Rationalizable Implementation of Correspondences*

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Abstract

We come close to characterizing the class of social choice correspondences that are implementable in rationalizable strategies. We identify a new condition, which we call set-monotonicity, and show that it is necessary and almost sufficient for rationalizable implementation. Set-monotonicity is much weaker than Maskin monotonicity, which is the key condition for Nash implementation and which also had been shown to be necessary for rationalizable implementation of social choice functions. Set-monotonicity reduces to Maskin monotonicity in the case of functions. We conclude that the conditions for rationalizable implementation are not only starkly different from, but also much weaker than those for Nash implementation, when we consider social choice correspondences.

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1 Introduction

The design of institutions to be used by rational agents has been an important research agenda in economic theory. As captured by the notion of Nash equilibrium, rationality is encapsulated in two aspects: these are (i) the best responses of agents to their beliefs, and (ii) that those beliefs are correct, the so-called rational

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expectations assumption. One can drop the latter and retain the former, moving then into the realm of rationalizability. One would conjecture that the design of institutions under rationalizable behavior, i.e., without insisting on rational expectations, should leave room for significantly different results than the theory based on equilibrium.[1] Settling this important question is our task in this paper.

The theory of Nash implementation has uncovered the conditions under which one can design a mechanism (or game form) such that the set of its Nash equilibrium outcomes coincides with a given *social choice correspondence* (henceforth, SCC). Indeed, Maskin (1999) proposes a well-known monotonicity condition, which we refer to as *Maskin monotonicity*. Maskin's (1999) main result shows that Maskin monotonicity is necessary and almost sufficient for Nash implementation.

Nash implementation is concerned with complete information environments, in which all agents know the underlying state and this fact is commonly certain among them. As a foundation of Nash equilibrium, Aumann and Brandenburger (1995) delineate the set of epistemic conditions under which the agents' strategic interaction always leads to a Nash equilibrium. Furthermore, Polak (1999) shows that when the agents' payoffs are common knowledge, as complete information environments prescribe, the Aumann-Brandenburger epistemic conditions imply common knowledge of rationality.

Bernheim (1984) and Pearce (1984) independently propose rationalizability, a weaker solution concept than Nash equilibrium, by asking what are the strategic implications that come solely from common knowledge of rationality. Brandenburger and Dekel (1987) allow for the agents' beliefs to be correlated and propose an even weaker version of rationalizability. Throughout the current paper, our discussion is entirely based upon Brandenburger and Dekel's version of rationalizability. In this case, the set of all rationalizable strategies is fully characterized in terms of the strategies that survive the iterative deletion of never best responses.

In a paper that was our starting point and motivation, Bergemann, Morris, and Tercieux (2011) –BMT in the sequel– recently consider the implementation of *social choice functions* (henceforth, SCFs) under complete information in rationalizable strategies. By an SCF we mean a single-valued SCC. They show that Maskin monotonicity is necessary and almost sufficient for rationalizable implementation. This essentially would imply that rationalizable implementation is similar to Nash implementation. However, their result has one important caveat: BMT focus only on SCFs in their analysis (we note that rationalizability and single-valuedness amount to uniqueness of Nash equilibrium). In any attempt to extend their result, one should ponder the following observations: (1) Maskin's characterization on

[1] On the one hand, from the existence point of view, since rationalizability is a weaker solution concept, one would conjecture a more permissive theory. On the other hand, uniqueness would be harder to establish. Hence, the answer, a priori, is far from clear.

Nash implementation holds true regardless of whether we consider SCFs or SCCs; (2) Maskin monotonicity can be quite restrictive in the case of SCFs (see, e.g., Mueller and Satterthwaite (1977) and Saijo (1987)); and (3) Many interesting SCCs are Maskin monotonic, including the Pareto, Core, envy-free, constrained Walrasian or Lindhal correspondences, while any SCF selected from a Maskin monotonic SCC no longer inherits the property.

