# The Evolutionary Robustness of Forgiveness and Cooperation

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**Abstract**

We study the evolutionary robustness of strategies in infinitely repeated prisoners' dilemma games in which players make mistakes with a small probability and are patient. The evolutionary process we consider is given by the replicator dynamics. We show that there are strategies with a uniformly large basin of attraction independent of the size of the population. Moreover, we show that those strategies forgive defections and, assuming that they are symmetric, they cooperate. We provide partial efficiency results for asymmetric strategies.

## 1 Introduction

The theory of infinitely repeated games has been very influential in the social sciences showing how repeated interaction can provide agents with incentives to overcome opportunistic behavior. However, a usual criticism of this theory is that there may

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be a multiplicity of equilibria. While cooperation can be supported in equilibrium when agents are sufficiently patient, there are also equilibria with no cooperation. Moreover, a variety of different punishments can be used to support cooperation.

To solve this multiplicity problem, we study what types of strategies will have a large basin of attraction regardless of what other strategies are considered in the evolutionary dynamic. More precisely, we study the replicator dynamic over arbitrary finite set of strategies in which the strategy makes a mistake with a small probability $1-p$ in every round of the game, following Fudenberg and Maskin [FM2]. We study which strategies have a non-vanishing basin of attraction with a uniform size regardless of the set of strategies being considered in the population. We say that a strategy has a uniformly large basin of attraction if it repels invasions of a given size for arbitrarily patient players and small probability of errors and for any possible combination of alternative strategies (see definition 2 for details).

We find that two well known strategies, "always defect" and "grim," do not have uniformly large basins of attraction. Moreover, any strategy that does not forgive cannot have a uniformly large basin either. The reason is that, as players become arbitrarily patient and the probability of errors becomes small, unforgiving strategies lose in payoffs relative to strategies that forgive and the size of the basins of attraction between these two strategies will favor the forgiving one. This is the case even when the inefficiencies happen off the equilibrium path (as it is the case for grim).

Also, we show that symmetric strategies leading to inefficient payoffs (on or off the path) cannot have uniformly large basins of attractions. We also provide some efficiency results for asymmetric strategies. First, we show that there is a relationship between the size of the basin of attraction and the frequency of cooperation. Second, we show that there is a relationship between the degree of asymmetry of a strategy and its efficiency. Third, we show that strategies with a uniformly large basin of attraction cannot have inefficient payoffs in all histories.

It could be the case that inefficient and unforgiving strategies do not have uniformly large basins since actually there may be no strategies with that property! We prove that that is not the case by showing that the strategy "win-stay-lose-shift" has a uniformly large basin of attraction, provided a probability of mistakes smaller than a large discount factor. As this strategy is efficient (and symmetric), we show that the concept of uniformly large basins of attraction provides a (partial) solution to the long studied problem of equilibrium selection in infinitely repeated games: only efficient equilibria survive for patient players if we focus on symmetric strategies.

Note that we not only provide equilibrium selection at the level of payoffs but also at the level of the type of strategies used to support those payoffs: the payoffs from mutual cooperation can only be supported by strategies that do not involve asymptotically inefficient punishments. This provides theoretical support to the claims of Axelrod [Ax], that successful strategies should be cooperative and forgiving.

In addition, we prove that our results are also valid in a general class of dynamics provided that it is still the case that the only growing strategies are those that perform better than the average.

In our study of the replicator dynamics (and its perturbations and generalizations) we develop tools that can be used to analyze the basins of attractions outside of the particular case of infinitely repeated games. In fact the results are based in a series of theorems about general replicator dynamics which can be used to study the robustness of steady states for games in general.

An extensive previous literature has addressed the multiplicity problem in infinitely repeated games. Part of this literature focuses on strategies of finite complexity with costs of complexity to select a subset of equilibria (see Rubinstein [R], Abreu and Rubinstein [AR], Binmore and Samuelson [BiS], Cooper [C] and Volij [V]). This literature finds that the selection varies with the equilibrium concept being used and the type of cost of complexity. Another literature appealed to ideas of evolutionary stability as a way to select equilibria and found that no strategy is evolutionary stable in the infinitely repeated prisoners' dilemma (Boyd and Lorberbaum [BL]). The reason is that for any strategy there exists another strategy that differs only after events that are not reached by this pair of strategies. As such, the payoff from both strategies is equal when playing with each other and the original strategy cannot be an attractor of an evolutionary dynamic. Bendor and Swistak [BeS] circumvent the problem of ties by weakening the stability concept and show that cooperative and retaliatory strategies are the most robust to invasions.

In a different approach to ties, Boyd [B] introduced the idea of errors in decision making. If there is a small probability of errors in every round, then all events in a game occur with positive probability destroying the certainty of ties allowing for some strategies to be evolutionary stable. However, as shown by Boyd [B] and Kim [Ki], many strategies that are sub-game perfect for a given level of patience and errors can also be evolutionary stable.

Fudenberg and Maskin [FM2] (see also Fudenberg and Maskin [FM]) show that evolutionary stability can have equilibrium selection implications if we ask that the size of invasions that the strategy can repel to be uniformly large with respect to any alternative strategy and for large discount factors and small probabilities of mistakes. They show that the only strategies with this characteristic must be cooperative. There are three main differences with our results. First, Fudenberg and Maskin [FM2] focus on strategies of finite complexity while we do not have that restriction. Second, our robustness concept does not only consider the robustness to invasion by a single alternative strategy but also robustness to invasion by any arbitrary combination of alternative strategies. In other words, we also look at the size of the basin of attraction inside the simplex. Third, our full efficiency result only applies to the case of symmetric strategies and we only provide partial efficiency results for the general case. We want to point out that to prove efficiency we use a similar approach to the one used in [FM2].

Our results also relate to Johnson, Levine and Pesendorfer [JLP], Volij [V] and Levine and Pesendorfer [LP] who use stochastic stability (Kandori, Mailath and Rob [KMR] and Young [YP]) to select equilibria in infinitely repeated games. As having large basin of attraction is a necessary condition (but not sufficient) for stochastic stability, the present results could help characterize strategies that are stochastically stable in any finite population.

There is a previous theoretical literature providing evolutionary support for the strategy win-stay-lose-shift (see Nowak and Sigmund [NS] and Imhof, Fudenberg and Nowak [IFN]). This strategy has received little support from experiments on infinitely repeated games (see Dal Bó and Fréchette [DBF], Fudenberg, Rand and Dreber [FRD] and Dal Bó and Fréchette [DBF2]). We hope that new experiments can be designed to test this strategy's robustness to invasions when it is already highly prevalent in the population.

Finally, our result linking the size of the basin of attraction and the frequency of cooperation relates to other experimental evidence provided by Dal Bó and Fréchette [DBF]. They find that the frequency of cooperation is increasing in the size of the basin of attraction of Grim versus the strategy Always Defect.

In section 2 we describe the model, the main concepts and results. Sections 3 to 5 present the main results and proofs are given in section 6. We provide some generalizations in the appendix.

# 2 Model, definitions and preliminary results

## 2.1 Infinitely repeated prisoners' dilemma with trembles

We state the definitions of the game first without trembles and later with trembles as in [FM2].

In each period $t = 0, 1, 2, ...$ the 2 agents play a symmetric stage game with action space $A = \{C, D\}$. At each period $t$ player one chooses action $a^t \in A$ and player two chooses action $b^t \in A$. We denote the vector of actions until time $t$ as $a_t = (a^0, a^1, \ldots, a^t)$ for player one and $b_t = (b^0, b^1, \ldots, b^t)$ for player two. The payoff from the stage game at time $t$ is given by utility function $u(a^t, b^t) : A \times A \to \Re$ for player one and $u(b^t, a^t) : A \times A \to \Re$ for player two such that $u(D, C) = T$, $u(C, C) = R$, $u(D, D) = P$, $u(C, D) = S$, with $T > R > P > S$ and $2R > T + S$.

Agents observe previous actions and this knowledge is summarized by histories. When the game begins we have the null history $h^0 = (a^0, b^0)$, afterwards $h_t = (a_{t-1}, b_{t-1}) = ((a^0, b^0), \ldots (a^{t-1}, b^{t-1}))$ and $H_t$ is the space of all possible $t$ histories. Let $H_\infty$ be the set of all possible histories. A pure strategy is a function $s : \cup_{t \geqslant 0} H_t \to A$.

It is important to remark, that given two strategies $s_1, s_2$ and a finite path $h_t = (a_{t-1}, b_{t-1})$, if $s_1$ encounter $s_2$ then $h^t = (s_1(h_t), s_2(\hat{h}_t))$, where $\hat{h}_t := (b_{t-1}, a_{t-1})$. Given a pair of strategies $(s_1, s_2)$ we denote the history that they generate as $h_{s_1,s_2}$. We denote with $h_{s_1,s_2 t}$ the path up to period $t-1$. Given a finite path $h_t$, with $h_{s_1,s_2/h_t}$ we denote the path between $s_1$ and $s_2$ with seed $h_t$.

For the case of trembles, we have the probability of making a mistake, more precisely, with a positive $p < 1$ we denote the probability that a strategy perform what intends. Now, given two strategies $s_1, s_2$ (they can be the same strategy) we define $U_{\delta,p}(s_1, s_2) = (1 - \delta) \sum_{t \geqslant 0, a_t, b_t} \delta^t p_{s_1,s_2}(a_t, b_t) u(a^t, b^t)$ where $u(a^t, b^t)$ denotes the usual payoff of the pair $(a^t, b^t)$ and $p_{s_1,s_2}(a_t, b_t)$ denote the probability that the strategies $s_1$ and $s_2$ go through the path $h_t = (a_t, b_t)$ when they are playing one to each other. Observe that $p_{s_1,s_2}(a_t, b_t) = p_{s_1,s_2}(a_{t-1}, b_{t-1})p^{i_t+j_t}(1-p)^{1-i_t+1-j_t}$ where $i_t = 1$ if $a^t = s_1(h_t)$, $i_t = 0$ otherwise, and $j_t = 1$ if $b^t = s_2(\hat{h}_{t-1})$, $j_t = 0$ otherwise. Therefore, $p_{s_1,s_2}(a_t, b_t) = p^{m_t+n_t}(1-p)^{2t+2-m_t-n_t}$ where $m_t = Cardinal\{0 \leqslant i \leqslant t : s_1(h_i) = a^i\}$ and $n_t = Cardinal\{0 \leqslant i \leqslant t : s_2(\hat{h}_i) = b^i\}$. Observe that $p_{s_1,s_2}(h_{s_1,s_2 t}) = p^{2t+2}$.

## 2.2 Replicator dynamics

In this section we introduce the notion of replicator dynamics and attractors. Let the payoff matrix $A = (a_{ij})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant n}$ and $\Delta$ be the $n-$dimensional simplex $\Delta = \{x = (x_1 \dots x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 1,\ x_i \geqslant 0, \forall i\}$ where $x_i \geqslant$ denotes the prevalence of strategy $i$ in the population. We consider the replicator dynamics $X$ associated to the payoff matrix $A$ on the $n$ dimensional simplex given by the equations $\dot{x}_j = X_j(x) := x_j F_j(x) = x_j(f_j - \bar{f})(x)$ where $f_j(x) = (Ax)_j$, $\bar{f}(x) = \sum_{l=1}^{n} x_l f_l(x)$, and $(AX)_j$ denotes the $j-$th coordinate of the vector $Ax$. We denote with $\varphi$ the associated flow that provides the solution of the replicator equation: $\varphi : \mathbb{R} \times \Delta \to \Delta$. Observe that any vertex is a singularity of the replicator equation, therefore, any vertex is a fixed point of the flow. Given a vertex $e_i$ and $\epsilon > 0$, $\Delta_\epsilon(e_i) = \{x : \sum_{l=1, l\neq i}^{n} x_l < \epsilon\}$ denotes the ball of radius $\epsilon$ and center $e$.

**Definition 1. Attracting fixed point and local basin of attraction**. *The fixed point $e$ is an attractor if there exists an open neighborhood $U$ of $e$ such that for any $x \in U$ follows that $\varphi_t(x) \to e$ as $t \to +\infty$. The global basin of attraction $B^s(e)$ is the set of points with forward trajectories converging to $e$. Moreover, given $\epsilon > 0$ and a vertex $e_i$, we say that $\Delta_\epsilon(e_i)$ is contained in the local basin of attraction of $e_i$ if $\Delta_\epsilon(e_i)$ is contained in the global basin of attraction and any forward trajectory starting in*

*$\Delta_\epsilon(e_i)$ remains inside $\Delta_\epsilon(e_i)$. This is denoted with $\Delta_\epsilon(e_i) \subset B^s_{loc}(e_i)$.*

We are interested in the size of the basin of attraction of strategies as this measures their robustness to invasion by other strategies. If a strategy can be invaded by one alternative strategy at a time, then it is easy to calculate the size of the basin of attraction of the original strategy relative to the alternative strategy under replicator dynamics. This size depends on the payoff matrix of the 2x2 game formed by these two strategies, strategy 1 is the original strategy and strategy 2 is the invading one. In this case the size of the basin of attraction of strategy 1 is $p_{12} = \frac{1}{1+\frac{a_{22}-a_{12}}{a_{11}-a_{21}}}$ if one strategy does not dominate the other in the 2x2 game. Note that the size of the invasion by strategy 2 that strategy 1 can resist is decreasing in the cost of miscoordinating when the other plays strategy 2, and increasing in the cost of miscoordinating when the other plays strategy 1.

However, strategies may not invade one at a time and calculations of the size of a basin of attraction need to consider invasions by any combination of alternative strategies. In other words, we need to consider the size of the basin of attraction also inside the simplex, not only on its sides. It is important to realize that the robustness to invasion by combination of strategies may be quite different to robustness to invasion by single strategies. It could well be that the border of the basin of attraction bends inward in the interior of the simplex, with the degree of bending depending on the payoffs that the invading strategies earn while interacting with each other. As such, calculating the size of the basin of attraction of a strategy would involve considering all the possible combinations of invading strategies what could be quite consuming if the number of possible strategies is large. In the next subsection we show that a simpler approach is possible under replicator dynamics.

## 2.3 Main theorem for replicator equations

In this subsection we show that it is enough to consider all invading pair of strategies to calculate the size of the basin of attraction of a given strategy.

Given matrix $A$, we define a matrix $M$ and the vector $N$ given by

$$N_i = a_{i1} - a_{11}, \;\; M_{ij} = a_{ji} - a_{1i} + a_{11} - a_{j1}, \;\; M_{ji} = a_{ij} - a_{1j} + a_{11} - a_{i1}.$$

Moreover, we assume that the vertex $\{e_2 \dots e_n\}$ are ordered in such a way that $a_{11} - a_{i1} \geqslant a_{11} - a_{j1}$ for any $2 \leqslant i < j$.

**Theorem 1.** *Let $A \in \mathbb{R}^{n\times n}$ (n arbitrary) such that $a_{j1} < a_{11}$. Let*

$$M_0 \;\; = \;\; \max_{i,j\geqslant i}\{\frac{M_{ij}+M_{ji}}{-N_i}, \;\; 0\}. \qquad (1)$$

*Then, $\Delta_{\frac{1}{M_0}} = \{x : \sum_{i\geqslant 2} x_i \leqslant \frac{1}{M_0}\} \subset B^s_{loc}(e_1)$.*

The proof of the theorem is given in section 6.1. The intuition behind this result is that the replicator dynamic is a quadratic equation and therefore only pairs of alternative strategies matter in calculating the differences in payoffs. In the appendix we show that this result also holds for more general evolutionary dynamics.

## 2.4 Uniformly large basin of attraction in infinitely repeated games

In the present section we recast the replicator dynamics when the matrix of payoffs is given by an infinitely repeated prisoners' dilemma game with discount factor $\delta$ and error probability $1-p$ for a finite set of strategies $\mathcal{S} = \{s_1, \dots, s_n\}$. With $B_{loc}(s, \delta, p, \mathcal{S})$ we denote the local basin of attraction of $s$ in any set of strategies $\mathcal{S}$ and identifying

$s$ with $s_1$. It is well known that any strict equilibrium strategy is an attractor in any population containing it. However, the size of the basin of attraction of such a strategy could in principle be made arbitrarily small by appropriately choosing the set of alternative strategies. Moreover, note that there is an infinite and uncountable number of alternative strategies in infinitely repeated games. To capture the idea of evolutionary robustness, in this paper we ask that a strategy have a "large" basin of attraction independently of the finite set of other participating strategies. See definition below.

**Definition 2.** *We say that a strategy $s$ has a uniformly large basin if there exists $K$ verifying that for any finite set of strategies $\mathcal{S}$ containing $s$ and any $\delta$ and $p$ close to one, it holds that $\{(x_1, \dots, x_n) : x_2 + \cdots + x_n \leqslant K\} \subset B_{loc}(s, p, \delta, \mathcal{S})$ where $n = cardinal(\mathcal{S})$.*

## 2.5 Having a large basin for populations of two strategies is not enough

In this section we give an example that shows that when a population of three strategies is considered it can happen that one of them has a uniformly large basin when we consider the subset of two strategies but it does not have a large basin when the three strategies are considered simultaneously. In what follows, given a population of three strategies $\mathcal{S} = \{s, s^*, s'\}$ and its replicator equation, the first strategy is identified with the point $(1, 0, 0)$.

**Theorem 2.** *For any $\lambda$ small, there exists a population of three strategies $\mathcal{S} = \{s, s^*, s'\}$ such that*

*(i) $s$ is an attractor in $\mathcal{S}$ and it always cooperate with itself;*

*(ii)* *in the population* $\{s, s^*\}$ *and in the population* $\{s, s'\}$, *s is a global attractor (in the terminology of the replicator equation, the interior of the one dimensional simplex is in the basin of attraction of s);*

*(iii)* *the point* $(1, \lambda, \lambda)$ *is not contained in the basin of attraction of s.*

# 3 Strategies with a uniformly large basin

In the present section we consider the problem of existence of strategies with uniformly large basin. Unfortunately, for technical reason that are clarified in section 6.3, we have to restrict the probabilities of mistakes in relation with the discount factor. In fact, we are going to consider $\delta > \frac{1}{2}$ and $p > p(\delta)$ where

$$p(\delta) := \sqrt{1 - K(1-\delta)^2}, \tag{2}$$

for some positive $K$ that depends on the payoff matrix of the stage game. Following that, we show that strategies like win-stay-lose-shift satisfy the conditions introduced in subsection 3.1 and hence has a uniformly large basin of attraction with the restriction that $p > p(\delta)$.

