1. Hence the enumeration procedure using G_{xy} succeeds.

For the other direction, assume that the enumeration procedure using G_{xy} succeeds and gives an enumeration $x_1, \ldots, x_{|X|}$ of X. By the definition of G_{xy}, x must appear earlier in the enumeration than y. Recall that G_{xy} is a selection of F. By the proof of Theorem 3, we use the enumeration to construct a filter Ψ'' according to equation (8) such that (Ψ'', P^*) generate a choice function $c'' : \mathcal{P}(X) \to X$ that coincides with c_{obs} on $\mathcal{D}$. Since $\{x, y\} \in \mathcal{D}$, we know that for any $S \in \mathcal{D}$ for which $x, y \in S$, $c_{obs}(S) \neq y$ (otherwise we would have that $x \succ y$). By (8), the choice from any $S \in \mathcal{P}(X)$ which contains $\{x, y\}$

cannot be y. This is because y is not the choice of any superset in the data, and cannot be the element of minimal index (x has a lower index). Hence c'' is a complete choice function where y is never chosen in the presence of x. But then by Cherepanov, Feddersen and Sandroni (2010, Proposition 4), there is no revealed preference relationship between x and y, and in particular, one can rationalize c_{obs} with $yP'x$ and some filter Ψ'.

Finally, suppose x is revealed preferred to y. Either $x \succ y$ (in which case we are done) or not $x \succ y$. In the latter case, we just showed the enumeration procedure using G_{xy} does not succeed. Conversely, if $x \succ y$ then x is revealed preferred to y by Cherepanov, Feddersen, and Sandroni (2010, Proposition 4). If $x \succ y$ does not hold and the enumeration procedure using G_{xy} does not succeed, then we have just shown that x is revealed preferred to y. □

Remark 1. *When P^* is incomplete and cyclic, success of the enumeration procedure does not ensure consistency. Take $X = \{a, b, c, d, w, x, y, z\}$, $\mathcal{D} = \{ab, ad, bc, cd, abdw, abdx, bcdy, bcdz\}$, and $c_{obs4} : \mathcal{D} \to X$ given by*

S	ab	ad	bc	cd	$abdw$	$abdx$	$bcdy$	$bcdz$
$c_{obs4}(S)$	b	a	c	d	a	d	b	c.

In this case, P^* coincides with $\succ$, and $aP^*dP^*cP^*bP^*a$. The enumeration procedure succeeds using this incomplete P^* (especially given that P^* has no cycle involving only two or three alternatives). Yet c_{obs4} is inconsistent with the theories of rationalization or choice by categorization. To check this, consider the problem $\{b, d\} \notin \mathcal{D}$. If consistency holds, then it must also hold when extending c_{obs4} by picking one of b or d out of $\{b, d\}$. Suppose b is picked, and let P' be the relation derived from P^* by adding $bP'd$. The enumeration procedure fails for P' given the extended choice function, a contradiction. Indeed, condition (2) in the definition of $F(\{a, b, d\})$ fails for a, b, and d, because P' is cyclic on these three elements, and two of them are chosen in observed choice problems that contain $\{a, b, d\}$. A similar contradiction holds when trying to extend c_{obs4} by picking d out of $\{b, d\}$ (with P' now being cyclic on $\{b, c, d\}$).

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