Therefore, what we set out to resolve here is the question of how close rationalizable implementation really is to Nash implementation. In this endeavor, dealing with correspondences is the main objective of this paper. We identify a new condition, which we call *set-monotonicity*.[2] We show that set-monotonicity is necessary (Theorem 1) and almost sufficient (Theorems 2 and 3) for rationalizable implementation of SCCs.[3] Our set-monotonicity requires the lower contour sets to be nested across states "uniformly" over all outcomes in the range of the SCC. This setwise definition of monotonicity exhibits a clear contrast with Maskin monotonicity, which is a "pointwise" condition, in the sense that it requires the nestedness of the lower contour sets across states at any fixed outcome in the range of the SCC. Set-monotonicity is logically weaker than Maskin monotonicity, and it is likely to be much weaker if the SCC contains many values in its range. However, they become equivalent in the case of SCFs. We also construct an example in which an SCC is rationalizably implementable by a finite mechanism, while it violates Maskin monotonicity at almost any outcome in the range of the SCC. In this sense, the SCC in the example is "very far from" being Nash implementable. Of course, as expected from our necessity result, we confirm that set-monotonicity is satisfied for this SCC (Lemma 1).

We conclude that rationalizable implementation is generally quite different from Nash implementation, and their aledged resemblance in BMT arose as the artifact of the assumption that only SCFs were being considered. In addition, our rationalizable implementation results are significantly more permissive than the Nash implementation counterparts in the sense that set-monotonicity is much weaker than Maskin monotonicity. In particular, we do not require the existence of Nash equilibrium in the mechanism, unlike BMT, who need to establish the existence of an equilibrium leading to the realization of the SCF in their canonical mechanism.

The rest of the paper is organized as follows. In Section 2, we introduce the general notation for the paper. Section 3 introduces rationalizability as our solution concept and defines the concept of rationalizable implementation. In Section

[2]A weaker version of this condition, based on the strict lower contour sets, first surfaced in Cabrales and Serrano (2011) under the name *weak quasimonotonicity;* see also its corrigendum, posted at http://www.econ.brown.edu/faculty/serrano/pdfs/2011GEB73-corrigendum.pdf.

[3]The term *set-monotonicity* is already used by Mezzetti and Renou (2012) for a different property. We apologize for the potential confusion, but believe that set-monotonicity is the appropriate term for our notion, as explained shortly.

4, we propose set-monotonicity and show it to be necessary for rationalizable implementation. Section 5 illustrates by an example the conditions for rationalizable implementation and Nash implementation. In Sections 6 and 7, we delineate the sufficient conditions for rationalizable implementation. Section 8 concludes.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ denote the finite set of agents and Θ be the finite set of states. It is assumed that the underlying state $\theta \in \Theta$ is common knowledge among the agents. Let A denote the set of social alternatives, which are assumed to be independent of the information state. Let $\mathcal{A}$ be a σ-algebra on A and $\triangle$ denote the set of probability measures on $(A, \mathcal{A})$. We shall assume that $\mathcal{A}$ contains all singleton sets. For ease in the presentation, we shall assume that A is finite, and denote by $\Delta(A)$ the set of probability distributions over A. Agent i's state dependent von Neumann-Morgenstern utility function is denoted $u_i : \Delta(A) \times \Theta \to \mathbb{R}$.[4] We can now define an *environment* as $\mathcal{E} = (A, \Theta, \{u_i\}_{i \in N})$, which is implicitly understood to be common knowledge among the agents.

A (stochastic) *social choice correspondence* $F : \Theta \rightrightarrows \Delta(A)$ is a mapping from Θ to a nonempty compact subset of $\Delta(A)$. The mapping F is called a *social choice function* if it is single-valued. In this case, we denote it by $f : \Theta \to \Delta(A)$. We use the acronimes SCC and SCF for both objects, respectively.