## 3.1 Sufficient conditions to have a uniformly large basin

In this section we provide general sufficient conditions to guarantee that a strategy has a uniformly large basin. The condition provided in this subsection is based on theorem 1. In subsection 6.5 we introduce an alternative sufficient condition. Given two strategies $s$ and $s^*$, we write $N_{\delta,p}(s, s^*) := U_{\delta,p}(s, s) - U_{\delta,p}(s^*, s)$. Let $s$ be a strict perfect public equilibrium strategy for $\delta$ and $p$ large. Given $s'$ and $s^*$ with $N_{\delta,p}(s, s^*) \geqslant N_{\delta,p}(s, s')$ we consider the following numbers

$$M_{\delta,p}(s,s^*,s') := \frac{N_{\delta,p}(s,s^*)+N_{\delta,p}(s,s')+U_{\delta,p}(s',s^*)-U_{\delta,p}(s,s^*)+U_{\delta,p}(s^*,s')-U_{\delta,p}(s,s'),}{N_{\delta,p}(s,s^*)},$$

$$M_{\delta,p}(s) := \sup_{N_{\delta,p}(s,s^*)\geqslant N_{\delta,p}(s,s')}\{M_{\delta,p}(s,s^*,s'),\ \ 0\}.$$

We also define $M(s) := \limsup_{\delta,p\to 1} M_{\delta,p(\delta)}(s)$ and observe that if $M(s) < \infty$ then $s$ has a large basin of attraction.

**Definition 3.** *We say that a strategy $s$ satisfies the "uniformly Large Basin condition" if it is a strict perfect public equilibrium strategy and $M(s) < \infty$.*

**Theorem 3.** *If $s$ satisfies the "uniformly Large Basin condition", then $s$ has a uniformly large basin. More precisely, let $\beta$ be small. Then, there exists $\delta_0$ such that for any $\delta > \delta_0$ ($p > p(\delta)$) and any finite set of strategies $\mathcal{S}$ containing $s$, follows that $s$ is an attracting point such that $B(s) \subset B^s_{loc}(s)$ where $B(s) = \{(x_1,\ldots,x_n) : x_2+\cdots+x_n \leqslant \frac{1}{M(s)+\beta}\}$ and $n = cardinal(\mathcal{S})$.*

## 3.2 Win-stay-lose-shift has a uniformly large basin of attraction

**Definition 4. win-stay-lose-shift** *The strategy that stays if it gets either $T$ or $R$ and shifts if not, is called win-stay-lose-shift. From now on, we denote it as $w$.*

**Theorem 4.** *If $2R > T+P$ then $w$ has a uniformly large basin provided that $\delta$ is large and $p > p(\delta)$ as defined in equality (2).*

For win-stay-lose-shift to be an equilibrium strategy it is required that $2R > T+P$ (see [NS]) and $\delta$ and $p$ large. This theorem shows that the set of strategies with a uniformly large basin of attraction is not empty.

To prove theorem 4 we use the sufficient conditions introduced in 6.5 and to calculate the quantities involved we develop a technical approach that is introduced in section 6.3.

# 4 The importance of forgiveness

In this section we show the importance of forgiveness for the evolutionary robustness of strategies. First we prove that neither the strategy Grim nor Always Defect has a uniformly large basin of attraction. Recall that Grim is the strategy that cooperates in the first period and then cooperates if there has been no defection before. We provide the proof for Grim and we observe that this proof is obviously adapted for Always Defect.

**Theorem 5.** *Grim does not have a uniformly large basin of attraction. More precisely, there exists a strategy $s$ such that for any population $\mathcal{S} = \{s, g\}$ and $\epsilon > 0$ small, there exist $p_0, \delta_0$ such that for any $p > p_0, \delta > \delta_0$, the size of the basin of attraction of grim is smaller than $\epsilon$.*

Theorem 5 shows that the well known strategy Grim does not have a uniformly large basin of attraction given that after a defection it behaves like always defect, which does not a uniformly large basin of attraction either. In a world with trembles unforgivingness is evolutionary costly. Relatedly, Myerson [M] proved that whenever the strategy Always Defect is compared with Grim (without tremble), its basin of attraction collapses as the discount factor converges to one.

We formalize next the idea of unforgivingness and provide a general result regarding the basin of attraction of unforgiving strategies.

**Definition 5.** *We say that a strategy $s$ is unforgiving if there exists a history $h_t$ such that $s(h_{t+\tau}/h_t) = D$ for all $h_{t+\tau}$ with $\tau = 0, 1, 2....$*

**Theorem 6.** *Unforgiving strategies do not have a uniformly large basin of attraction.*

# 5 Efficiency and size of the basin of attraction

In the present section we study the relationship between efficiency of a strategy and the size of its basin of attraction. Given a history $h_t$, and a pair of strategies $s, s^*$ we define $U(s, s/h_t) = \lim_{\delta\to 1} \lim_{p\to 1} U_{\delta,p}(s, s/h_t)$.

**Definition 6.** *We say that a strategy $s$ is asymptotically efficient if for any finite path $h_t$ it follows that $U(s, s/h_t) = R$.*

In next subsection we prove that strategies having a uniformly large basin of attraction are assymptotically efficient provided that the strategies are symmetric. In subsection 5.2 we discuss the case of non-symmetric strategy proving some form of weak efficiency.

## 5.1 The symmetric case

**Definition 7.** *We say that a strategy $s$ is symmetric if for any finite path $h_t$ it follows that $s(h_t) = s(\hat{h}_t)$.*

If the strategy $s$ is symmetric, the pair $(s, s)$ would be a strongly symmetric profile as it is usually defined in the literature (see Fudenberg and Tirole [FT]). We drop the use of the word “strongly" for simplicity.

**Theorem 7.** *If $s$ has a uniformly large basin of attraction and is symmetric, then it is asymptotically efficient.*

The proof for this theorem makes the most out of the large number of alternative strategies. While the large number of strategies was a usual hurdle for the study of evolutionary stability in infinitely repeated games, we make the most of it in the proof of efficiency. We construct a sequence of alternative strategies against which an inefficient and symmetric strategy cannot have a uniformly large basin of attraction. For a strategy $s$ to have a uniformly large basin of attraction, it must be that the ratio of cost of miscoordination $\frac{U_{\delta p}(s',s')-U_{\delta p}(s,s')}{U_{\delta p}(s,s)-U_{\delta p}(s',s)}$ is uniformly bounded for large $\delta$ and $p$ for any alternative strategy $s'$. We construct the alternative strategy $s'$ by making it cooperate forever against itself starting from a history $h_t$ in which $s$ is inefficient but imitates $s$ in other histories. Then, the ratio of miscoordination costs that must be bounded is $\frac{U_{\delta p}(s',s'|h_t)+U_{\delta p}\left(s',s'|\widehat{h}_t\right)-U_{\delta p}(s,s'|h_t)-U_{\delta p}\left(s,s'|\widehat{h}_t\right)}{U_{\delta p}(s,s|h_t)+U_{\delta p}\left(s,s|\widehat{h}_t\right)-U_{\delta p}(s',s|h_t)-U_{\delta p}\left(s',s|\widehat{h}_t\right)}$. The difference in payoffs must include the history $\widehat{h}_t$ as difference of the two strategies for the other player also affects payoffs. The symmetry of $s$ implies that $U_{\delta p}\left(s,s'|h_t\right)+U_{\delta p}\left(s,s'|\widehat{h}_t\right) = U_{\delta p}\left(s',s|h_t\right)+U_{\delta p}\left(s',s|\widehat{h}_t\right)$. For the ration of miscoordination cost to be bounded we must have that the subtracting terms must be even lower than the inefficient payoff of $s$ at history $h_t$. Since $s'$ imitates $s$ outside of $h_t$, this implies that there exists another history $h'_t$ in which $s$ obtains even lower payoffs than in the original history. Repeating the previous reasoning across a sequence of histories and alternative strategies, we find that $s$ should be increasingly inefficient up to an impossibly low continuation payoff, reaching a contradiction.

An easy corollary is the following:

**Corollary 1.** *If $s$ has a uniformly large basin of attraction and is symmetric, then for any $R_0 < R$ there exists $\delta_0 := \delta_0(s)$ such that for any $\delta > \delta_0$ there exists $p_0(\delta)$ verifying that if $\delta > \delta_0, p > p_0(\delta)$ then $U_{\delta,p}(s, s/h_t) > R_0$ for any history $h_t$.*

Here it is important to compare theorem 5 with corollary 1. First, observe that the conclusion of theorem 5 is obtained for any $\delta > \delta_0$ and $p > p_0$; instead, in corollary 1 the result is for $\delta > \delta_0$ but $p > p(\delta)$ with $p(\delta)$ strongly depending of $\delta$. Second, a weaker version of theorem 5 can be concluded from corollary 1.

## 5.2 Non-symmetric strategies

We provide a series of results about weak forms of efficiency for strategies having a uniformly large basin of attraction without assuming that the strategies are symmetric. The first result (theorem 8) gives a lower estimate of the size of the basin related to a quantity that measure the non-symmetry of a strategy

The next definition is related to the asymmetry of a strategy. In few words, it measures how frequent $s(h_t) \neq s(\hat{h}_t)$ is along a path.

**Definition 8.** *A strategy $s$ is $c-$asymmetric if for any $h_t$ holds $\sum_{j:u^j(s,s/h_t)=T} \delta^j + \sum_{j':u^{j'}(s,s/h_t)=S} \delta^j \leqslant c$ and there are paths such that $\sum_{j:u^j(s,s/h_t)=T} \delta^j$ $+ \sum_{j':u^{j'}(s,s/h_t)=S} \delta^j$ is arbitrary close to $c$. In particular, if $s$ is $0-$asymmetric, then it follows that it is symmetric.*

**Theorem 8.** *If $s$ has a uniformly large basin of attraction is $c-$asymmetric, then for any $h_t$ follows that $U(s, s/h_t) \geqslant R - 2c(T - S)$.*

In the next theorem we show that if there is a history such that a strategy is not fully efficient for any subsequent path, then it cannot have a uniformly large basin of attraction. The proof is based on theorem 10 where we analyze the dynamics of a population of three strategies, while in the proof of theorem 8 only pairs of strategies are used.

**Theorem 9.** *Given a strategy $s$, if there exists $h_t$ such that $U(s, s/h_k) < R_1$ for some $R_1 < R$ for any $h_k$ containing either $h_t$ or $\hat{h}_t$ (either $h_t \subset h_k$ or $\hat{h}_t \subset h_k$), then $s$ does not have a uniformly large basin of attraction.*

# 6 Proofs

## 6.1 Replicator equation: Proof of theorem 1

We consider an affine change of coordinates to define the dynamics in the positive quadrant of $\mathbb{R}^{n-1}$ instead of the simplex $\Delta$. The affine change of coordinates is given by $\bar{x}_1 = 1 - \sum_{j\geqslant 2} x_j,\ \ \bar{x}_j = x_j\ \ \forall\ j \geqslant 2$ and so, the replicator equation is defined as $\dot{x}_j = F_j(\bar{x})x_j$ for $j = 2, \dots, n$ where $\bar{x} = (\bar{x}_1, \bar{x}_2 \dots, \bar{x}_n)$ with $x_i \geqslant 0, x_2 + \dots + x_n \leqslant 1$ and $F_j(\bar{x}) = (f_j - \bar{f})(1 - \sum_{i\geqslant 2} x_i, x_2, \dots, x_n)$. Observe that in these coordinates the point $e_1 = (1, 0, .., 0)$ corresponds to $(0, .., 0)$ and in these coordinates the simplex $\Delta$ is $\{(x_2, .., x_n) : x_i \geqslant 0, \sum_{i=2}^{n} x_i \leqslant 1\}$.

We can rewrite $F_j$ in the following way:

$$F_j(\bar{x}) \ = \sum_{l\neq j, l\geqslant 1} (f_j - f_l)(\bar{x})\bar{x}_l = f_j - f_1)(\bar{x}) + \sum_{l\geqslant 2} (f_1 - f_l)(\bar{x})x_l.$$

Denoting $R(\bar{x}) := \sum_{l\geqslant 2} (f_1 - f_l)(\bar{x})x_l$, it follows that $F_j(\bar{x}) = (f_j - f_1)(\bar{x}) + R(\bar{x})$ where

$$\begin{aligned}(f_j - f_l)(\bar{x}) &= \sum_{k\geqslant 1} (a_{jk} - a_{lk})\bar{x}_k = (a_{j1} - a_{l1})\bar{x}_1 + \sum_{k\geqslant 2} (a_{jk} - a_{lk})\bar{x}_k \\ &= a_{j1} - a_{l1} + \sum_{k\geqslant 2} (a_{jk} - a_{lk} - a_{j1} + a_{l1})x_k.\end{aligned}$$

Observe that if we take the matrix $M \in \mathbb{R}^{(n-1)\times(n-1)}$ and the vector $N \in \mathbb{R}^{n-1}$ such that $M_{jk} = a_{kj} - a_{1j} + a_{11} - a_{k1}$, $N_j = a_{j1} - a_{11}$ then the replicator equation on affine coordinates is given by $\dot{x}_j = x_j[(N + Mx)_j - x^t(N + Mx)]$, for $j = 2, \dots, n$, where $(v + Mx)_j$ is the $j - th$ coordinate of $v + Mx$.

The proof of theorem 1 is based on the next lemma about quadratic polynomials.

**Lemma 1.** *Let $Q : \mathbb{R}^n \to \mathbb{R}$ given by $Q(x) = Nx + x^t Mx$ with $x \in \mathbb{R}^n$, $N \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$. Let us assume that $N_i < 0$ for any $i$ and for any $j > i$, $|N_i| \geqslant |N_j|$. Let $M_0 = \max_{i,\, j>i}\{\frac{M_{ij}+M_{ji}}{-N_i},\ 0\}$. Then, the set $\Delta_{\frac{1}{M_0}} = \{x \in \mathbb{R}^n : x_i \geqslant 0, \sum_{i=1}^n x_i < \frac{1}{M_0}\}$ is contained in $\{x : Q(x) < 0\}$. In particular, if $M_0 = 0$ then $\frac{1}{M_0}$ is treated as $\infty$ and this means that $\{x \in \mathbb{R}^n : x_i \geqslant 0\} \subset \{x : Q(x) \leqslant 0\}$.*

*Proof.* For any $v \in \mathbb{R}^n$ such that $v_i \geqslant 0$ and $\sum_i v_i = 1$, we consider the following one dimensional quadratic polynomial, $Q^v : \mathbb{R} \to \mathbb{R}$ given by $Q^v(s) := Q(sv) = sNv + s^2 v^t Mv$. To prove the thesis of the lemma is enough to show the following claim: for any positive vector $v$ with norm equal to 1, if $0 < s < \frac{1}{M_0}$ then $Q^v(s) < 0$; in fact, the claim implies the lemma, otherwise, arguing by contradiction, if there is a point $x_0 \in \Delta_{\frac{1}{M_0}}$ different than zero (i.e.: $0 < |x_0| < \frac{1}{M_0}$) such that $Q(x_0) = 0$, then taking $v = \frac{x_0}{|x_0|}$ and $s = |x_0|$ follows that $Q^v(s) = Nx_0 + x_0^t Mx_0 = 0$, but $|v| = 1, s < \frac{1}{M_0}$, a contradiction.