A *mechanism* (or *game form*) $\Gamma = ((M_i)_{i \in N}, g)$ describes a nonempty countable message space M_i for each agent $i \in N$ and an outcome function $g : M \to \Delta(A)$ where $M = M_1 \times \cdots \times M_n$.

3 Implementation in Rationalizable Strategies

We adopt *correlated rationalizability* as a solution concept and investigate the implications of implementation in *rationalizable* strategies. This is the version of rationalizability used by Brandenburger and Dekel (1987). We fix a mechanism $\Gamma = (M, g)$ and define a message correspondence profile $S = (S_1, \ldots, S_n)$, where each $S_i \in 2^{M_i}$, and we write $\mathcal{S}$ for the collection of message correspondence profiles. The collection $\mathcal{S}$ is a lattice with the natural ordering of set inclusion: $S \leq S'$ if $S_i \subseteq S_i'$ for all $i \in N$. The largest element is $\bar{S} = (M_1, \ldots, M_n)$. The smallest element is $\underline{S} = (\emptyset, \ldots, \emptyset)$.

We define an operator $b^\theta : \mathcal{S} \to \mathcal{S}$ to iteratively eliminate never best responses

[4] With finite environments, expected utility takes the form of sums, not integrals. This would also hold in a separable space, due to its countable dense subset.

with $b^\theta = (b_1^\theta, \ldots, b_n^\theta)$ and b_i^θ is now defined as:

$$b_i^\theta(S) \equiv \left\{ m_i \in M_i \;\middle|\; \begin{array}{l} \exists \lambda_i \in \Delta(M_{-i}) \text{ such that} \\ (1)\lambda_i(m_{-i}) > 0 \Rightarrow m_j \in S_j \; \forall j \neq i; \\ (2) m_i \in \arg\max_{m_i'} \sum_{m_{-i}} \lambda_i(m_{-i}) u_i(g(m_i', m_{-i}); \theta) \end{array} \right\}$$

Observe that b^θ is increasing by definition: i.e., $S \leq S' \Rightarrow b^\theta(S) \leq b^\theta(S')$. By Tarski's fixed point theorem, there is a largest fixed point of b^θ, which we label $S^{\Gamma(\theta)}$. Thus, (i) $b^\theta(S^{\Gamma(\theta)}) = S^{\Gamma(\theta)}$ and (ii) $b^\theta(S) = S \Rightarrow S \leq S^{\Gamma(\theta)}$. We can also construct the fixed point $S^{\Gamma(\theta)}$ by starting with $\bar{S}$ – the largest element of the lattice – and iteratively applying the operator b^θ. If the message sets are finite, we have

$$S_i^{\Gamma(T)}(\theta) \equiv \bigcap_{k \geq 1} b_i^\theta \left(\left[b^\theta \right]^k (\bar{S}) \right)$$

In this case, the solution coincides with iterated deletion of strictly dominated strategies. But because the mechanism Γ may be infinite, transfinite induction may be necessary to reach the fixed point. It is useful to define

$$S_{i,k}^{\Gamma(\theta)} \equiv b_i^\theta \left(\left[b^\theta \right]^{k-1} (\bar{S}) \right),$$

using transfinite induction if necessary. Thus $S_i^{\Gamma(\theta)}$ is the set of messages surviving (transfinite) iterated deletion of never best responses of agent i.

Next, we provide the definition of *weak rationalizable implementation.*

Definition 1 (Weak Rationalizable Implementation) *An SCC F is **weakly implementable in rationalizable strategies** if there exists a mechanism $\Gamma = (M, g)$ such that for each $\theta \in \Theta$, the following two conditions hold: (1) $S^{\Gamma(\theta)} \neq \emptyset$; and (2) for each $m \in M$, $m \in S^{\Gamma(\theta)} \Rightarrow g(m) \in F(\theta)$.*

We strengthen the requirement of weak implementation into the following:

Definition 2 (Full Rationalizable Implementation) *An SCC F is **fully implementable in rationalizable strategies** if there exists a mechanism $\Gamma = (M, g)$ such that for each $\theta \in \Theta$,*

$$\bigcup_{m \in S^{\Gamma(\theta)}} g(m) = F(\theta).$$

We observe that, when we consider only SCFs, these two concepts of implementation become equivalent.

4 Necessary Conditions for Implementation in Rationalizable Strategies

In complete information environments, Maskin (1999) proposes a monotonicity condition for *Nash implementation* where the set of Nash equilibrium outcomes is required to coincide with the SCC. This condition is often called *Maskin monotonicity.*

Definition 3 *An SCC F satisfies* ***Maskin monotonicity*** *if, for any states $\theta, \theta' \in \Theta$ and any $a \in F(\theta)$, whenever*

$$u_i(a, \theta) \geq u_i(z, \theta) \Rightarrow u_i(a, \theta') \geq u_i(z, \theta') \ \ \forall i \in N, \ \forall z \in \Delta(A),$$

then $a \in F(\theta')$.

Let D denote a subset of $\Delta(A)$ with a generic element d being a lottery over A. We denote the convex hull of D by

$$\text{co}(D) = \left\{ \{\alpha_d\}_{d \in D} \middle| \ \alpha_d \geq 0 \ \forall d \in D \ \text{ and } \sum_{d \in D} \alpha_d = 1 \right\}.$$

Definition 4 *An SCC F satisfies* ***weak set-monotonicity*** *if, for every pair of states $\theta, \theta' \in \Theta$, whenever*

$$u_i(a; \theta) \geq u_i(z; \theta) \Rightarrow u_i(a; \theta') \geq u_i(z; \theta') \quad \forall a \in co(F(\theta)), \forall i \in N, \ \forall z \in \Delta(A),$$

then, $F(\theta) \subseteq F(\theta')$.

Remark: When we consider SCFs, co$(F(\theta))$ becomes a singleton set. Therefore, in this case, the condition just defined reduces to Maskin monotonicity.

We slightly strengthen weak set-monotonicity into the following:

Definition 5 *An SCC F satisfies* ***set-monotonicity*** *if, for every pair of states $\theta, \theta' \in \Theta$, whenever*

$$u_i(a; \theta) \geq u_i(z; \theta) \Rightarrow u_i(a; \theta') \geq u_i(z; \theta') \quad \forall a \in F(\theta), \forall i \in N, \ \forall z \in \Delta(A),$$

then, $F(\theta) \subseteq F(\theta')$.

Remark: Note how, under expected utility, both conditions amount to the same thing, as requiring the nestedness of the lower contour sets over all $a \in F(\theta)$ or their convex hull is equivalent. However, it will be convenient to use the weak version for the proof of the necessity result, and the strong version for the proof of sufficiency.

Theorem 1 *If an SCC F is (weakly or fully) implementable in rationalizable strategies, it satisfies weak set-monotonicity.*

Proof: Suppose F is weakly implementable in rationalizable strategies by a mechanism $\Gamma = (M, g)$. Fix two states $\theta, \theta' \in \Theta$ satisfying the following property:

$$u_i(a;\theta) \geq u_i(z;\theta) \Rightarrow u_i(a;\theta') \geq u_i(z;\theta') \ \forall a \in \text{co}(F(\theta)), \ \forall i \in N, \quad (*)$$

Then, due to the hypothesis that F is implementable by Γ, we fix $m^* \in S^{\Gamma(\theta)}$, and we have that $g(m^*) \in F(\theta)$.

Fix $i \in N$. Since $m_i^* \in S_i^{\Gamma(\theta)}$, there exists $\lambda_i^{m_i^*,\theta} \in \Delta(M_{-i})$ satisfying the following two properties: (i) $\lambda_i^{m_i^*,\theta}(m_{-i}) > 0 \Rightarrow m_{-i} \in S_{-i}^{\Gamma(\theta)}$ and $g(m_i^*, m_{-i}) \in F(\theta)$; and (ii) $\sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i});\theta) \geq \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i', m_{-i});\theta)$ for each $m_i' \in M_i$.