Now we proceed to show the above claim. Observe that the roots of $Q^v(s)$ are given by $s = 0$ and $s = \frac{-Nv}{v^t Mv}$. Observe that $-Nv = \sum(-N_i)v_i > 0$. If $v^t Mv < 0$ then it follows that $Q^v$ is a one dimensional quadratic polynomial with negative quadratic term and two non-positive roots, so for any $s > 0$ holds that $Q^v(s) < 0$ and therefore proving the claim in this case. So, it remains to consider the case that $v^t Mv > 0$. In this case, since $Q^v$ is a one dimensional quadratic polynomial with positive quadratic term $(v^t Mv)$, therefore for any $s$ between both roots $(0, \frac{-Nv}{v^t Mv})$ follows that $Q < 0$ so to finish we have to prove that $\frac{-Nv}{v^tMv} \geqslant \frac{1}{M_0}$ which follows from the next inequalities:

$$\begin{aligned} v^tMv &= \sum_{ij} v_iv_jM_{ij} = \sum_i [v_i^2M_{ii} + \sum_{j>i} v_iv_j(M_{ij}+M_{ji})] \\ &\leqslant \sum_i [v_i^2(-N_i)M_0 + \sum_{j>i} v_iv_j(-N_i)M_0] = M_0\sum_i(-N_i)v_i[\sum_{j\geqslant i} v_j] \\ &\leqslant M_0\sum_i(-N_i)v_i = M_0(-Nv). \end{aligned}$$

□

*Proof of theorem 1:* We consider the change of coordinates: $\bar{x}_1 = 1 - \sum_{j\geqslant 2} x_j, \bar{x}_j = x_j, j = 2,\ldots,n$ introduced before. Let $X = (X_2,\ldots,X_n)$ the vector field in these coordinates, where $X_j = \bar{x}_jF_j(\bar{x})$. For any $k < 1$ we denote $\Delta_k := \{\bar{x} : \sum_{i\geqslant 2} x_i \leqslant k\}$ and $\partial\Delta_k = \{\bar{x} : \sum_{i\geqslant 2} x_i = k\}$. We want to show that for any initial condition $\bar{x}$ in the region $\Delta_{\frac{1}{M_0}}$ follows that the map $t \to \bar{x}(t) = \sum_{i\geqslant 2} \bar{x}_k(t)$ is a strictly decreasing function and so the trajectories remains inside $\Delta_{\frac{1}{M_0}}$ and since it can not escape $\Delta$ it follows that $\bar{x}(t) \to 0$. To do that, we prove $\dot{\bar{x}} < 0$. Therefore, we have to show

$$Q(\bar{x}) := \dot{\bar{x}} = \sum_{j\geqslant 2} X_j = \sum_{j\geqslant 2} x_jF_j(\bar{x}) < 0. \tag{3}$$

Recall that $F_j = (f_j - f_1)(\bar{x}) + R(\bar{x})$ where $R(\bar{x}) = \sum_{l\geqslant 2}(f_1 - f_l)(\bar{x})x_l$. Therefore,

$$Q(\bar{x}) = \sum_{j\geqslant 2}(f_j - f_1)(\bar{x})x_j + \sum_{j\geqslant 2} R(\bar{x})x_j = \sum_{j\geqslant 2}(f_j - f_1)(\bar{x})x_j + R(\bar{x})\sum_{j\geqslant 2} x_j.$$

Since $\sum_{j\geqslant 2} x_j = k$ (with $k < 1$) follows that $Q(\bar{x}) = \sum_{j\geqslant 2}(f_j - f_1)(\bar{x})x_j + R(\bar{x})k$. Recalling the expression of $R$ we get that $Q(\bar{x}) = (1-k)\sum_{j\geqslant 2}(f_j - f_1)(\bar{x})x_j$. So, to prove inequality (3) is enough to show that $Q(\bar{x}) = (1-k)\sum_j x_j(f_j - f_1)(\bar{x}) < 0$ for any $\bar{x} \in \Delta_k$ and $k < \frac{1}{M_0}$. First we rewrite $Q$. Observe that

$$(f_j - f_1)(\bar{x}) \;=\; \sum_i (a_{ji} - a_{1i})\bar{x}_i = a_{j1} - a_{11} + \sum_{i\geqslant 2}(a_{ji} - a_{1i} + a_{11} - a_{j1})\, x_i.$$

If we note the vector $N := (a_{j1} - a_{11})_j$ and the matrix $M := (M_{ij}) = a_{ji} - a_{1i} + a_{11} - a_{j1}$. Therefore, $Q(\bar{x}) = N\bar{x} + \bar{x}^t M\bar{x}$. So we have to find the region given by $\{\bar{x} : Q(\bar{x}) < 0\}$. To deal with it, we apply lemma 1 and we use equation (1) and the theorem is concluded. □

**Remark 1.** *If we apply the proof of lemma 1 to the particular case that $v = e_j$, we are considering the map $Q^v(s) = s[a_{j1} - a_{11} + (a_{jj} - a_{1j} + a_{11} - a_{j1})\, s]$ and $Q(s) = 0$ if and only if $s = 0$ or $s = \frac{a_{11} - a_{j1}}{a_{11} - a_{j1} + a_{jj} - a_{1j}} = \frac{1}{1 + \frac{a_{jj} - a_{1j}}{a_{11} - a_{j1}}} = p_{1j}$ and so $Q(s) < 0$, for any $0 < s < p_{1j}$. In particular, if we apply this to theorem 1, it follows that the whole segment $[0, p_{1j})$ is in the basin of attraction of $e_1$.*

## 6.2 Comparing strategies by pairs is not enough

In this section we show that to guarantee that a strategy has a uniformly large basin of attraction is not enough to compare it with every other single strategy one at the time. In other words, it is not enough to bound by below the size of the basin of attraction when only considering populations of two strategies. More precisely, we provide an example of a set of three strategies where only one is an attractor (and therefore its basin is large in each one dimensional simplex) but it has a small local basin.

We consider a replicator dynamics in dimension two and we write the equation in affine coordinates $\{(x_1, x_2) : 0 \geqslant x_2 \leqslant 1, 0 \geqslant x_3 \leqslant 1, x_2 + x_3 \leqslant 1\}$. Given $\lambda > 0$ and close to zero, we consider the almost horizontal and vertical lines given by

$H_\lambda(x_2) = (x_2, \lambda(1-x_2))$, and $V_\lambda(x_3) = (\lambda(1-x_3), x_3)$.

**Theorem 10.** *Given $\lambda > 0$ close to zero and $a > 0$, there exist $A \in \mathbb{R}^{3\times 3}$ such that $0 < a_{ij} < a$, satisfying that*

*(i) $(0,0)$ is an attractor and the horizontal line $(x_1, 0), 0 \leqslant x_1 < 1$ and vertical line $(0, x_2), 0 \leqslant x_2 < 1$ are contained in the basin of attraction of $(0,0)$;*

*(ii) there are no points in the region bounded by $H_\lambda, V_\lambda$ and $x_1 + x_2 = 1$ contained in the basin of attraction of $(0,0)$.*

*Proof.* To prove the result, we choose $A \in \mathbb{R}^{3\times 3}$ such that for any $(x_2, x_3) \in H_\lambda$ and $(x_2, x_3) \in V_\lambda$ follows that $X(x_2, x_3)$ points towards the region bounded by $H_\lambda, V_\lambda$ and $x_1 + x_2 = 1$. For that, it is enough to show that

$$\frac{X_3(H_\lambda(x_2))}{X_2(H_\lambda(x_2))} = \frac{\lambda(1-x_2)F_3(H(x_2))}{|x_2 F_2(H(x_2))|} > \frac{1}{4}, \quad F_3(H(x_2)) > 0 \ \text{ for} \frac{\lambda}{1-\lambda} < x_2 < 1, \quad (4)$$

$$\frac{X_2(V_\lambda(x_3))}{X_3(V_\lambda(x_3))} = \frac{\lambda(1-x_3)F_2(V(x_3))}{|x_3 F_3(V(x_3))|} > \frac{1}{4}, \quad F_2(V(x_3)) > 0 \ \text{ for} \frac{\lambda}{1-\lambda} < x_3 < 1, \quad (5)$$

where $(\frac{\lambda}{1-\lambda}, \frac{\lambda}{1-\lambda})$ is the intersection point of $H_\lambda$ and $V_\lambda$. Recall the definition of $N \in \mathbb{R}^2, M \in \mathbb{R}^{2\times 2}$ that induce the replicator dynamics in affine coordinates. Given $\lambda$ we assume that $N_2 = N_3$, $\frac{M_{32}}{N_3} = \frac{M_{23}}{N_3} = \frac{1}{\lambda}$, and $\frac{M_{22}}{N_2} = \frac{M_{33}}{N_2} = 2$. To get that, and recalling the relation between the coordinates of $M$ and $A$, we choose the matrix $A$ such that $\frac{a_{33}-a_{13}}{N_3} = 3$, $\frac{a_{22}-a_{12}}{N_2} = 3$, $a_{32} > a_{22}$, $a_{23} > a_{33}$ and $\frac{a_{32}-a_{22}}{N_2} = \frac{a_{23}-a_{33}}{N_2} = \frac{1}{\lambda} - 2$. With this assumption, now we prove that inequality (4) is satisfied: Let us denote $x := x_2$ and we first calculate $F_3(x, \lambda(x-1))$ and $F_2(x, \lambda(x-1))$ and its quotient with $N_3$ and $N_2$

respectively:

$$F_3(x,\lambda(1-x)) = N_3 + M_{32}x + M_{33}\lambda(1-x) -$$
$$[x(N_2 + M_{22}x + M_{23}\lambda(1-x)) + \lambda(1-x)(N_3 + M_{32}x + M_{33}\lambda(1-x))],$$
$$\frac{F_3(x,\lambda(1-x))}{N_3} = 1 + \frac{M_{32}}{N_3}x + \frac{M_{33}}{N_3}\lambda(1-x) -$$
$$[x(\frac{N_2}{N_3} + \frac{M_{22}}{N_3}x + \frac{M_{23}}{N_3}\lambda(1-x)) + \lambda(1-x)(1 + \frac{M_{32}}{N_3}x + \frac{M_{33}}{N_3}\lambda(1-x))]$$
$$= 1 + \lambda - 2\lambda^2 + (\frac{1}{\lambda} - \lambda + 4\lambda^2 - 3)x - 2\lambda^2x^2,$$
$$F_2(x,\lambda(1-x)) = N_2 + M_{22}x + M_{23}\lambda(1-x) -$$
$$[x(N_2 + M_{22}x + M_{23}\lambda(1-x)) + \lambda(1-x)(N_3 + M_{32}x + M_{33}\lambda(1-x))]$$
$$\frac{F_2(x,\lambda(1-x))}{N_2} = 1 + \frac{M_{22}}{N_2}x + \frac{M_{23}}{N_2}\lambda(1-x) -$$
$$[x(1 + \frac{M_{22}}{N_2}x + \frac{M_{23}}{N_2}\lambda(1-x)) + \lambda(1-x)(1 + \frac{M_{32}}{N_2}x + \frac{M_{33}}{N_2}\lambda(1-x))]$$
$$= (1-x)[1 - \lambda - 2\lambda^2 + (1 + 2\lambda^2)x].$$

Therefore, on one hand observe that $1+\lambda-2\lambda^2+(\frac{1}{\lambda}-\lambda+4\lambda^2-3)x-2\lambda^2x^2$ is a quadratic polynomial with negative leading term that is positive at 1 and $\frac{\lambda}{1-\lambda}$ (provided that $|\lambda|$ is small) so is positive for $\frac{\lambda}{\lambda-1} < x < 1$, on the other hand $(1-x)[1-\lambda-2\lambda^2+(1+2\lambda^2)x]$ is positive in the same range, so

$$\frac{\lambda(x-1)F_3(x,\lambda(x-1))}{|xF_2(x,\lambda(x-1))|} = \frac{\lambda[1 + \lambda - 2\lambda^2 + (\frac{1}{\lambda} - \lambda + 4\lambda^2 - 3)x - 2\lambda^2x^2]}{x[1 - \lambda - 2\lambda^2 + (1 + 2\lambda^2)x]};$$

since the minimum of the numerator is attained at $\frac{\lambda}{1-\lambda}$ getting a value close to 1 and the maximum of the denominator is attained at 1 getting a value close to 2, follows that in the range $\frac{\lambda}{\lambda-1} < x < 1$ holds $\frac{\lambda(x-1)F_3(x,\lambda(x-1))}{|xF_2(x,\lambda(x-1))|} \geqslant \frac{1}{3}$, and therefore the inequality (4) is proved. The proof of inequality (5) is similar and left for the reader. $\square$

**Remark 2.** *Observe that under the hypothesis of theorem 10 the point* $(\frac{\lambda}{\lambda+1}, \frac{\lambda}{\lambda+1})$ *is* *not in the basin of attraction of* $e_1$.

*Proof of theorem 2.* Given any small $\lambda > 0$, we build three strategies such that identifying $s$ with $(0,0)$, $s^*$ with $(1,0)$ and $s'$ with $(0,1)$ satisfy the hypothesis of theorem 10. This is done as it follows: First, we assume that the strategies $s'$ and $s^*$ deviate from $s$ at the $0-$history, $s$ plays always cooperate with itself and so $s'(0) = s^*(0) = D$. We fix $\gamma > 0$ and we take $\epsilon$ small. Observe that provided any $\epsilon > 0$ small, taking $\delta$ large, follows that there exist different $b'_1, b'_2, b'_3, b'_4$ and $b^*_1, b^*_2, b^*_3, b^*_4$ such that $0 < R - (b'_1 R + b'_2 T + b'_3 S + b'_4 P) = R - (b^*_1 R + b^*_2 T + b^*_3 S + b^*_4 P) = \epsilon$ but $R - (b'_1 R + b'_2 S + b'_3 T + b'_4 P) = R - (b^*_1 R + b^*_2 S + b^*_3 T + b^*_4 P) > \gamma$. Now, from $(C, D)$ we choose $s, s', s^*$ such that $U_\delta(s, s^*) = U_\delta(s, s') = b'_1 R + b'_2 T + b'_3 S + b'_4 P$ but in such a way that $s' \neq s^*$. To show that it is possible to choose $s'$ independently of $s^*$ against $s$ is enough to take $s'(C, D) \neq s^*(C, D)$. Now, we take $s^*$ and $s'$ from $(D, D)$ such that $s^*(D, D) \neq s'(D, D)$ and $U_\delta(s^*, s^*) - U_\delta(s, s^*) = U_\delta(s^*, s^*) - (b^*_1 R + b^*_2 S + b^*_3 T + b^*_4 P) = -\epsilon$, $U_\delta(s', s') - U_\delta(s, s') = U_\delta(s', s') - (b'_1 R + b'_2 S + b'_3 T + b'_4 P) = -\epsilon$. Moreover, we can take $s', s^*$ such that $U_\delta(s', s^*) = U_\delta(s', s^*) = R$ therefore, $U_\delta(s', s^*) - U_\delta(s^*, s^*) = U_\delta(s', s^*) - U_\delta(s', s') > \gamma$. So, $\frac{U_\delta(s', s^*) - U_\delta(s^*, s^*)}{U_\delta(s, s) - U_\delta(s^*, s)} > \frac{\gamma}{\epsilon}$ and so choosing $\epsilon$ properly we can assume that the quotient is equal to $\frac{1}{\lambda}$. □

## 6.3 Recalculating payoff with trembles

With $U_{\delta,p,h_{s_1,s_2}}(s_1, s_2)$ we denote the utility along the path $h_{s_1,s_2}$. With $U_{\delta,p,h^c_{s_1,s_2}}(s_1, s_2)$ we denote the difference, i.e., $U_{\delta,p}(s_1, s_2) - U_{\delta,p,h_{s_1,s_2}}(s_1, s_2)$. With $U_{\delta,p}(s_1, s_2/h_t)$ we denote the utility with seed $h_t$ and with $U_{\delta,p}(h_{s_1,s_2/h_t})$ we denote the utility only along the path with seed $h_t$ for the pair $s_1, s_2$. In the same way, with $U_{\delta,p}(h^c_{s_1,s_2/h_t})$ we denote $U_{\delta,p}(s_1, s_2/h_t) - U_{\delta,p}(h_{s_1,s_2/h_t})$. Also, with $\mathcal{NE}$ we denote the set of path which are not $h_{s_1,s_2}-$paths; usually those paths are called second order paths.

**Definition 9.** *We say that $s$ is a perfect public equilibrium strategy if for any $s'$ different than $s$ it follows that $U_{\delta,p}(s,s/h_t) - U_{\delta,p}(s',s/h_t) \geqslant 0$. It is also said that $s$ is a strict perfect public equilibrium strategy if $U_{\delta,p}(s,s/h_t) - U_{\delta,p}(s',s/h_t) > 0$.*

Let us consider two strategies $s_1$ and $s_2$ and let

$$\mathcal{R}_{s_1,s_2} := \{h \in H_0 : \exists\ k \geqslant 0,\ s_1(h_t) = s_2(h_t)\ \forall\, t < k;\ s_1(h_k) \neq s_2(h_k)\}.$$

Observe that if $s_1(h_0) \neq s_2(h_0)$ then any path $h \in H_\infty$ belongs to $\mathcal{R}_{s_1,s_2}$. In other words, we consider all the paths where $s_1$ and $s_2$ differ at some moment, including the first move. Observe that $k$ depends on $h$, and it is defined as the first time that $s_1$ differs with $s_2$ along $h$, i.e. $k_h(s_1,s_2) = \min\{t \geqslant 0 : s_1(h_t) \neq s_2(h_t)\}$. From now on, to avoid notation we drop the dependence on the path, and with $h_k$ we denote the $k-$finite truncation of $h$ where $k$ is the first time that $s_1$ and $s_2$ deviate along $h$. Observe that for $h \in \mathcal{R}_{s_1,s_2}$, the fact that $s_1(h_t) = s_2(h_t)$ for any $t < k$ does not imply that $h^t = s_1(h_t)$. Moreover, observe also that if $s_1 \neq s_2$ then $\mathcal{R}_{s_1,s_2} \neq \emptyset$.