We focus on the best response property of m_i^* summarized by inequality (ii). Fix $m_i' \in M_i$. Due to the construction of $\lambda_i^{m_i^*,\theta}$, we have that

$$\begin{aligned} \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i});\theta) &\geq \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i', m_{-i});\theta) \\ u_i(a;\theta) &\geq u_i(z^a;\theta), \end{aligned}$$

where the two lotteries a and z^a are defined as

$$a = \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) g(m_i^*, m_{-i}) \ \text{ and } \ z^a = \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) g(m_i', m_{-i}).$$

Since $g(m_i^*, m_{-i}) \in F(\theta)$ for each m_{-i} with $\lambda_i^{m_i^*,\theta}(m_{-i}) > 0$, we have $a \in \text{co}(F(\theta))$. Using Property $(*)$, we also obtain

$$u_i(a;\theta') \geq u_i(z^a;\theta').$$

Due to the choice of a and z^a and the hypothesis that $u_i(\cdot)$ is a von Neumann-Morgenstern expected utility, we obtain the following:

$$\sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i});\theta') \geq \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i', m_{-i});\theta').$$

Since this argument does not depend upon the choice of m_i', this shows that m_i^* is a best response to $\lambda_i^{m_i^*,\theta}$ in state θ' as well. Therefore, $m_i^* \in S_i^{\Gamma(\theta')}$. Since the choice of agent i is arbitrary, we can conclude that $m^* \in S^{\Gamma(\theta')}$. Furthermore, since

the choice of $m^* \in S^{\Gamma(\theta)}$ is also arbitrary, we have $S^{\Gamma(\theta)} \subseteq S^{\Gamma(\theta')}$. Finally, by weak implementability, this implies that

$$\bigcup_{m \in S^{\Gamma(\theta)}} g(m) \subseteq \bigcup_{m \in S^{\Gamma(\theta')}} g(m) \subseteq F(\theta').$$

Observing that full implementability is stronger than its weak counterpart, the proof is complete.■

5 An Example

In this section, we show by example that rationalizable implementation can be very different from Nash implementation. We consider the following example. There are two agents $N = \{1, 2\}$; two states $\Theta = \{\alpha, \beta\}$; and a finite number K of pure outcomes $A = \{a_1, a_2 \ldots, a_K\}$ where $K \geq 4$.[5] Assume that it is commonly certain that both agents know the state, i.e., it is a complete information environment. Agent 1's utility function is given as follows: for each $k = 1, \ldots, K$,

$$u_1(a_k, \alpha) = u_1(a_k, \beta) = \begin{cases} 1 + K\varepsilon & \text{if } k = K \\ 1 + (K - k)\varepsilon & \text{if } k \neq K \end{cases}$$

where $\varepsilon \in (0, 1)$. Hence, agent 1 has state-uniform preferences over A and a_K is the best outcome in both states; a_1 is the second best outcome in both states; ...; and a_{K-1} is the worst outcome in both states for agent 1.

Agent 2's utility function in state α is defined as follows: for each $k = 1, \ldots, K$,

$$u_2(a_k, \alpha) = \begin{cases} 1 + (K + 1)\varepsilon & \text{if } k = K \\ 1 + K\varepsilon & \text{if } k = 2 \\ 1 + k\varepsilon & \text{otherwise} \end{cases}$$

In state β, agent 2's utility function is defined as follows: for each $k = 1, \ldots, K$,

$$u_2(a_k, \beta) = \begin{cases} 1 + (K + 1)\varepsilon & \text{if } k = K \\ 1 & \text{if } k = 2 \\ 1 + k\varepsilon & \text{otherwise} \end{cases}$$

Note that a_K is the best outcome for agent 2 in both states; a_2 is her second best outcome in state α but it is her worst outcome in state β; and a_{K-1} is her third best outcome in state α and it is her second best outcome in state β.

We consider the following SCC F: $F(\alpha) = \{a_1, a_2, \ldots, a_K\}$ and $F(\beta) = \{a_K\}$.

[5] This example builds upon the one discussed in the Concluding Remarks section of BMT (2011).

Claim 1 *For every outcome $a_k \in A$ such that $a_k \neq a_2$ or $a_k \neq a_K$,*

$$u_i(a_k, \alpha) \geq u_i(y, \alpha) \Rightarrow u_i(a_k, \beta) \geq u_i(y, \beta) \ \ \forall i = \{1, 2\}, \ \forall y \in \Delta(A).$$

Proof: Since agent 1 has state-uniform preferences, this claim is trivially true for agent 1. Thus, in what follows, we focus on agent 2. Take any lottery in the lower contour set of $a_k \in A \setminus \{a_2\}$ in state α. If that lottery did not contain a_2 in its support, it is still in the lower contour set of a_k in state β as no utilities have changed, and if it did contain a_2 in its support, since the utility of a_2 has decreased, it will also be in the lower contour set at β. This completes the proof.■

Fix $a_k \in A \backslash \{a_2, a_K\}$ arbitrarily. If F were to satisfy Maskin monotonicity, we must have $a_k \in F(\beta)$, which is not the case. Therefore, we confirm the violation of Maskin monotonicity by the SCC F at every $a_k \in A \backslash \{a_2, a_K\}$. As is clear from the construction, we can choose K arbitrarily large. Therefore, the violation of Maskin monotonicity is severe, measured by the number of alternatives that should remain in the social choice in state β given the relevant nestedness of agents' preferences across the two states. In this sense, this correspondence is "very far" from being Maskin monotonic.

Nevertheless, we claim that the SCC F is implementable in rationalizable strategies using a finite mechanism. Consider the following mechanism $\Gamma = (M, g)$ where $M_i = \{m_i^1, m_i^2, \ldots, m_i^K\}$ for each $i = 1, 2$ and the deterministic outcome function $g(\cdot)$ is given in the table below:

$g(m)$		Agent 2						
		m_2^1	m_2^2	m_2^3	m_2^4	$\cdots$	m_2^{K-1}	m_2^K
Agent 1	m_1^1	a_1	a_1	a_{K-2}	a_{K-3}	$\cdots$	a_2	a_{K-1}
	m_1^2	a_2	a_1	a_1	a_{K-2}	$\cdots$	a_3	a_{K-1}
	m_1^3	a_3	a_2	a_1	a_1	$\cdots$	a_4	a_{K-1}
	m_1^4	a_4	a_3	a_2	a_1	$\cdots$	a_5	a_{K-1}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
	m_1^{K-1}	a_1	a_{K-2}	a_{K-3}	a_{K-4}	$\cdots$	a_1	a_{K-1}
	m_1^K	a_{K-1}	a_{K-1}	a_{K-1}	a_{K-1}	$\cdots$	a_{K-1}	a_K

Claim 2 *The SCC F is fully implementable in rationalizable strategies by the mechanism Γ.*

Proof: In state α, all messages can be best responses. Therefore, no message can be discarded via the iterative elimination of never best responses. That is,

the set of rationalizable message profiles $S^{\Gamma}(\alpha) = M$. This implies that the set of rationalizable outcomes in state α is $F(\alpha) = \{a_1, a_2, \ldots, a_K\}$.

In state β, message m_2^K strictly dominates all other messages, $m_2^1, \ldots, m_2^{K-1}$ for agent 2. On the other hand, all messages for agent 1 can be a best response. In the second round of elimination of never best responses, m_1^K strictly dominates all other messages $m_1^1, \ldots, m_1^{K-1}$ for agent 1. Thus, we have $S^{\Gamma(\beta)} = (m_1^K, m_2^K)$. This implies that we have $F(\beta) = a_K$ as the unique rationalizable outcome in state β. This completes the proof.■

BMT (2011) show in their Proposition 1 that strict Maskin monotonicity is necessary for implementation in rationalizable strategies under complete information. It follows from the previous example that this crucially relies on the assumption that only SCFs were considered in BMT's main result. More specifically, we show that, while the failure of Maskin monotonicity is severe, implementation in rationalizable strategies is still possible by a finite mechanism. For completeness, we provide the following lemma.