**Lemma 2.** *It follows that*

$$U_{\delta,p}(s_1,s_1) - U_{\delta,p}(s_2,s_1) = \sum_{h \in \mathcal{R}_{s_1,s_2}} \delta^k p_{s_1,s_1}(h_k)(U_{\delta,p}(s_1,s_1/h_k) - U_{\delta,p}(s_2,s_1/h_k)). \quad (6)$$

*Proof.* If $s_1(h_0) \neq s_2(h_0)$ then $\mathcal{R}_{s_1,s_2} = H_\infty$, $h_k = h_0$ and in this case there is nothing to prove. If $s_1(0) = s_2(0)$, the result follows from the next claim that states that given a history path $h$ then

$$p_{s_1,s_1}(h_t) = \begin{cases} p_{s_2,s_1}(h_t) & \text{if } t \leqslant k \\ p_{s_2,s_1}(h_k)p_{s_2,s_1/h_k}(\sigma^k(h)_{t-k}) = p_{s_1,s_1}(h_k)p_{s_2,s_1/h_k}(\sigma^k(h)_{t-k}) & \text{if } t > k \end{cases}$$

(where $\sigma^k(h)$ is a history path that verifies $\sigma^k(h)_j = h_{j+k}$). To prove the claim in the case that $t \leqslant k$ we proceed by induction: it follows that $p_{s_1,s_1}(a_t, b_t) = p_{s_1,s_1}(a_{t-1}, b_{t-1})p^{i_t^1+j_t^1}(1-p)^{2-i_t^1-j_t^1}$ where $i_t^1 = 1$ if $a_t = s_1(h_{t-1}) = s_1(a_{t-1}, b_{t-1})$, $i_t^1 = 0$ otherwise, and $j_t^1 = 1$ if $b_t = s_1(\hat{h}_{t-1}) = s_1(b_{t-1}, a_{t-1})$, $j_t^1 = 0$ otherwise; in the same way $p_{s_2,s_1}(a_t, b_t) = p_{s_2,s_1}(a_{t-1}, b_{t-1})p^{i_t^2+j_t^2}(1-p)^{2-i_t^2-j_t^2}$ where $i_t^2 = 1$ if $a_t = s_2(h_{t-1}) = s_2(a_{t-1}, b_{t-1})$, $i_t^2 = 0$ otherwise, and $j_t^2 = 1$ if $b_t = s_1(\hat{h}_{t-1}) = s_1(b_{t-1}, a_{t-1})$, $j_t^2 = 0$ otherwise. Now, by induction follows that $p_{s_1,s_1}(a_{t-1}, b_{t-1}) = p_{s_2,s_1}(a_{t-1}, b_{t-1})$ and from $s_1(h_{t-1}) = s_2(h_{t-1})$ follows that $i_t^1 = i_t^2, j_t^1 = j_t^2$. □

**Lemma 3.** *For any pair of strategies $s_1, s_2$ it follows that $|U_{\delta,p}(h^c_{s_2,s_1/h_t})| < \frac{1-p^2}{p^2(1-\delta)}M$ where $M = \max\{T, |S|\}$.*

*Proof.* Observe that fixed $t$ then $\sum_{h_t \in H_t} p_{s_1,s_2}(h_t) = 1$, since in the equilibrium path at time $t$ the probability is $p^{2t+2}$ it follows that $\sum_{h_t \notin H_t \cap \mathcal{NE}} p_{s_1,s_2}(h_t) = 1 - p^{2t+2}$. Therefore, and recalling that $u(h^t) \leqslant M$,

$$\begin{aligned} & |U_{\delta,p}(h^c_{s_2,s_1/h_t})| = |\frac{1-p^2\delta}{p^2} \sum_{t\geqslant 0, h_t \notin \mathcal{NE}} \delta^t p_{s_1,s_2}(h_t) u(h^t)| \\ \leqslant \; & (1-\delta)\sum_{t\geqslant 0} \delta^t \sum_{h_t \notin \mathcal{NE}} p_{s_1,s_2}(h_t)|u(h^t)| \leqslant (1-\delta)M\sum_{t\geqslant 0}\delta^t(1-p^{2t+2}) \\ = \; & M[(1-\delta)\sum_{t\geqslant 0}\delta^t - (1-\delta)\sum_{t\geqslant 0}\delta^t p^{2t+2}] = M[1 - p^2\frac{1-\delta}{1-p^2\delta}] = \frac{1-p^2}{(1-p^2\delta)}M. \end{aligned}$$

□

From previous lemma, we can conclude the next two lemmas:

**Lemma 4.** *Given two strategies $s_1$ and $s_2$* $\lim_{p\to 1} \sum_{h_t \in \mathcal{NE}} U_{\delta,p}(s_1, s_2/h_t) = 0$.

**Lemma 5.** *Given two strategies $s_1$ and $s_2$ then*

$$lim_{p\to 1} U_{\delta,p}(s_2, s_2) - U_{\delta,p}(s_1, s_2) = \sum_{h_k, h \in \mathcal{R}_{s_1,s_2}} \delta^k [\, U_\delta(h_{s_2,s_2/h_k}) - U_\delta(h_{s_1,s_2/h_k})].$$

Now, we are going to rewrite the equation (6) considering at the same time the paths $h$ and $\hat{h}$.

**Remark 3.** *Observe that given a strategy $s$ if $\hat{h}_t \neq h_t$ it could hold that $s(\hat{h}_t) \neq s(h_t)$. Also, given two strategies $s_1, s_2$ it also could hold that $k_h(s_1,s_2) \neq k_{\hat{h}}(s_1,s_2)$. However, it follows that if $k_h(s_1,s_2) \leqslant k_{\hat{h}}(s_1,s_2)$ then $p_{s_1,s_1}(h_k) = p_{s_1,s_1}(\hat{h}_k) = p_{s_1,s_2}(h_k) = p_{s_1,s_2}(\hat{h}_k) = p_{s_2,s_1}(h_k) = p_{s_2,s_1}(\hat{h}_k) = p_{s_2,s_2}(h_k) = p_{s_2,s_2}(\hat{h}_k)$*

Using the previous remark, we define the set $\mathcal{R}^*_{s_1,s_2}$ as the set

$$\mathcal{R}^*_{s_1,s_2} = \{h \in \mathcal{R}_{s_1,s_2} : k_h(s_1,s_2) \leqslant k_{\hat{h}}(s_1,s_2)\}$$

and therefore the differences $U_{\delta,p}(s_2,s_2) - U_{\delta,p}(s_1,s_2)$ can be written in the following way (denoting $k$ as $k_h(s_1,s_2)$)

$$\begin{aligned} U_{\delta,p}(s_1,s_1) - U_{\delta,p}(s_2,s_1) = \sum_{h_k, h \in \mathcal{R}^*_{s_1,s_2}} \delta^k p_{s_1,s_1}(h_k)[U_{\delta,p}(s_1,s_1/h_k) - U_{\delta,p}(s_2,s_1/h_k)] \\ + \sum_{h_k, h \in \mathcal{R}^*_{s_1,s_2}} \delta^k p_{s_1,s_1}(h_k)[U_{\delta,p}(s_1,s_1/\hat{h}_k) - U_{\delta,p}(s_2,s_1/\hat{h}_k)]. \end{aligned}$$

We give now a lemma that compares equilibrium paths with seeds $h_t$ and $\hat{h}_t$; later, we also compare the payoff along those paths. The proofs of the next two next lemmas are straightforward and left to the reader.

**Lemma 6.** *Given two strategies $s_1$ and $s_2$ and a path $h_t$ follows that $\widehat{h_{s_2,s_1/h_t}} = h_{s_1,s_2/\hat{h}_t}$.*

Now, we compare the payoffs. Given two strategies $s_1$ and $s_2$ and a path $h_k$, we take $b_1 = (1-\delta)\sum_{j:u^j(s_2,s_1/h_k)=R} \delta^j$, $b_2 = (1-\delta)\sum_{j:u^j(s_2,s_1/h_k)=S} \delta^j$, $b_3 = (1-\delta)\sum_{j:u^j(s_2,s_1/h_k)=T} \delta^j$, $b_4 = (1-\delta)\sum_{j:u^j(s_2,s_1/h_k)=P} \delta^j$. Observe that $b_1 + b_2 + b_3 +$

$b_4 = 1$ and $U(s_2, s_1) = b_1 R + b_2 S + b_3 T + b_4 P$. In the same way, for $\hat{h}_k$ we define $\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4$, $\hat{b}_1 = (1-\delta)\sum_{j:u^j(s_2,s_1/\hat{h}_k)=R} \delta^j$, $\hat{b}_2 = (1-\delta)\sum_{j:u^j(s_2,s_1/\hat{h}_k)=S} \delta^j$, $\hat{b}_3 = (1-\delta)\sum_{j:u^j(s_2,s_1/\hat{h}_k)=T} \delta^j$, $b_4 = (1-\delta)\sum_{j:u^j(s_2,s_1/\hat{h}_k)=P} \delta^j$. Observe that $\hat{b}_1+\hat{b}_2+\hat{b}_3+\hat{b}_4 = 1$. Now we define $B_1 = b_1 + \hat{b}_1$, $B_2 = b_2 + \hat{b}_2$, $B_3 = b_3 + \hat{b}_3$, $B_4 = b_4 + \hat{b}_4$.

The next two lemmas are trivial and left for the reader.

**Lemma 7.** *Given two strategies $s_1$ and $s_2$ and a path $h_k$, if $U_\delta(h_{s_2,s_1/h_k}) = b_1R+b_2S+b_3T+b_4PS$ then $U_\delta(h_{s_1,s_2/\hat{h}_k}) = b_1R + b_2T + b_3S + b_4P$. Moreover, if $U_\delta(h_{s_2,s_1/h_k}) + U_\delta(h_{s_2,s_1/\hat{h}_k}) = B_1R + B_2T + B_3S + B_4P$, then $U_\delta(h_{s_1,s_2/h_k}) + U_\delta(h_{s_1,s_2/\hat{h}_k}) = B_1R + B_2S + B_3T + B_4P$.*

**Lemma 8.** *Provided that $T + S \leqslant 2R$, given two strategies $s$ and $s_2$ and a path $h_k$, follows that $U_\delta(h_{s_1,s_2/h_k}) + U_\delta(h_{s_2,s_1/\hat{h}_k}) \leqslant 2R$.*

From now one, given a pair of strategies $s_1$ and $s_2$ ($s_2$ could be equal to $s_1$) we use the following notations,

$$U_{\delta,p}(s_1, s_2/\tilde{h}_k) := U_{\delta,p}(s_1, s_2/h_k) + U_{\delta,p}(s_1, s_2/\hat{h}_k), \tag{7}$$

$$U_{\delta,p}(h_{s_1,s_2/\tilde{h}_k}) := U_{\delta,p}(h_{s_1,s_2/h_k}) + U_{\delta,p}(h_{s_1,s_2/\hat{h}_k}). \tag{8}$$

## 6.4 Recalculating payoffs with trembles for small probability of mistakes

Now, we develop a way to calculate payoffs for strategies by, roughly speaking, approximating the payoff using equilibrium paths, when the probability of mistake is small. This first order approximation allows to prove that the asymptotic bounded condition (see inequalities (15) and (16)) for certain types of strategies (namely strict perfect public equilibrium strategies, see definition 10). In few words, the difference in utility between two strategies can be estimated in the following way (provided that $p$ is sufficiently close to 1): first, we consider all the paths up to its first node of divergence between two strategies, $h_k, \hat{h}_k$ (see equalities (9, 10, 12)); secondly, from the node of divergence we consider equilibrium payoffs (lemma 10). If $s(h_0) \neq s^*(h_0)$, $U_{\delta,p}(s,s) - U_{\delta,p}(s^*,s)$ is close to $U_{\delta,p,h_{s,s}}(s,s) - U_{\delta,p,h_{s^*,s}}(s^*,s)$. From now on, we denote $\bar{N}_{\delta,p}(s,s^*) := U_{\delta,p}(s,s) - U_{\delta,p}(s,s^*)$ and $B_{\delta,p}(s,s^*,s') := U_{\delta,p}(s',s^*) + U_{\delta,p}(s^*,s') - 2U_{\delta,p}(s,s)$. Recall that (and recall notation (7) and (8))

$$N_{\delta,p}(s,s^*) = \sum_{h_k, h \in \mathcal{R}^*_{s,s^*}} \delta^k p_{s,s}(h_k)[\,U_{\delta,p}(s, s/\tilde{h}_k) - U_{\delta,p}(s^*, s/\tilde{h}_k)], \tag{9}$$

$$\bar{N}_{\delta,p}(s,s^*) = \sum_{h_k, h \in \mathcal{R}^*_{s,s^*}} \delta^k p_{s,s}(h_k)[\,U_{\delta,p}(s, s/\tilde{h}_k) - U_{\delta,p}(s, s^*/\tilde{h}_k)]. \tag{10}$$

We define $N^e_{\delta,p}(s,s^*) := \sum_{h_k, h \in \mathcal{R}^*_{s,s^*}} \delta^k p_{s,s}(h_k)[\,U_{\delta,p}(h_{s,s/\tilde{h}_k}) - U_{\delta,p}(h_{s^*,s/\tilde{h}_k})]$ and $\bar{N}^e_{\delta,p}(s,s^*) := \sum_{h_k, h \in \mathcal{R}^*_{s,s^*}} \delta^k p_{s,s}(h_k)[\,U_{\delta,p}(h_{s,s/\tilde{h}_k}) - U_{\delta,p}(h_{s,s^*/\tilde{h}_k})]$ where
$U_{\delta,p}(h_{s,s^*/\tilde{h}_k}) := U_{\delta,p}(h_{s,s^*/h_k}) + U_{\delta,p}(h_{s,s^*/\hat{h}_k})$.

We look for conditions such that there exists a uniform constant $C$ satisfying that

$$\frac{\bar{N}_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} \leqslant \frac{\bar{N}^e_{\delta,p}(s,s^*)}{N^e_{\delta,p}(s,s^*)} + C. \tag{11}$$

We develop a similar approach for $B_{\delta,p}(s,s',s^*)$ that consists in comparing different paths for three strategies $s, s^*, s'$. Given any pair of paths $h, \hat{h}$ where $s, s', s^*$ differ (meaning that at least two of the strategies differ at some finite paths contained either in $h$ or $\hat{h}$), there exist $k' = k(s,s',h), \hat{k}' = \hat{k}(s,s',\hat{h})$, $k^* = k(s,s^*,h), \hat{k}^* = \hat{k}(s,s^*,\hat{h})$, such that $s(h_{k'}) \neq s'(h_{k'}), s(\hat{h}_{k'}) \neq s'(\hat{h}_{k'})$ and $s(\hat{h}_{k^*}) \neq s^*(\hat{h}_{k^*})$. Observe that some of them could be infinity.

We take $k(s,s',s^*) := \min\{k', \hat{k}', k^*, \hat{k}^*\}$ which is finite and observe that $p_{ss}(h_k) =$

$p_{s's^*}(h_k) = p_{s^*s'}(h_k) = p_{s^*s}(h_k) = p_{s's}(h_k)$ and $p_{ss}(\hat{h}_k) = p_{s's^*}(\hat{h}_k) = p_{s^*s'}(\hat{h}_k) = p_{s^*s}(\hat{h}_k) = p_{s's}(\hat{h}_k)$, so

$$B_{\delta,p}(s,s^*,s') = \sum_{k(s,s',s^*)} \delta^k p_{ss}(h_k)[U_{\delta,p}(s',s^*/\tilde{h}_k) + U_{\delta,p}(s^*,s'/\tilde{h}_k) - 2U_{\delta,p}(s,s/\tilde{h}_k)] \quad (12)$$

Now we define

$$B^e_{\delta,p}(s,s^*,s') = \sum_{h:k(s,s',s^*)} \delta^k p_{ss}(h_k)[U_{\delta,p}(h_{s',s^*/\tilde{h}_k}) + U_{\delta,p}(h_{s^*,s'/\tilde{h}_k}) - 2U_{\delta,p}(h_{s,s/\tilde{h}_k})].$$

So, we look for conditions such that there exists a uniform constant $C$ such that

$$\frac{B_{\delta,p}(s,s^*,s')}{N_{\delta,p}(s,s^*)} \leqslant \frac{B^e_{\delta,p}(s,s^*,s')}{N^e_{\delta,p}(s,s^*)} + C. \quad (13)$$

We are going to restrict a relation between $p$ and $\delta$. From now on we take $p \geqslant \sqrt{\delta}$. To simplify calculations we change the usual renormalization factor $1-\delta$ by $\frac{1-p^2\delta}{p^2}$ and we calculate the payoff as follows: $U_{\delta,p}(s_1,s_2) = \frac{1-p^2\delta}{p^2}\sum_{t\geqslant 0, a_t, b_t} \delta^t p_{s_1,s_2}(a_t,b_t)u(a^t,b^t)$. Both ways calculating the payoff (either with renormalization $1-\delta$ or $\frac{1-p^2\delta}{p^2}$) are equivalent as they rank histories in the same way. In addition it holds that: $\frac{1}{2} < \frac{1-\delta}{1-\delta p^2} < 1$. If $s_1 = s_2$ along the equilibrium then $U_{\delta,p}(h_{s,s}) = \frac{1-\delta p^2}{p^2}\sum_{t\geqslant 0} p^{2t+2}\delta^t u(a^t,a^t) \leqslant R$.

**Lemma 9.** *It holds* $N_{\delta,p}(s,s^*) \leqslant N^e_{\delta,p}(s,s^*) + 2\frac{1-p^2}{p^2(1-\delta)}M$; $\bar{N}_{\delta,p}(s,s^*) \leqslant \bar{N}^e_{\delta,p}(s,s^*) + 2\frac{1-p^2}{p^2(1-\delta)}M$; $B_{\delta,p}(s,s^*,s') \leqslant B^e_{\delta,p}(s,s^*,s') + 3\frac{1-p^2}{p^2(1-\delta)}M$.

The next definition is an extension of the definition of perfect public equilibrium strategies.

**Definition 10.** *We say that $s$ is a uniformly strict sub game perfect if for any $s^*$ and $h \in \mathcal{R}_{s,s^*}$, it follows that $(1-p^2\delta)C_0 < U_{\delta,p}(h_{s,s/h_k}) - U_{\delta,p}(h_{s^*,s/h_k})$, for $p > p_0, \delta > \delta_0$ where $C_0$, $\delta_0, p_0$ are positive constants that only depend on $T, R, P, S$.*

Given $\delta$ we take $p$ such that $3\frac{1-p^2}{p^2(1-\delta)}\frac{M}{C_0(1-p^2\delta)} < 1$. Since $p < 1$ it follows that $1 - p^2\delta > 1 - \delta$ and taking $p > \frac{1}{2}$ then to satisfy above equation we require that $\frac{3}{4}\frac{1-p^2}{(1-\delta)^2}\frac{M}{C_0} < 1$. Therefore, we take

$$p(\delta) = \sqrt{1 - \frac{4}{3}\frac{C_0}{M}(1-\delta)^2} \tag{14}$$

and observe that it is a function smaller than 1 for $\delta < 1$ and larger than $\sqrt{\delta}$.