Lemma 1 *The SCC F satisfies set-monotonicity.*

Proof: Since agent 1 has state-uniform preferences, we only focus on agent 2 in the following argument. First, we set $\theta = \alpha$ and $\theta' = \beta$ in the definition of set-monotonicity. We know that $F(\alpha) = \{a_1, \ldots, a_K\}$ and by Claim 1, for any $a \in F(\alpha) \backslash \{a_2, a_K\}$ and $i \in \{1, 2\}$, we have the corresponding monotonic transformation from α to β. For $a_2 \in F(\alpha)$, however, we have

$$u_2(a_2; \alpha) > u_2(a_3; \alpha) \text{ and } u_2(a_2; \beta) < u_2(a_3; \beta).$$

Therefore, the condition needed for the monotonic transformation from α to β under set-monotonicity is not satisfied. Hence, in this case, set-monotonicity imposes no conditions on SCCs.

Second, we set $\theta = \beta$ and $\theta' = \alpha$ in the definition of set-monotonicity. Since $F(\beta) = a_K$ and a_K is the best outcome for agent 2 in both states, we have that for any $y \in \Delta(A)$,

$$u_2(a_K; \beta) \geq u_2(y; \beta) \Rightarrow u_2(a_K; \alpha) \geq u_2(y; \alpha).$$

In this case, set-monotonicity implies that $a_K \in F(\alpha)$, which is indeed the case. Thus, F satisfies set-monotonicity.■

6 Sufficient Conditions for Weak Implementation in Rationalizable Strategies

We turn in this section to our first general sufficiency result. Before that, we introduce an additional condition.

Definition 6 *An SCC F satisfies the* ***strong no-worst-alternative*** *condition (henceforth, SNWA) if, for each $\theta \in \Theta$ and $i \in N$, there exists $z_i^\theta \in \Delta(A)$ such that, for each $a \in F(\theta)$,*

$$u_i(a;\theta) > u_i(z_i^\theta;\theta).$$

Remark: This condition is introduced by Cabrales and Serrano (2011). In words, SNWA says that the SCC never assign the worst outcome to any agent at any state. BMT (2011) use its SCF-version and call it the no-worst-alternative condition (NWA).

Lemma 2 *If an SCC F satisfies SNWA, then for each $i \in N$, there exists a collection of lotteries $\{z_i(\theta,\theta')\}_{\theta,\theta'\in\Theta}$ such that for all $\theta,\theta' \in \Theta$:*

$$u_i(a;\theta') > u_i(z_i(\theta,\theta');\theta') \ \ \forall a \in F(\theta')$$

and whenever $\theta \neq \theta'$,

$$u_i(z_i(\theta,\theta');\theta) > u_i(z_i(\theta',\theta');\theta).$$

Proof: This is a straightforward extension of Lemma 2 in BMT (2011) to SCCs. So, we omit the proof.■

We are now ready to state the main sufficiency result for weak implementation in rationalizable strategies.

Theorem 2 *Suppose that there are at least three agents ($n \geq 3$). If an SCC F satisfies set-monotonicity and SNWA, it is* ***weakly*** *implementable in rationalizable strategies.*