**Lemma 10.** *If $s^*$ is strict perfect public equilibrium and $p > p(\delta)$ (giving by equality 14) then $\frac{\bar{N}_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} \leqslant \frac{\bar{N}^e_{\delta,p}(s,s^*)}{N^e_{\delta,p}(s,s^*)} + 1$ and $\frac{B_{\delta,p}(s,s^*,s')}{N_{\delta,p}(s,s^*)} \leqslant \frac{B^e_{\delta,p}(s,s^*,s')}{N^e_{\delta,p}(s,s^*)} + 1$.*

*Proof.* From lemma 9, $s$ being a perfect public equilibrium and equality (14), we have

$$\frac{\bar{N}_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} \leqslant \frac{\bar{N}^e_{\delta,p}(s,s^*) + 2\frac{1-p^2}{p^2(1-\delta)}M}{N^e_{\delta,p}(s,s^*)(1 + 2\frac{1-p^2}{p^2(1-\delta)}M\frac{1}{N^e_{\delta p}(s,s^*)})} \leqslant$$
$$\frac{\bar{N}^e_{\delta,p}(s,s^*)}{N^e_{\delta,p}(s,s^*)(1 + 2M\frac{1-p^2}{(1-\delta)p^2C_0(1-p^2\delta)})} + 2M\frac{1-p^2}{(1-\delta)p^2C_0(1-p^2\delta)}) \leqslant \frac{\bar{N}^e_{\delta,p}(s,s^*)}{N^e_{\delta,p}(s,s^*)} + 1.$$

The result for $\frac{B_{\delta,p}(s,s^*,s')}{N_{\delta,p}(s,s^*)}$ can be shown in a similar way. $\square$

Now we will bound $\frac{U_{\delta,p}(s,s)-U_{\delta,p}(s,s^*)}{U_{\delta,p}(s,s)-U_{\delta,p}(s^*,s)}$ based on lemma 10.

**Lemma 11.** *If $p > p(\delta)$ (giving by equality 14) and $s$ is a uniform strict and there exists $D$ such that for any $h \in \mathcal{R}^*_{s,s^*}$ it holds $\frac{U_{\delta,p}(h_{s,s/\tilde{h}_k})-U_{\delta,p}(h_{s,s^*/\tilde{h}_k})}{U_{\delta,p}(h_{s,s/\tilde{h}_k})-U_{\delta,p}(h_{s^*,s/\tilde{h}_k})} < D$ then $\frac{U_{\delta,p}(s,s)-U_{\delta,p}(s,s^*)}{U_{\delta,p}(s,s)-U_{\delta,p}(s^*,s)} < D + 1$.*

*Proof.* It is enough to estimate $\frac{\bar{N}_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)}$

$$\frac{\bar{N}_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} = \frac{\sum_{h\in\mathcal{R}_{s,s^*}\delta,p} \delta^k p_{s,s}(h_k)(U_{\delta,p}(h_{s,s/\tilde{h}_k}) - U_{\delta,p}(h_{s,s^*/\tilde{h}_k}))}{\sum_{h\in\mathcal{R}_{s,s^*},\delta,p} \delta^k p_{s,s}(h_k)(U_{\delta,p}(h_{s,s/\tilde{h}_k}) - U_{\delta,p}(h_{s^*,s/\tilde{h}_k}))}$$
$$= \frac{\sum_{k,h_k} \delta^k p_{s,s}(h_k)\frac{U_{\delta,p}(h_{s,s/\tilde{h}_k})-U_{\delta,p}(h_{s,s^*/\tilde{h}_k})}{U_{\delta,p}(s,s/\tilde{h}_k)-U_{\delta,p}(s^*,s/\tilde{h}_k)}(U_{\delta,p}(h_{s,s/\tilde{h}_k}) - U_{\delta,p}(h_{s,s^*/\tilde{h}_k})}{\sum_{k,h_k} \delta^k p_{s,s}(h_k)(U_{\delta,p}(h_{s,s/\tilde{h}_k}) - U_{\delta,p}(h_{s^*,s/\tilde{h}_k})}) \leqslant$$
$$D\frac{\sum_{h\in\mathcal{R}_{s,s^*}\delta,p} \delta^k p_{s,s}(h_k)(U_{\delta,p}(h_{s,s/\tilde{h}_k}) - U_{\delta,p}(h_{s^*,s/\tilde{h}_k}))}{\sum_{h\in\mathcal{R}_{s,s^*},\delta,p} \delta^k p_{s,s}(h_k)(U_{\delta,p}(h_{s,s/\tilde{h}_k}) - U_{\delta,p}(h_{s^*,s/\tilde{h}_k}))} = D.$$

□

## 6.5 Proof of theorem 3 and new conditions to have a uniformly large basin

*Proof of theorem 3.* The proof follows immediately from theorem 1 and the definition of $M(s)$. In fact, ordering the strategies in such a way that $s$ corresponds to the first one and $N(s, s_i) \geqslant N(s, s_j)$ if $j > i$ then it follows that for $\delta$ large, then the constant $M_0 = \sup\{\frac{M_{ij}+M_{ji}}{-N_{ii}}, 0\} < M(s) + \beta$ and therefore $B(s)$ is contained in the basin of attraction of $e_1$. □

We provide now a condition that implies that $s$ is has a uniformly Large Basin of attraction. This new conditions is based on the conditions defined in subsection 3.1 but it is easier to calculate.

**Definition 11.** *We say that a strict perfect public equilibrium strategy $s$ satisfies the asymptotic bounded condition if*

– *there exists $R_0$ such that for any $s^*$ holds*

$$\limsup_{\delta\to 1, p\to 1, p>p(\delta)} \sup_{s^*:N_{\delta,p}(s,s^*)>0} \frac{U_{\delta,p}(s,s) - U_{\delta}(s,s^*)}{N_{\delta,p}(s,s^*)} < R_0, \tag{15}$$

– *there exists* $R_1$ *such that for any* $s^*, s'$ *for which* $N_{\delta,p}(s,s^*) \geqslant N_{\delta,p}(s,s^*)$ *holds, then*

$$\limsup_{\delta\to 1, p\to 1, p>p(\delta)} \sup_{s^*:N_\delta(s,s^*)>0} \frac{U_{\delta,p}(s',s^*) + U_{\delta,p}(s^*,s') - 2U_{\delta,p}(s,s)}{N_{\delta,p}(s,s^*)} < R_1. \qquad (16)$$

**Theorem 11.** *Let s be a strict perfect public equilibrium strategy satisfying the asymptotic bounded condition. Then, s satisfies the " uniformly Large Basin condition" and therefore has a uniformly large basin of attraction.*

**Remark 4.** *From the proof of theorem 11, it follows that* $M(s) \leqslant 2 + 2R_0 + R_1$.

*Proof of theorem 11.* To prove the theorem we need to show that $M_{\delta,p}(s) < +\infty$.

$$\begin{aligned} & M_{\delta,p}(s,s^*,s') = \\ = \; & \frac{N_{\delta,p}(s,s^*) + N_{\delta,p}(s,s') + U_{\delta,p}(s',s^*) - U_{\delta,p}(s,s^*) + U_{\delta,p}(s^*,s') - U_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)} \\ = \; & 1 + \frac{N_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)} + \frac{U_{\delta,p}(s',s^*) - U_{\delta,p}(s,s^*) + U_{\delta,p}(s^*,s') - U_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)}. \end{aligned}$$

Recalling that $N_{\delta,p}(s,s') \leqslant N_{\delta,p}(s,s^*)$ we need to bound by above the following expression $\frac{U_{\delta,p}(s',s^*) - U_{\delta,p}(s,s^*) + U_{\delta,p}(s^*,s') - U_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)}$. So,

$$\begin{aligned} & \frac{U_{\delta,p}(s',s^*) - U_{\delta,p}(s,s^*) + U_{\delta,p}(s^*,s') - U_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)} = \\ = \; & \frac{U_{\delta,p}(s',s^*) + U_{\delta,p}(s^*,s') - 2U_{\delta,p}(s,s)}{N_{\delta,p}(s,s^*)} + \\ + \; & \frac{U_{\delta,p}(s,s) - U_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} + \frac{U_{\delta,p}(s,s) - U_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)} \leqslant \\ & \leqslant R_1 + \frac{U_{\delta,p}(s,s) - U_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} + \frac{U_{\delta,p}(s,s) - U_{\delta,p}(s,s')}{N_{\delta,p}(s,s')} \frac{N_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)} \leqslant R_1 + 2R_0. \end{aligned}$$

□

### 6.6 $w$ has a uniformly large basin of attraction: proof of theorem 4

First we prove that $w$ is a uniformly strict perfect public equilibrium (this is done in subsection 6.6.1), and later we show that $w$ satisfies the "Asymptotic bounded condition". For the latter we need to bound

$$\frac{\bar{N}_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)}, \tag{17}$$

$$\frac{B_{\delta,p}(s,s^*,s')}{N_{\delta,p}(s,s^*)} \tag{18}$$

which is done in subsection 6.6.2 and 6.6.3, respectively.

#### 6.6.1 The profile $(w,w)$ is a uniformly strict perfect public equilibrium

Given $h_k$ we have to estimate $U_{\delta,p}(h_{w,w/h_k}) - U_{\delta,p}(h_{s,w/h_k})$ where $h_{w,w/h_k}$ is the equilibrium path for $w,w$ starting with $h_k$ and $h_{s,w/h_k}$ is the equilibrium path for $s,w$ starting with $h_k$.

In what follows, to avoid notation, with $U(.,.)$ we denote $U_{\delta,p}(h_{.,./h_k})$. Following that, we take $b_1 = \frac{1-p^2\delta}{p^2}\sum_{j:u^j(s,w/h_k)=R} p^{2j+2}\delta^j$, $b_2 = \frac{1-p^2\delta}{p^2}\sum_{j:u^j(s,w/h_k)=S} p^{2j+2}\delta^j$, $b_3 = \frac{1-p^2\delta}{p^2}\sum_{j:u^j(s,w/h_k)=T} p^{2j+2}\delta^j$, and $b_4 = \frac{1-p^2\delta}{p^2}\sum_{j:u^j(s,w/h_k)=P} p^{2j+2}\delta^j$. Observe that $b_1+b_2+b_3+b_4 = 1$ and $U(s,w) = b_1R + b_2S + b_3T + b_4P$. If for each $T$ in a period ($s$ plays $D$ and $w$ plays $C$), in the next period $s$ earns either $S$ or $P$ given that $w$ will play $D$. Therefore,

$$b_2 + b_4 \geqslant p^2\delta b_3. \tag{19}$$

To calculate $U(w,w)$ we have to consider either $s(h_k) = C$, $w(h_k) = D$ or $s(h_k) = D$,

$w(h_k) = C$. So, from the fact that $w$ is symetric (i.e. $w(h_t) = w(\hat{h}_t)$) follows that

$$U(w,w) = \begin{cases} R & \text{if } w(h_k) = C \\ \frac{1-p^2\delta}{p^2} P + p^2\delta R & \text{if } w(h_k) = D \end{cases}$$

To calculate $U(w,w) - U(s,w)$ in case that $s(h_k) = D, w(h_k) = C$, we write $R = b_1R + b_2R + b_3R + b_4R$ by inequality (19) it follows that

$$\begin{aligned} U(w,w) - U(s,w) &= b_2(R-S) + b_3(R-T) + b_4(R-P) \\ &\geqslant (b_2+b_4)(R-P) + b_3(R-T) \geqslant \delta p^2 b_3(R-P) + b_3(R-T) \\ &\geqslant b_3[(1+p^2\delta)R - (T+P)]. \end{aligned}$$

Observing that if $s(h_k) = D, w(h_k) = C$, then $b_3 \geqslant 1 - p^2\delta$, and since $2R - (T+P) > 0$ it follows that, for $\delta$ and $p$ large, $[(1+p^2\delta)R - (T+P)] > C_0$ for a positive constant smaller than $2R - (T+P)$. Therefore, it follows that $U(w,w) - U(s,w) > (1-p^2\delta)C_0$, (provided that $\delta$ and $p$ large are large) concluding that $(w,w)$ is a uniformly strict perfect public equilibrium if $2R > T + S$.

In the case that $s(h_k) = C, w(h_k) = D$, observe that $b_2 \geqslant 1 - \delta$ and calculating again the quantities $b_1, b_2, b_3, b_4$ but starting from $j \geqslant 1$, we get that $U(s,w) = (1 - p^2\delta)S + p^2\delta[b_1R + b_2S + b_3T + b_4P]$. Therefore, writing $p^2\delta R = p^2\delta[b_1R + b_2R + b_3R + b_4R]$ and arguing as before,

$$\begin{aligned} U(w,w) - U(s,w) &= (1-p^2\delta)(P-S) + \delta[b_2(R-S) + b_3(R-T) + b_4(R-P)] \\ &\geqslant (1-p^2\delta)(P-S) + \delta[(b_2+b_4)(R-P) + b_3(R-T)] \\ &\geqslant (1-p^2\delta)(P-S) + \delta[\delta b_3(R-P) + b_3(R-T)] \\ &\geqslant (1-p^2\delta)(P-S) + \delta b_3[(1+\delta)R - (T+P)]. \end{aligned}$$

Since $2R-(T+P)>0$, it follows that for $\delta$ large ($b_3$ now can be zero) $U(w,w)-U(s,w)>(1-p^2\delta)(P-S)$, proving that $(w,w)$ is a uniformly strict perfect public equilibrium in this case.

**Remark 5.** *For $\epsilon$ small and $\delta$ large then $C_0=\min\{P-S, 2R-(T+S)-\epsilon\}$.*

**Remark 6.** *To prove that $w$ is a uniformly strict perfect public perfect, the main two properties of $w$ used are: it cooperates after seeing cooperation and so $U(w,w)=R$ after $w(h_k)=C$, after getting $P$ it goes back to cooperate, so $U(w,w)=(1-\delta p^2)P+\delta p^2 R$ after $w(h_k)=D$, it punishes after getting $S$,$2R>T+P$. Observe, that the previous calculation does not use the fact that $w$ keeps defecting after obtaining $T$.*

### 6.6.2 Bounding (17)

First we estimate $U_{\delta,p}(w,w)-U_{\delta,p}(s,w)$ and $U_{\delta,p}(w,w)-U_{\delta,p}(w,s)$. Recall that from lemma 11 is enough to bound for any $h\in\mathcal{R}^*_{w,s}$ : $\frac{U_{\delta,p}(h_{w,w/\tilde{h}_k})-U_{\delta,p}(h_{w,s/\tilde{h}_k})}{U_{\delta,p}(h_{w,w/\tilde{h}_k})-U_{\delta,p}(h_{s,w/\tilde{h}_k})}$. To avoid notation, we denote $U(s,s'):=U_{\delta,p}(h_{s,s'/h_k\hat{h}_h})=U_{\delta,p}(h_{s,s'/h_k})+U(h_{s,s'/\hat{h}_k})$. Observe that if $U(w,w)-U(s,w)=b_2(R-S)+b_3(R-T)+b_4(R-P)$, then $U(w,w)-U(w,s)=b_2(R-T)+b_3(R-S)+b_4(R-P)$. To avoid notation, let us denote $L=U(w,w)-U(s,w)=b_2(R-S)+b_3(R-T)+b_4(R-P)$ so, $b_4(R-P)=L-[b_2(R-S)+b_3(R-T)]$ and therefore

$$
\begin{aligned}
& U(w,w)-U(w,s)=b_2(R-T)+b_3(R-S)+L-[b_2(R-S)+b_3(R-T)] \\
= \quad & L+b_2(S-T)+b_3(T-S)=L+(b_3-b_2)(T-S)\leqslant L+b_3(T-S).
\end{aligned}
$$

Recalling that in case that $b_3 \neq 0$, $L = U(w,w) - U(s,w) \geqslant b_3[(1+\delta)R - (T+P)]$ (if $b_3 = 0$ then $\frac{U(w,w)-U(w,s)}{U(w,w)-U(s,w)} \leqslant 1$) it follows that

$$\begin{aligned}\frac{U(w,w)-U(w,s)}{U(w,w)-U(s,w)} &\leqslant \frac{L+b_3(T-S)}{L} \leqslant 1 + \frac{b_3(T-S)}{b_3[(1+\delta)R-(T+P)]} \\ &= 1 + \frac{T-S}{(1+\delta)R-(T+P)}.\end{aligned}$$

Therefore, $\frac{U_{\delta,p}(h_{w,w/\tilde{h}_k})-U_{\delta,p}(h_{w,s/\tilde{h}_k})}{U_{\delta,p}(h_{w,w/\tilde{h}_k})-U_{\delta,p}(h_{s,w/\tilde{h}_k})} \leqslant 1 + \frac{T-S}{(1+\delta)R-(T+P)}$, and applying lemma 11 it follows that $\frac{U_{\delta,p}(w,w)-U_{\delta,p}(w,s)}{U_{\delta,p}(w,w)-U_{\delta,p}(s,w)} \leqslant 2 + \frac{T-S}{(1+\delta)R-(T+P)}$.

**Remark 7.** *The main property of $w$ used to bound (17) is that if $b_3 \neq 0$ then $U(w,w) - U(s,w) \geqslant b_3[(1+\delta)R - (T+P)]$ and this follows from the properties listed in remark 6.*

### 6.6.3 Bounding (18)

By lemma 10 we need to bound $\frac{B^e_{\delta,p}(s,s^*,s')}{N^e_{\delta,p}(s,s^*)}$. Recall that

$$B^e_{\delta,p}(s,s^*,s') = \sum_{h:k(s,s',s^*)} \delta^k p_{ss}(h_k)[U_{\delta,p}(h_{s',s^*/\tilde{h}_k}) + U_{\delta,p}(h_{s^*,s'/\tilde{h}_k}) - 2U_{\delta,p}(h_{s,s/\tilde{h}_k})].$$

If $s = w$ we divide the paths in two cases: either $w(h_k) = C$ or $w(h_k) = D$. For the first case, we claim $U_{\delta,p}(h_{s',s^*/\tilde{h}_k}) + U_{\delta,p}(h_{s^*,s'/\tilde{h}_k}) - 2U_{\delta,p}(h_{w,w/\tilde{h}_k}) \leqslant 0$ : since $U_{\delta,p}(h_{w,w/\tilde{h}_k}) = 2R$ and by lemma 8 follows the assertion above. Therefore,

$$B^e_{\delta,p}(s,s^*,s') \leqslant \sum_{h:k(s,s',s^*),w(h_k)=D} U_{\delta,p}(h_{s',s^*/\tilde{h}_k}) + U_{\delta,p}(h_{s^*,s'/\tilde{h}_k}) - 2U_{\delta,p}(h_{w,w/\tilde{h}_k}).$$

In the second, observe that $U(h_{w,w/\tilde{h}_k}) = 2\frac{1-p^2\delta}{p^2}P + 2R\delta$. To deal with this situation we consider two subcases: $i)$ $s'(h_k) = C$ or $s'(\hat{h}_k) = C$, and $ii)$ $s^*(h_k) = C$ or $s^*(\hat{h}_k) = C$.

So,

$$B^e_{\delta,p}(s,s^*,s') \leqslant \sum_{h:s'(h_k)=C \text{ or } s'(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/\tilde{h}_k}) + U_{\delta,p}(h_{s^*,s'/\tilde{h}_k}) - 2U_{h_{\delta,p}(w,w/\tilde{h}_k)} + \sum_{h:s^*(h_k)=C \text{ or } s^*(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/\tilde{h}_k}) + U_{\delta,p}(h_{s^*,s'/\tilde{h}_k}) - 2U_{\delta,p}(h_{w,w/\tilde{h}_k}).$$

*Subcase i) $s'(h_k)=C$ or $s'(\hat{h}_k)=C$:* In this situation follows that $h \in \mathcal{R}^*(s',w)$. We rewrite

$$\sum_{h:s'(h_k)=C \text{ or } s'(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/\tilde{h}_k}) + U_{\delta,p}(h_{s^*,s'/\tilde{h}_k}) - 2U_{\delta,p}(h_{w,w/\tilde{h}_k}) = \sum_{h:s'(h_k)=C \text{ or } s'(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/h_k}) + U_{\delta,p}(h_{s^*,s'/\tilde{h}_k}) - U_{\delta,p}(h_{w,w/\tilde{h}_k}) + \sum_{h:s'(h_k)=C \text{ or } s'(\hat{h}_k)=C} U_{\delta,p}(h_{s^*,s'/h_k}) + U_{\delta,p}(h_{s',s^*/\tilde{h}_k}) - U_{\delta,p}(h_{w,w/\tilde{h}_k}).$$

From $h \in \mathcal{R}^*(s',w)$, and lemma 8 we have that

$$\sum_{h:s'(h_k)=C \text{ or } s'(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/h_k}) + U_{\delta,p}(h_{s^*,s'/\hat{h}_k}) - U_{\delta,p}(h_{w,w/\tilde{h}_k}) \leqslant \frac{1-p^2\delta}{p^2} \sum_{h:h\in\mathcal{R}^*(s',w)} p_{ws'}(h_k)\delta^k[S+T-2P]$$

and

$$\sum_{h:s'(h_k)=C \text{ or } s'(\hat{h}_k)=C} U_{\delta,p}(h_{s^*,s'/h_k}) + U_{\delta,p}(h_{s',s^*/\hat{h}_k}) - U_{\delta,p}(h_{w,w/\tilde{h}_k}) \leqslant \frac{1-p^2\delta}{p^2} \sum_{h:h\in\mathcal{R}^*(s',w)} p_{ws'}(h_k)\delta^k[S+T-2P].$$

Since

$$U_{\delta,p}(w,w) - U_{\delta,p}(s',w) \geqslant \frac{1-p^2\delta}{p^2} \sum_{h:h\in\mathcal{R}^*(s',w)} p_{ws'}(h_k)\delta^k[2P-(S+P)]$$

it follows that

$$\sum_{h:s'(h_k)=C\,\text{or}\,s'(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/h_k}) + U_{\delta,p}(h_{s^*,s'/\hat{h}_k}) - U_{\delta,p}(h_{w,w/\tilde{h}_k}) \leqslant$$
$$U_{\delta p}(w,w) - U_{\delta p}(s',w),$$
$$\sum_{h:s'(h_k)=C\,\text{or}\,s'(\hat{h}_k)=C} U_{\delta,p}(h_{s^*,s'/h_k}) + U_{\delta,p}(h_{s',s^*/\hat{h}_k}) - U_{\delta,p}(h_{w,w/\tilde{h}_k}) \leqslant$$
$$U_{\delta,p}(w,w) - U_{\delta,p}(s',w).$$

*Subcase ii)* $s^*(h_k) = C$ *or* $s^*(\hat{h}_k) = C$*:* From $h \in \mathcal{R}^*(s^*,w)$, and in a similar fashion to subcase $i$) we have that

$$\sum_{h:s^*(h_k)=C\,\text{or}\,s^*(\hat{h}_k)=C} U_{\delta,p}(h_{s^*,s'/h_k}) + U_{\delta,p}(h_{s^*,s'/\hat{h}_k}) - U_{\delta,p}(h_{w,w/\tilde{h}_k}) \leqslant$$
$$U_{\delta,p}(w,w) - U_{\delta,p}(s^*,w)$$
$$\sum_{h:s^*(h_k)=C\,\text{or}\,s^*(\hat{h}_k)=C} U_{\delta,p}(h_{s^*,s'/h_k}) + U_{\delta,p}(h_{s',s^*/\hat{h}_k}) - U_{\delta,p}(h_{w,w/\tilde{h}_k}) \leqslant$$
$$\leqslant U_{\delta,p}(w,w) - U_{\delta,p}(s^*,w).$$

Therefore, recalling that $U_{\delta,p}(w,w) - U_{\delta,p}(s^*,w) \geqslant U_{\delta,p}(w,w) - U_{\delta,p}(s',w)$ we conclude that $\frac{B_{\delta,p}(s,s^*,s')}{U_{\delta,p}(w,w)-U_{\delta,p}(s^*,w)}$ is uniformly bounded and therefore bounding (18). This proof that $w$ has a uniformly large basin of attraction.

## 6.7 Forgiveness: proof of theorems 5 and 6

To prove that Grim ($g$ from now on) does not have a uniformly large basin of attraction, we are going to find a strategy $s$ such that the basin of attraction of $g$ is arbitrary small provided that $\delta$ and $p$ are close to 1. In fact, we use the equation provided in remark 1 to determine the boundary point $p_{g,s} = \frac{1}{1+\frac{U_{\delta,p}(s,s)-U_{\delta,p}(g,s)}{U_{\delta,p}(g,g)-U_{\delta,p}(s,g)}}$ of the basin of attraction of $g$ (the smaller $p_{g,s}$ is, the smaller the basin of attraction of $g$ is).

*Proof of theorem 5.* We consider the strategy $s$ that behaves like g but forgives mutual defections in the first period ($t = 0$). We need to show that for any $\epsilon > 0$ small, there exist $p_0, \delta_0$ such that for any $p > p_0, \delta > \delta_0$, follows that $\frac{1}{1+\frac{U_{\delta,p}(s,s)-U_{\delta,p}(g,s)}{U_{\delta,p}(g,g)-U_{\delta,p}(s,g)}} < \epsilon$. From the definition of $s$, for any $h$ verifying that $h^0 \neq (D,D)$ and any $t$ it follows that $p_{g,g}(h_t) = p_{s,g}(h_t) = p_{g,s}(h_t) = p_{s,s}(h_t)$. Therefore,

$U_{\delta,p}(s,s/(C,C)) = U_{\delta,p}(s,g/(C,C)) = U_{\delta,p}(g,g/(C,C)) = U_{\delta,p}(g,s/(C,C))$,

$U_{\delta,p}(s,s/(D,C)) = U_{\delta,p}(s,g/(D,C)) = U_{\delta,p}(g,g/(D,C)) = U_{\delta,p}(g,s/(D,C))$,

$U_{\delta,p}(s,s/(C,D)) = U_{\delta,p}(s,g/(C,D)) = U_{\delta,p}(g,g/(C,D)) = U_{\delta,p}(g,s/(C,D))$, so

$U_{\delta,p}(s,s) - U_{\delta,p}(g,s) = U_{\delta,p}(s,s/(D,D))p_{s,s}(D,D) - U_{\delta,p}(g,s/(D,D))p_{g,s}(D,D)$,

$U_{\delta,p}(g,g) - U_{\delta,p}(s,g) = U_{\delta,p}(g,g/(D,D))p_{g,g}(D,D) - U_{\delta,p}(s,g/(D,D))p_{s,g}(D,D)$.

Recalling that $s$ after $(D,D)$ behaves as $g$ and $g$ after $(D,D)$ behaves as the strategy always defect (denoted as $a$) and $p_{s,s}(D,D) = p_{s,g}(D,D) = p_{g,s}(D,D) = p_{g,g}(D,D) = (1-p)^2$, then $U_{\delta,p}(s,s) - U_{\delta,p}(g,s) = (1-p)^2\delta[U_{\delta,p}(g,g) - U_{\delta,p}(a,g)]$ and $U_{\delta,p}(g,g) - U_{\delta,p}(s,g) = (1-p)^2\delta[U_{\delta,p}(a,a) - U_{\delta,p}(g,a)]$. Therefore, it remains to calculate the payoffs involving $a$ and $g$. Also observe that for any path $h$ if we take $k$ as the first non-negative integer such that $h_k \neq (C,C)$ then for any $t > k$ $p_{s_1,s_2}(h_t) = p_{s_1,s_2}(h_k)p_{a,a}(\sigma^k(h)_{t-k})$ where $s_1$ and $s_2$ is either $g$ or $a$ and $\sigma^k(h)$ is a history path that verifies $\sigma^k(h)_j = h_{j+k}$. Therefore $U_{\delta,p}(g,g/h_k) = U_{\delta,p/h_k}(a,g) = U_{\delta,p}(g,a/h_k) = U_{\delta,p/h_k}(a,a)$. So, noting with $(C,C)^t$ a path of $t$ consecutive simultaneous cooperation and $L = \sum_{t\geqslant 0,h_t} \delta^t p_{a,a}(h_t)u(h_t) = \frac{1}{1-\delta}[(1-p)^2 R + (S+T)(1-p)p + p^2 P]$, follows that

$$\begin{aligned}
&U_{\delta,p}(g,g) - U_{\delta,p}(a,g) = (1-\delta)\{\sum_{t\geqslant 0} \delta^t u(C,C)[p_{g,g}((C,C)^t) - p_{a,g}((C,C)^t)] + \\
&\sum_{t\geqslant 0} \delta^t [u(C,D) + \delta L][p_{g,g}((C,C)^t(C,D) - p_{a,g}((C,C)^t(C,D))] + \\
&\sum_{t\geqslant 0} \delta^t [u(D,C) + \delta L][p_{g,g}((C,C)^t(D,C)) - p_{a,g}((C,C)^t(D,C))] + \\
&\sum_{t\geqslant 0} \delta^t [u(D,D) + \delta L][p_{g,g}((C,C)^t(D,D)) - p_{a,g}((C,C)^t(D,D))]\}
\end{aligned}$$

Therefore $U_{\delta,p}(g,g) - U_{\delta,p}(a,g) = (1-\delta)GA(\delta,p)$ where

$$\begin{aligned}
&GA(\delta,p) = R[\frac{p^2}{1-p^2\delta} - \frac{p(1-p)}{1-p(1-p)\delta}] + [S+\delta L][\frac{p(1-p)}{1-p^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] + \\
&[T+\delta L][\frac{(1-p)p}{1-p^2\delta} - \frac{p^2}{1-p(1-p)\delta}] + [P+\delta L][\frac{(1-p)^2}{1-p^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}].
\end{aligned}$$

and we write $GA(\delta,p) = GA^0(\delta,p) + GA^1(\delta,p)$ where

$$\begin{aligned}
&GA^0(\delta,p) = R[\frac{p^2}{1-p^2\delta} - \frac{p(1-p)}{1-p(1-p)\delta}] + S[\frac{p(1-p)}{1-p^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] + \\
&T[\frac{(1-p)p}{1-p^2\delta} - \frac{p^2}{1-p(1-p)\delta}] + P[\frac{(1-p)^2}{1-p^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}] = \\
&[Rp^2 + (S+T)p(1-p) + P(1-p)^2][\frac{1}{1-p^2\delta} - \frac{1}{1-p(1-p)\delta}],
\end{aligned}$$

$$\begin{aligned}
&GA^1(\delta,p) = \delta L[\frac{p(1-p)}{1-p^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] + \\
&\delta L[\frac{(1-p)p}{1-p^2\delta} - \frac{p^2}{1-p(1-p)\delta}] + \delta L[\frac{(1-p)^2}{1-p^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}] = \\
&\delta L[\frac{1-p^2}{1-p^2\delta} - \frac{1-(1-p)p}{1-p(1-p)\delta}].
\end{aligned}$$

Observe that when $p,\delta \to 1$ then $Rp^2 + (S+T)p(1-p) + P(1-p)^2 \to R$, $\frac{1}{1-p(1-p)\delta} \to 1$,

$\frac{1-(1-p)p}{1-p(1-p)\delta} \to 1$ and recalling that $(1-\delta)L = \hat{P} = (1-p)^2R + (S+T)(1-p)p + p^2P$ then for $\delta, p$ large follows that

$$(1-\delta)GA^0(\delta,p) \geqslant \frac{R}{2}\frac{1-\delta}{(1-p^2\delta)}, \quad (1-\delta)GA^1(\delta,p) \geqslant \frac{\hat{P}}{2}\frac{1-p^2}{(1-p^2\delta)}.$$

In the same way

$$U_{\delta,p}(a,a) - U_{\delta,p}(g,a) = (1-\delta)\{\sum_{t\geqslant 0}\delta^t u(C,C)[p_{a,a}((C,C)^t) - p_{g,a}((C,C)^t)] +$$
$$\sum_{t\geqslant 0}\delta^t[u(C,D)+\delta L][p_{a,a}((C,C)^t(C,D) - p_{g,a}((C,C)^t(C,D))] +$$
$$\sum_{t\geqslant 0}\delta^t[u(D,C)+\delta L][p_{a,a}((C,C)^t(D,C)) - p_{g,a}((C,C)^t(D,C))] +$$
$$\sum_{t\geqslant 0}\delta^t[u(D,D)+\delta L][p_{a,a}((C,C)^t(D,D)) - p_{g,a}((C,C)^t(D,D))]\}$$

Therefore $U_{\delta,p}(a,a) - U_{\delta,p}(g,a) = (1-\delta)AG(\delta,p)$ where

$$AG(\delta,p) = R[\frac{(1-p)^2}{1-(1-p)^2\delta} - \frac{p(1-p)}{1-p(1-p)\delta}] +$$
$$[S+\delta L][\frac{p(1-p)}{1-(1-p)^2\delta} - \frac{p^2}{1-p(1-p)\delta}] +$$
$$[T+\delta L][\frac{(1-p)p}{1-(1-p)^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] +$$
$$[P+\delta L][\frac{p^2}{1-(1-p)^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}]$$

and we write $AG(\delta,p) = AG^0(\delta,p) + AG^1(\delta,p)$ where

$$AG^0(\delta,p) = R[\frac{(1-p)^2}{1-(1-p)^2\delta} - \frac{p(1-p)}{1-p(1-p)\delta}] +$$
$$S[\frac{p(1-p)}{1-(1-p)^2\delta} - \frac{p^2}{1-p(1-p)\delta}] +$$
$$T[\frac{(1-p)p}{1-(1-p)^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] + P[\frac{p^2}{1-(1-p)^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}]$$

$$AG^1(\delta,p) = \delta L[\frac{p(1-p)}{1-(1-p)^2\delta} - \frac{p^2}{1-p(1-p)\delta}] +$$
$$\delta L[\frac{(1-p)p}{1-(1-p)^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] + \delta L[\frac{p^2}{1-(1-p)^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}] =$$
$$\delta L(1-p)[\frac{2p}{1-(1-p)^2\delta} - \frac{1-p}{1-p(1-p)\delta} + \frac{p^2\delta}{(1-(1-p)^2\delta)(1-p(1-p)\delta)}]$$

Observe that when $p, \delta \to 1$ then $AG^0(\delta,p) \to AG^0(1,1) = P - S$,

$$\frac{2p}{1-(1-p)^2\delta} - \frac{1-p}{1-p(1-p)\delta} + \frac{p^2\delta}{(1-(1-p)^2\delta)(1-p(1-p)\delta)} \to 3$$

and recalling that $(1-\delta)L = \hat{P} = (1-p)^2R + (S+T)(1-p)p + p^2P$ then for $\delta, p$ large follows that

$$(1-\delta)AG^0(\delta,p) \leqslant 2(1-\delta)(P-S), \quad (1-\delta)AG^1(\delta,p) \leqslant 4(1-p)\hat{P}.$$

Recall now that the size of the basin of attraction of $a$ is given by $E(\delta,p) := \frac{1}{1+\frac{(1-\delta)GA(\delta,p)}{(1-\delta)AG(\delta,p)}}$. Observe that for any $\epsilon > 0$ for $p, \delta$ large then $(1-\delta)AG^0(\delta,p) \leqslant \epsilon(1-\delta)GA^0(\delta,p)$ and $(1-\delta)AG^1(\delta,p) \leqslant \epsilon(1-\delta)GA^1(\delta,p)$, therefore, for $p, \delta$ large $E(\delta,p) \leqslant \frac{1}{1+\frac{1}{\epsilon}} = \frac{\epsilon}{1+\epsilon}$ and so the theorem is concluded. $\square$

The proof of theorem 6 is similar to the proof of theorem 5 with the difference that the first point of divergence may not be $t = 1$.

## 6.8 Efficiency: proofs of theorems 7, 8 and 9

### 6.8.1 The symmetric case

Before the proof of theorem 7 we need the next two easy lemmas.

**Lemma 12.** *If $s$ has a uniformly large basin then there exists $C_0$ such that for any strategy $s^*$ and for any $p, \delta$ large (independently of $s^*$) follows that* $\frac{U_{\delta,p}(s^*,s^*)-U_{\delta,p}(s,s^*)}{U_{\delta,p}(s,s)-U_{\delta,p}(s^*,s)} <$

$C_0$. *In particular,* $\lim_{\delta\to 1}\lim_{p\to 1}\frac{U_{\delta,p}(s^*,s^*)-U_{\delta,p}(s,s^*)}{U_{\delta,p}(s,s)-U_{\delta,p}(s^*,s)} < C_0$.

**Lemma 13.** *If $s$ has a uniformly large basin of attraction, then there exists $C_0$ such that for any $s^*$ and $h_t$ follows that*

$$\lim_{\delta\to 1}\lim_{p\to 1}\frac{U_{\delta,p}(s^*,s^*/h_t)-U_{\delta,p}(s,s^*/h_t)+U_{\delta,p}(s^*,s^*/\hat{h}_t)-U_{\delta,p}(s,s^*/\hat{h}_t)}{U_{\delta,p}(s,s/h_t)-U_{\delta,p}(s^*,s/h_t)+U_{\delta,p}(s,s/\hat{h}_t)-U_{\delta,p}(s^*,s/\hat{h}_t)} < C_0.$$

*Proof.* It follows immediately from lemma 12 considering a strategy $s^*$ such that the first deviation from $s$ occurs at $h_t$ (and obviously also at $\hat{h}_t$). $\square$

*Proof of theorem 7:* Let us assume that there exists a path $h_t$ and $\lambda_0 < 1$ such that $U(s,s/h_t) = \lambda_0 R$ and $(s,s)$ is a strict perfect public equilibrium. We start assuming that $h_t$ is not symmetric. Then we show how to deal with the symmetric case using the asymmetric one.

From the fact that $s$ is symmetric, it follows that $U(s,s/h_t) = U(s,s/\hat{h}_t)$ and therefore $U(s,s/h_t)+U(s,s/\hat{h}_t) = 2\lambda_0 R$. Moreover, since $U(s,s/h_t) < R$, we can assume that $s(h_t) = D$. We are going to build a strategy $s^*$ such that $U(s^*,s^*/h_t) = U(s^*,s^*/\hat{h}_t) = R$ and $s^*$ acts like $s$ after meeting $s$ at $h_t$ and $\hat{h}_t$. To build that strategy $s^*$, first we take $s^*$ such that $s^*(h_t) = s^*(\hat{h}_t) = C$ and then we consider all the paths that follow after $h_t, \hat{h}_t$ for the pairs $s,s; s^*,s; s,s^*; s^*,s^*$, i.e.: $h_t(D,D)$, $\hat{h}_t(D,D)$ for $s,s$; $h_t(C,D)$, $\hat{h}_t(C,D)$ for $s^*,s$; $h_t(D,C)$, $\hat{h}_t(D,C)$ for $s,s^*$; $h_t(C,C)$, $\hat{h}_t(C,C)$ for $s^*,s^*$. Observe that the paths involving $h_t$ are all different and the same holds for the paths involving $\hat{h}_t$.

Now we make $s^*$ play $C$ for ever after $h_t(C,C)$ and $\hat{h}_t(C,C)$, so $h_{s^*,s^*/h_t} = (C,C)..(C,C)..$, $h_{s^*,s^*/\hat{h}_t} = (C,C)..(C,C)..$, and so $U(s^*,s^*/h_t) = U(s^*,s^*/\hat{h}_t) = R$. We also make $s^*(h_t(C,D)) = s(h_t(C,D))$, $s^*(\hat{h}_t(C,D)) = s(\hat{h}_t(C,D))$, and observe that both requirement can be satisfied simultaneously and inductively we get that

$h_{s^*,s/h_t(C,D)} = h_{s,s/h_t(C,D)}$, $h_{s^*,s/\hat{h}_t(C,D)} = h_{s,s/\hat{h}_t(C,D)}$. From the fact that $s$ is symmetric, it follows that each entry of $h_{s^*,s/h_t(C,D)} = h_{s,s/h_t(C,D)}$ and $h_{s^*,s/\hat{h}_t(C,D)} = h_{s,s/\hat{h}_t(C,D)}$ is $(C,C)$ or $(D,D)$ and, recalling lemma 6, it follows that $U(s^*, s/h) = U(s, s^*/\hat{h})$ and $U(s^*, s/\hat{h}) = U(s, s^*/h)$. Therefore, $U(s^*, s/h)+U(s^*, s/\hat{h}) = U(s, s^*/h)+U(s, s^*/\hat{h})$. Since, $(s, s)$ is a strict perfect public equilibrium (otherwise it would not have a uniform large basin of attraction) then $U(s^*, s/h_t) + U(s^*, s/\hat{h}_t) < 2\lambda_0 R$ and therefore $U(s, s^*/h_t) + U(s, s^*/h_t) < 2\lambda_0 R$. By lemma 13 it follows that if we denote $U(s^*, s/h_t) + U(s^*, s/\hat{h}_t) = 2\lambda_1 R$, then $\frac{1-\lambda_1}{\lambda_0-\lambda_1} < C_0$, and taking a positive constant $C_1 < 1 - \lambda_0 < 1 - \lambda_1$ it follows that $\lambda_1$ satisfies inequality $\frac{C_1}{\lambda_0-\lambda_1} < C_0$. Therefore, it follows that there exists $\gamma > 0$ such that $\lambda_1 < \lambda_0 - \gamma$. Now, we consider the path $h_t(C, D)$ and we denote it as $h_{t_2}$ and as before we construct a new strategy $s_2^*$ that satisfies the same type of properties as the one satisfied by $s^*$ respect to $s$ but on the path $h_{t_2}$ instead on the path $h_t$. Inductively, we construct a sequences of paths $h_{t_i}$, strategies $s_i^*$ and constants $\lambda_i$ such that $U(s_i^*, s/h_{t_i}) = \lambda_i R$ and they satisfy the following equation $\frac{1-\lambda_{i+1}}{\lambda_i-\lambda_{i+1}} < C_0$, and since $\lambda_{i+1} < \lambda_i$ then also satisfy $\frac{C_1}{\lambda_i-\lambda_{i+1}} < C_0$, and therefore $\lambda_{i+1} < \lambda_i - i\gamma$ but this implies that $\lambda_i \to -\infty$ and so $U(s^*, s/h_{t_i}) \to -\infty$, a contradiction because utilities are bounded by $S$.

To finish, we have to deal with the case that $h_t$ is symmetric and $U(s, s/h_t) < R$. Recall that we can assume that $s(h_t) = s(\hat{h}_t) = D$. Now, let us consider the sequel path $h_t(C, D)$. We claim that if $U(s, s/h_t) < R$ then $U(s, s/h_t(C, D)) < R$. In fact, we can consider the strategy $s^*$ such that only differs on $h_t$ and after that plays the same as $s$ plays. Since $s$ is a sub game perfect (otherwise it would not have a uniform large basin of attraction), it follows that $U_{\delta,p}(s, s/h_t) \geqslant U_{\delta,p}(s^*, s/h_t)$ therefore, $U(s, s/h_t) = \lim_{\delta\to1} \lim_{p\to1} U_{\delta,p}(s, s/h_t) \geqslant \lim_{\delta\to1} \lim_{p\to1} U_{\delta,p}(s^*, s/h_t)$, but since $\lim_{\delta\to1} \lim_{p\to1} U_{\delta,p}(s^*, s/h_t) = \lim_{\delta\to1} \lim_{p\to1} U_{\delta,p}(s, s/h_t(C, D)) = U(s, s/h_t(C, D))$ the claim follows. Observe that the new path $h_t(C, D)$ now is not symmetric and since

$U(s, s/h_t(C, D)) < R$, to conclude the proof of theorem 7 we argue as above. $\square$

### 6.8.2 The non-symmetric case

Before entering in the proofs of theorems 8 and 9, we give a series of results involving pairs of strategies.

**Proposition 1.** *If $s$ has a uniformly large basin of attraction, then there exists $\epsilon > 0$ such that for any $h_t$ follows that $U(s, s/h_t) > P + \epsilon$.*

*Proof.* Choosing $s^*$ such that $s^*(h_t) \neq s(h_t)$ and for any $h_k$ containing $h_t(s^*(h_t), s(\hat{h}_t))$ then $s(h_k) = D$ follows that $U(s^*, s) \geqslant P$ and $U(s, s^*) \leqslant P$. Since also we can chose $s^*$ such that $h_{s^*, s^*/h_t((s^*(h_t), s^*(\hat{h}_t))}$ is a path of full cooperation, then by lemma 12 the conclusion of the proposition follows. $\square$

Observe that previous result is stronger than theorem 6 (provided that $p$ is much closer to one than $\delta$) since here it is shown that strategies with uniformly large basin of attraction have a payoff uniformly away from $P$. Next result goes in the same direction but relating payoff with the size of the basin of attraction.

**Proposition 2.** *If $s$ has a uniformly large basin of attraction and for any $p$ and $\delta$ large follows that there exists $k$ verifying $B_k(s) \subset B^s(s)$, then $U(s, s/h_t) > P + (R - P)k$.*

*Proof.* The proof is similar to the proof of proposition 1 and using remark 1 that allow us to estimate the size of the basin of attraction when only two strategies are involved. $\square$

The next lemmas relates the payoff of $s$ with $s$ starting at $h_t$ and starting at $\hat{h}_t$.

**Lemma 14.** *For any $s$ and a history $h_t$ then $U_\delta(s, s/\hat{h}_t) = U_\delta(s, s/h_t) + (x - y)(T - S)$, where $x = \sum_{j: u^j(s, s/h_t) = S} \delta^j$ and $y = \sum_{j': u^{j'}(s, s/h_t) = T} \delta^j$.*

*Proof.* If $U_\delta(s,s/h_t) = aR + xS + yT + bP$ where $a = \sum_{j:u^j(s,s/h_t)=R} \delta^j$ and $b = \sum_{j':u^{j'}(s,s/h_t)=P} \delta^j$ then $U_\delta(s,s/\hat{h}_t) = aR + xT + yS + bP = U_\delta(s,s/h_t) + xT + yS - xS - yT = U_\delta(s,s/h_t) + (x-y)(T-S)$. □

**Lemma 15.** *Given a strategy $s$ and a path $h_t$ it follows that if $c = x + y$ then $U_\delta(s,s/\hat{h}_t) \leqslant U_\delta(s,s/h_t) + c(T-S)$ if $c < \frac{R-U_\delta(s,s/h_t)}{R-S}$ and $U_\delta(s,s/\hat{h}_t) \leqslant -U_\delta(s,s/h_t) + 2R + c(T+S-2R)$ otherwise.*

*Proof.* From lemma 14 follow that $U_\delta(s,s/\hat{h}_t) = U_\delta(s,s/h_t) + (2x-c)(T-S)$. To conclude, observe that under the restriction $x+y=c, a+c+b=1$, $a,b,x,y$ are in $[0,1]$ and $U_\delta(s,s/h_t) = aR + xS + yT + bP$, the maximum of $2x - c$ is equal to $c$ if $c < \frac{R-U_\delta(s,s/h_t)}{R-S}$ and is equal to $2\frac{R-U_\delta(s,s/h_t)}{T-S} + c\frac{T+S-2R}{T-S}$ otherwise. □

*Proof of theorem 8.* Let us assume by contradiction that $U(s,s/h_t) < R_1 - 2c(T-S)$ for some $R_1 < R$. We can assume also that if $h_{t+1}$ (or $\hat{h}_{t+1}$) is the deviation from $h_t$ (or $\hat{h}_t$), i.e, the first coordinate of $h^{t+1}$ (or $\hat{h}^{t+1}$) is different to $s(h_t)$ (or $s(\hat{h}_t)$) but the second one is equal to $s(\hat{h}_t)$ (or $s(h_t)$) follows that $U(s,s/h_t) - U(s,s/h_{t+1})$ ($U(s,s/\hat{h}_t) - U(s,s/\hat{h}_{t+1})$, respectively) is smaller than $\varepsilon$ with $\varepsilon$ chosen arbitrary small (provided $\delta$ large and $1-p$ small). Now we take $s^*$ such that $s^*(h_t) \neq s(h_t)$, $s^*(\hat{h}_t) \neq s(\hat{h}_t)$ and after that deviation $s^*$ is like $s$ (observe that at this point we are using that $s/h_t$ is not symmetric; we can do that since in the symmetric case we can argue as in the proof of theorem 7). Moreover, we also assume that $U(s^*,s^*/h_t) = R$ and $U(s^*,s^*/\hat{h}_t) = R$. From the assumption, follows $c < \frac{R-U(s,s/h_t)}{T-S}$ and so by lemma 15 $U(s,s/\hat{h}_t) < R_1 - c(T-S)$. Therefore, $U(s^*,s/h_t) < R_1 - 2c(T-S)$ and $U(s^*,s/\hat{h}_t) < R_1 - c(T-S)$, so $U(s,s^*/h_t) < R_1 - c(T-S)$ and $U(s,s^*/\hat{h}_t) < R_1$ and therefore, $U(s^*,s^*/\hat{h}_t) + U(s^*,s^*/h_t) - [U(s,s^*/\hat{h}_t) + U(s,s^*/h_t)] > R - R_1 + c(T-S) \geqslant R - R_1$, and since $U(s,s/h_t) + U(s,s/\hat{h}_t) - [U(s^*,s/h_t) + U(s^*,s/\hat{h}_t)]$ is arbitrarily small, by lemma 12 follows that $s$ does not have a uniformly large basin of attraction. □

*Proof of theorem 9.* We are going to use theorem 10 which gives conditions, on group of three strategies, that implies that one of the strategies is an attractor but has an arbitrary small basin of attraction. More precisely, for $s$ not to have a uniform large basin of attraction, it has to be shown that there exists $C_0 > 0$ such that for any $\varepsilon > 0$, there is $s^*$ and $s'$ satisfying: $0 < U(s,s) - U(s^*,s) = U(s,s^*) - U(s^*,s^*) < \varepsilon$; $0 < U(s,s) - U(s',s) = U(s,s') - U(s',s') < \varepsilon$; $U(s^*,s') - U(s',s') > C_0$; $U(s',s^*) - U(s^*,s^*) > C_0$. Observe that under that above conditions, it follows from theorem 10 and remark 2 that once we identify $s$ with the vertex $e_1$, the point $(\frac{\varepsilon}{C_0+2\varepsilon}, \frac{\varepsilon}{C_0+2\varepsilon})$ is not in the basin of $e_1$.

Given $h_t$ such that for any $h_k$ that contains either $h_t$ or $\hat{h}_t$ follows that $U(s, s/h_k) < R_1$ for some $R_1 < R$, we can also take $h_t$ such that for the deviation $h'_{t+1}$ from $h_t$ (i.e, the first coordinate of $h^{t+1}$ is different to $s(h_t)$ but the second one is equal to $s(\hat{h}_t)$) follows that $U(s, s/h_t) - U(s, s/h_{t+1})$ is smaller than $\varepsilon$ with $\varepsilon$ chosen arbitrary small, provided $\delta$ large and $1-p$ small. Moreover, the election can be done in such a way that the same holds for for $\hat{h}_t$.

Now we build two strategies $s^*, s'$, that upset $s$. The strategy $s^*$ deviate respect to $s$ at $h_t$ but coincide with $s$ on $\hat{h}_t$. On the other hand, the strategy $s'$ deviate respect to $s$ at $\hat{h}_t$ but coincide with $s$ on $h_t$. Both strategies coincide with $s$ after the first deviation with $s$. In other words: $s^*(h_t) \neq s(h_t)$ and $s^*(\hat{h}_t) = s(\hat{h}_t)$; $s'(h_t) = s(h_t)$ and $s'(\hat{h}_t) \neq s(\hat{h}_t)$; $h_{s^*,s/h_t(s^*(h_t),s(\hat{h}_t))} = h_{s,s/h_t(s^*(h_t),s(\hat{h}_t))}$ and $h_{s^*,s/\hat{h}_t(s^*(\hat{h}_t),s(h_t))} = h_{s,s/\hat{h}_t(s^*(\hat{h}_t),s(h_t))}$; $h_{s',s/h_t(s'(h_t),s(\hat{h}_t))} = h_{s,s/h_t(s'(h_t),s(\hat{h}_t))}$ and $h_{s',s/\hat{h}_t(s'(\hat{h}_t),s(h_t))} = h_{s,s/\hat{h}_t(s'(\hat{h}_t),s(h_t))}$.

Observe that from that properties follows:

$U(s, s^*/h_t(s(h_t), s^*(\hat{h}_t))) = U(s, s/h_t)$;

$U(s^*, s^*/h_t(s^*(h_t), s^*(\hat{h}_t))) = U(s, s/h_t(s^*(h_t), s(\hat{h}_t)))$;

$U(s^*, s/\hat{h}_t) = U(s, s/\hat{h}_t)$, $U(s, s^*/\hat{h}_t) = U(s^*, s^*/\hat{h}_t)$;

$U(s, s'/\hat{h}_t(s(\hat{h}_t), s'(h_t))) = U(s, s/\hat{h}_t)$;

$U(s', s'/\hat{h}_t(s'(\hat{h}_t), s'(h_t))) = U(s, s/h_t(s(\hat{h}_t), s(h_t)));$

$U(s', s/h_t) = U(s, s/h_t)$, $U(s, s'/h_t) = U(s', s'/h_t)$.

Therefore it follows that:

$U(s, s/h_t) - U(s^*, s/h_t) < \varepsilon$ and $U(s, s/\hat{h}_t) - U(s^*, s/\hat{h}_t) = 0;$

$U(s, s/h_t) - U(s', s/\hat{h}_t) < \varepsilon$ and $U(s, s/h_t) - U(s', s/h_t) = 0;$

$U(s^*, s^*/h_t) - U(s, s^*/h_t) = -[U(s, s/h_t) - U(s^*, s/h_t)]$ and $U(s^*, s^*/\hat{h}_t) = U(s, s^*/\hat{h}_t);$

$U(s', s'/\hat{h}_t) - U(s, s'/\hat{h}_t) = -[U(s, s/\hat{h}_t) - U(s', s/\hat{h}_t)]$ and $U(s, s/h_t) = U(s, s/h_t)$.

Now we have to compare $s'$ and $s^*$. Observe that:

$h_{s's^*/h_t(s'(h_t), s^*(\hat{h}_t))} = h_{ss/h_t(s(h_t), s(\hat{h}_t))}$ so $U(s', s^*/h_t)$ is close to $U(s^*, s^*/h_t)$,

$h_{ss'/\hat{h}_t(s^*(h_t), s'(h_t))} = h_{ss/\hat{h}_t(s(\hat{h}_t), s(h_t))}$ so $U(s^*, s'/\hat{h}_t)$ is close to $U(s', s/\hat{h}_t)$.

Since $s^*$ and $s'$ deviate from $s$ at $h_t$ and $\hat{h}_t$ respectively and $h_t(s^*(h_t), s'(\hat{h}_t))$ is not one of the paths previously listed, we can assume that $U(s^*s'/h_t(s^*(h_t), s'(\hat{h}_t)) = R$. In the same way, $U(s's^*/\hat{h}_t(s^*(\hat{h}_t), s'(h_t)) = R$. Therefore, and from the assumption that $U(s, s/h_k) < R_1$ follows that $U(s's^*/h_t) + U(s's^*/\hat{h}_t) - [U(s^*s^*/h_t) + U(s^*s^*/\hat{h}_t)] > R - R_1 - \varepsilon$ and $U(s^*s'/h_t) + U(s^*s'/\hat{h}_t) - [U(s's'/h_t) + U(s's'/\hat{h}_t)] > R - R_1 - \varepsilon$ and therefore the theorem is proved. □

# References


[AR] Abreu, D. and A. Rubinstein (1988). *The Structure of Nash Equilibrium in Repeated Games with Finite Automata.* Econometrica, 56(6): 1259-81.

[Ax] Axelrod, R. (1984). *The Evolution of Cooperation.* Basic Books.

[BiS] Binmore, K.G. and L. Samuelson (1992). *Evolutionary Stability in Repeated Games Played by Finite Automata.* Journal of Economic Theory, 57(2): 278-305.

[BeS] Bendor, J. and P. Swistak (1997). *The Evolutionary Stability of Cooperation.* The American Political Science Review, 91(2): 290-307

[B] Boyd, R. (1989). *Mistakes allow evolutionary stability in the repeated prisoner's dilemma game*, Journal of Theoretical Biology 136(1): 47-56.

[BL] Boyd, R. and J.P. Lorberbaum (1987). *No Pure Strategy Is Evolutionarily Stable in the Repeated Prisoner's Dilemma Game.* Nature, 327: 58-59.

[C] Cooper, D.J. (1996). *Supergames Played by Finite Automata with Finite Costs of Complexity in an Evolutionary Setting.* Journal of Economic Theory, 68(1): 266-275.

[DBF] Dal Bó, P. and G.R. Fréchette (2011). *The Evolution of Cooperation in Infinitely Repeated Games: Experimental Evidence.* American Economic Review 101(1).

[DBF2] Dal Bó, P. and G.R. Fréchette (2011). *Strategy Choice In The Infinitely Repeated Prisoners Dilemma.* Unpublished.

[FM] Fudenberg, D. and E. Maskin (1990). *Evolution of cooperation in noisy repeated game*, The American Economic Review, 90(2): 274-279.

[FM2] Fudenberg, D. and E. Maskin (1993). *Evolution and Repeated Games.* Unpublished.

[FRD] Fudenberg, D., D.G. Rand, and A. Dreber (2012). *Slow to Anger and Fast to Forget: Cooperation in an Uncertain World.* American Economic Review, 102(2): 720-749.

[FT] Fudenberg, D. and J. Tirole (1991). *Game Theory.* Cambridge: MIT Press.

[IFN] Imhof, L., D. Fudenberg, M. Nowak (2007). *Tit-for-tat or win-stay, lose-shift?* Journal of Theoretical Biology, 247: 574-580.

[JLP] Johnson, P., D.K. Levine, and W. Pesendorfer (2001). *Evolution and Information in a Gift-Giving Game.* Journal of Economic Theory, 100(1): 1-21.

[KMR] Kandori, M., G.J. Mailath, and R. Rob (1993). *Learning, Mutation, and Long Run Equilibria in Games.* Econometrica, 61(1): 29-56.

[Ki] Kim, Y. (1994). *Evolutionarily Stable Strategies in the Repeated Prisoner Dilemma.* Mathematical Social Sciences, 28(3): 167-97.

[LP] Levine, D.K. and W. Pesendorfer (2007). *The Evolution of Cooperation through Imitation.* Games and Economic Behavior, 58(2): 293-315.

[M] Myerson, R.B. (1991). *Game Theory: Analysis of Conflict.* Cambridge, MA: Harvard University Press.

[NS] Nowak, M. and K. Sigmund (1993). *A strategy of win-stay, lose-shift that outperforms tit-for-tat in the Prisoner's Dilemma game.* Nature, 364: 56-58.

[R] Rubinstein, A. (1986). *Finite Automata Play the Repeated Prisoner's Dilemma.* Journal of Economic Theory, 39(1): 83-96.

[V] Volij, O. (2002). *In Defense of DEFECT.* Games and Economic Behavior, 39(2): 309-21.

[YP] Young, H.P. (1993). *The Evolution of Conventions.* Econometrica, 61(1): 57-84.

# 7 Appendix A. Perturbed Replicator Dynamics

We consider more general equations than the replicator dynamics with the restrictions that *individuals with low scores die off and the ones with high ones flourish.* More precisely, given a payoff matrix $A$ we consider equations defined in the usual $n-$dimensional simplex $\sum$, of the form $\dot{x}_i = x_i G_i(x)$ such that $G_i(x) > 0$ if and only if$(Ax)_i - x^t Ax > 0$ and $G_i(x) < 0$ if and only if $(Ax)_i - x^t Ax < 0$. In this case, it follows that $G_i(x) = [(Ax)_i - x^t Ax] H_i(x)$ where $H_i : \sum \to \mathbb{R}$. Moreover, form previous assumption it holds that $H_i$ is always positive in the simplex $\sum$. We require a slightly strong condition: $C^+ = \max\{H_i(x), x \in \sum, i = 1 \ldots m\} < +\infty$, and $C^- = \min\{H_i(x), x \in \sum, i = 1 \ldots m\} > 0$, therefore $0 < C^- \leqslant H_i < C^+$. The goal is to show that a version of theorem 1 can be obtained in the present case. More precisely, provided the hypothesis of theorem 1 and assuming equations as above, it is shown that $\Delta_{\frac{1}{M_0}} \cap \Delta_{\frac{C^-}{2C^+}}$ is contained in the local basin of attraction of $e_1$. The proof, goes through the same strategy: we shows that for any $k \leqslant \min\{\frac{1}{M_0}, \frac{C^-}{2C^+}\}$, re-writing the equations in affine coordinates follows that $\sum_{i \geqslant 2} x_i G_i = \sum_{i \geqslant 2} x_i F_i H_i < 0$ where $F_i$ is $(Ax)_i - x^t Ax$ in affine coordinates. From the inequalities $0 < C^- \leqslant H_i < C^+$, it follows that $x_i F_i(x) H_i(x) < C^+ x_i F_i(x)$ if $F_i(x) > 0$ and $x_i F_i(x) H_i(x) < C^- x_i F_i(x)$ if $F_i(x) < 0$. Recalling that $F_j(x) = (f_j - f_1)(x) + R(x)$ with $R(x) = \sum_l (f_1 - f_l)(x) x_l$ (the variable $x$ is already assumed in affine coordinates) follows that

$$
\begin{aligned}
\sum_i x_i F_i(x) H_i(x) &\leqslant \sum_{\{i: F_i(x) > 0\}} C^+ x_i F_i(x) + \sum_{\{i: F_i(x) < 0\}} C^- x_i F_i(x) \\
&= \sum_{\{i: F_i(x) > 0\}} x_i C^+ (f_i - f_1)(x) + \sum_{\{i: F_i(x) < 0\}} x_i C^- (f_i - f_1)(x) \\
&+ R(x) [\sum_{\{i: F_i(x) > 0\}} C^+ x_i + \sum_{\{i: F_i(x) < 0\}} C^- x_i].
\end{aligned}
$$

If $x \in \Delta_k$ with $k < \frac{C^-}{2C^+}$ it follows that $\sum_{\{i:F_i(x)>0\}} C^+x_i + \sum_{\{i:F_i(x)<0\}} C^-x_i] \leqslant \frac{C^-}{2}$ and recalling the definition of $R_0$ follows that

$$\sum_{\{i:F_i>0\}} x_iC^+(f_i - f_1)(x) + \sum_{\{i:F_i<0\}} x_iC^-(f_i - f_1) + R(x)[\sum C^+x_i + \sum C^-x_i]$$
$$\leqslant \sum_{\{i:F_i>0\}} x_i\hat{C}^+(f_i - f_1)(x) + \sum_{\{i:F_i<0\}} x_i\hat{C}^-(f_i - f_1)$$

where $\hat{C}^+ = C^+ - \frac{C^-}{2}, \hat{C}^- = \frac{C^-}{2}$. Therefore, rewriting the equation as it was done in the proof of theorem 1 to finish we have to prove that $N(cx) + x^tM(cx) < 0$ where $cx = (c_1x_1, c_2x_2, \ldots, c_nx_n)$ and $c_i$ is either $\hat{C}^+$ or $\hat{C}^-$ and $N, M$ are the vector and matrix induce by $A$ and so. To prove above inequality, we need a more general version of lemma 1. The proofs are similar.

**Lemma 16.** *Let $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$ such that each coordinate is positive. Let $Q_c : \mathbb{R}^m \to \mathbb{R}$ given by $Q(x) = N(cx) + x^tM(cx)$ with $x \in \mathbb{R}^m$, $N \in \mathbb{R}^m$, $M \in \mathbb{R}^{m\times m}$ and $cx := (c_1x_1, \ldots, c_mx_m)$. Let us assume that $N_i < 0$ for any $i$ and for any $j > i$, $|N_i| \geqslant |N_j|$. Let $M_0 = \max_{i,\, j>i}\{\frac{M_{ij}+M_{ji}}{-N_i},\ 0\}$. Then, the set $\Delta_{\frac{1}{M_0}} = \{x \in \mathbb{R}^m : x_i \geqslant 0, \sum_{i=1}^m x_i < \frac{1}{M_0}\}$, is contained in $\{x : Q_c(x) < 0\}$. In particular, if $M_0 = 0$ then $\frac{1}{M_0}$ is treated as $\infty$ and this means that $\{x \in \mathbb{R}^m : x_i \geqslant 0\} \subset \{x : Q_c(x) \leqslant 0\}$.*

# 8 Appendix B. Generalized $w$ for any payoff system

Recall that $w$ has uniformly large basin, provided that $2R > S+T$. Now, we consider $w-$type strategies that have a uniformly large basin for any payoff system.

**Definition 12. n-win-stay-lose-shift** *$n-$win-stay lose-shift. If it gets either $T$ or $R$ stays; if it gets $S$, shifts to $D$ and stays for $n-$period and then acts as $w$. We denote it with $w^n$.*

**Theorem 12.** *For any payoff set there exists $n$ such that $w^n$ is has a uniformly large basin.*

*Proof.* The proof follows the same steps that we used to prove that $w$ has a uniformly large basin of attraction when $2R-(T+P)>0$ but using the fact that for any payoff matrix there exists $n$ such that $nR > T+(n-1)P$.

To show that $w^n$ has a uniformly large basin of attraction, we calculate the quantities $b_1, b_2, b_3, b_4$ for $u(s, w^n)$ as it was done for $w$ in subsection 6.6.1. In addition, observe that for $w^n$ it follows that $b_2 + b_4 \geqslant \delta p^2 \frac{1-(\delta p^2)^n}{1-\delta p^2} b_3$ and if $n$ is large enough then $\frac{1-(\delta p^2)^n}{1-\delta p^2} > n-1$ and therefore, $b_2 + b_4 \geqslant (n-1)b_3$. Repeating the same calculation done for $w$, in case $w^n(h_k) = C, s(h_k) = D$ follows that $U(w^n, w^n) - U(s, w^n) \geqslant (n-1)b_3(R-P) + b_3(R-T) \geqslant (1-\delta p^2)[nR - T - (n-1)P]$. In case $w^n(h_k) = D, s(h_k) = C$, the calculation is similar.

To bound uniformly the quantities (17) and (18) for $w^n$, we proceed in a same way that was done for $w$ and it is only changed the upper bound $2R-(T+S)$ by $nR - T - (n-1)P$. □

**Examples of strategies with low frequency of cooperation which have large basin but they do not have uniformly large basin**

In what follows, we give examples of strategies with arbitrary low frequency of cooperation which have large basins (with size depending on $\delta$ and $p$), however, those strategies do not have uniformly large basin of attraction. In other words, the lower bounds of their basin shrinks to zero when $\delta, p \to 0$. More precisely, they can not have uniformly large basin due to theorem 7. Those strategies are built combining $w$ with $a$. Moreover, we establish some relation between the frequency of cooperation and the lower bounds of the size of their local basin (but depending on $\delta$ and $p$).

**Definition 13.** *We take $n$ large and $b_0 < 1$, we define the strategy $aw^{n,b_0}$ as the* *strategy that in blocks of times $I_w^l = [l(n + m_0 n), l(n + m_0 n) + n - 1]$ behaves as $w$ and in the blocks of times $I_a^l = [l(n + m_0 n) + n, (l + 1)(n + m_0 n) - 1]$ behaves as $a$, where $m_0$ denotes the integer part of $\frac{1}{b_0}$ and $l$ is a non-negative integer.*

**Theorem 13.** *For any $n$ large, and any positive $b_0$ the strategy $aw^{n,b_0}$ has a large basin of attraction, but not a uniformly large basin of attraction.*

*Proof.* From now on, and to avoid notation, we denote $aw^{n,d_0}$ with $aw$. First we are going to prove that $aw$ is a strict sub game perfect.

*The strategy $aw$ is a uniform strict subgame perfect:* The proof is similar to the one performed for $w$. Let $s$ be another strategy and given a path $h$ let $k$ be the first deviation ($s(h_h) \neq aw(h_k)$). Either $k \in I_w^l$ or $k \in I_a^l]$ for some non-negative $l$. It follows that $U_{\delta,p}(h_{aw,aw}/h_k) = b_0 R + (1 - b_0)P$ where

$$b_0 = \frac{1 - p^2\delta}{p^2} \sum_{j \geqslant 0 : u^j(aw,aw/h_k) = R} = \frac{1 - p^2\delta}{p^2} \sum_{j \geqslant 0, I_w^l} . \tag{20}$$

Observe that provided $\delta$ large, then $b_0$ is close to $d_0$. Now we take $s$ and assuming that it differs in $h_k$ and $aw(h_k) = R, s(h_k) = D$. In what follows, to avoid notation, with $U(.,.)$ we denote $U_{\delta,p,h_{.,.}}(.,./h_k)$. Following that, we take

$$b_1 = \frac{1 - p^2\delta}{p^2} \sum_{j : u^j(s,aw/h_k) = R} p^{2j+2}\delta^j, \quad b_2 = \frac{1 - p^2\delta}{p^2} \sum_{j : u^j(s,aw/h_k) = S} p^{2j+2}\delta^j,$$

$$b_3 = \frac{1 - p^2\delta}{p^2} \sum_{j : u^j(s,aw/h_k) = T} p^{2j+2}\delta^j, \quad b_4 = \frac{1 - p^2\delta}{p^2} \sum_{j : u^j(s,aw/h_k) = P} p^{2j+2}\delta^j.$$

Observe that $b_1 + b_2 + b_3 + b_4 = 1$ and $U(s, w) = b_1 R + b_2 S + b_3 T + b_4 P$. Moreover, since in blocks $I_a^l$ $aw$ behaves as $a$ then

$$b_4 \geqslant 1 - b_0 \tag{21}$$

From the property that $aw$ behaves as $w$ in blocks of the form $[l(n+m_0n),(l+1)(n+m_0n)+n]$, for each $T$ that $s$ can get on those blocks ($s$ plays $D$ and $w$ plays $C$) follows that in the next move $s$ may get either $S$ or $P$ because $w$ plays $D$, so, noting

$$b_4^w = \frac{1-p^2\delta}{p^2} \sum_{j\in I_w^l : u^j(s,w/h_k)=P} p^{2j+2}\delta^j$$

then

$$b_4 \geqslant 1 - b_0 + b_u^w \tag{22}$$

$$b_2 + b_4^w \geqslant p^2\delta b_3. \tag{23}$$

Writing $U(aw,aw) = b_0R+(1-b_0)P = [b_0-(1-b_4)]R+b_1R+b_2R+b_3R+(1-b_0)R$ by inequalities (21, 22, 23) it follows that

$$\begin{aligned} U(aw,aw)-U(s,aw) &= [b_0-(1-b_4)]R+b_2(R-S)+b_3(R-T) \\ &\quad +(1-b_0-b_4)(R-P) \\ &\geqslant (b_0+b_4-1+b_2)(R-P)+b_3(R-T) \\ &\geqslant (b_4^w+b_2)(R-P)+b_3(R-T) \\ &\geqslant \delta p^2 b_3(R-P)+b_3(R-T) \geqslant b_3[(1+p^2\delta)R-(T+P)]. \end{aligned}$$

Observing that if $s(h_k)=D, aw(h_k)=C$, then $b_3 \geqslant 1-p^2\delta$ and since $2R-(T+P)>0$ it follows that for $\delta$ and $p$ large (meaning that they are close to one), then $[(1+p^2\delta)R-(T+P)] > C_0$ for a positive constant smaller than $2R-(T+P)$ and therefore (provided that $\delta$ and $p$ large are large) follows that $U(aw,aw)-U(s,aw) > (1-p^2\delta)C_0$,