Proof: We construct a mechanism $\Gamma = (M, g)$ such that each agent i sends a message $m_i = (m_i^1, m_i^2, m_i^3, m_i^4, m_i^5, m_i^6)$ where $m_i^1 \in \Theta, m_i^2 = \{m_i^2[\theta]\}_{\theta\in\Theta}$ where $m_i^2[\theta] \in F(\theta)$, $m_i^3 = \{(m_i^3[\theta,1], m_i^3[\theta,2])\}_{\theta\in\Theta}$ where $m_i^3[\theta,1] \in \Delta(A)$ and $m_i^3[\theta,2] \in F(\theta)$, $m_i^4 \in \Delta(A), m_i^5 \in N$, and $m_i^6 \in \mathbb{N}$. The outcome function $g : M \to \Delta(A)$ is defined as follows: for each $m \in M$:

Rule 1: If there exists $\theta' \in \Theta$ such that $m_i^1 = \theta'$ and $m_i^6 = 1$ for all $i \in N$, then $g(m) = m_t^2[\theta']$ where $t = (\sum_{j\in N} m_j^5) \pmod{n+1}$.

Rule 2: If there exist $\theta' \in \Theta$ and $i \in N$ such that $m_i^6 > 1$ and $m_j^1 = \theta'$ and $m_j^6 = 1$ for all $j \neq i$, then the following subrules apply:

Rule 2-1: If $u_i(m_t^2[\theta']; \theta') \geq u_i(m_i^3[\theta', 1]; \theta')$ and $m_t^2[\theta'] = m_i^3[\theta', 2]$ where $t = (\sum_{j \in N} m_j^5) \pmod{n+1}$, then

$$g(m) = \begin{cases} m_i^3[\theta', 1] & \text{with probability } m_i^6/(m_i^6+1) \\ z_i(\theta', \theta') & \text{with probability } 1/(m_i^6+1) \end{cases}$$

Rule 2-2: Otherwise,

$$g(m) = \begin{cases} m_t^2[\theta'] & \text{with probability } m_i^6/(m_i^6+1) \\ z_i(\theta', \theta') & \text{with probability } 1/(m_i^6+1) \end{cases}$$

where $t = (\sum_{j \in N} m_j^5) \pmod{n+1}$.

Rule 3: In all other cases,

$$g(m) = \begin{cases} m_1^4 & \text{with probability } \frac{m_1^6}{n(m_1^6+1)} \\ m_2^4 & \text{with probability } \frac{m_2^6}{n(m_2^6+1)} \\ \vdots & \vdots \\ m_n^4 & \text{with probability } \frac{m_n^6}{n(m_n^6+1)} \\ \underline{z} & \text{with the remaining probability,} \end{cases}$$

where

$$\underline{z} = \frac{1}{n} \sum_{i \in N} \underline{z}_i \text{ and } \underline{z}_i = \frac{1}{|\Theta|} \sum_{\theta \in \Theta} z_i^\theta.$$

Throughout the proof, we denote the true state by θ. The proof consists of Steps 1 through 4.

Step 1: $m_i \in S_i^{\Gamma(\theta)} \Rightarrow m_i^6 = 1$.

Proof of Step 1: Let $m_i = (m_i^1, m_i^2, m_i^3, m_i^4, m_i^5, m_i^6) \in S_i^{\Gamma(\theta)}$. Suppose by way of contradiction that $m_i^6 > 1$. Then, for any profile of messages m_{-i} that agent i's opponents may play, (m_i, m_{-i}) will trigger either Rule 2 or Rule 3. We can partition the message profiles of all agents but i as follows:

$$M_{-i}^2 \equiv \left\{ m_{-i} \in M_{-i} \middle| \ \exists \theta' \in \Theta \text{ s.t. } m_j^1 = \theta', m_j^2[\theta'] \in F(\theta'), \text{ and } m_j^6 = 1 \ \forall j \neq i \right\}$$

denotes the set of messages of all agents but i in which Rule 2 is triggered, and

$$M_{-i}^3 \equiv M_{-i} \backslash M_{-i}^2$$

denotes the set of messages of all agents but i in which Rule 3 is triggered.

Suppose first that agent i has a belief $\lambda_i \in \Delta(M_{-i})$ under which Rule 3 is triggered with positive probability, so that $\sum_{m_{-i} \in M^3_{-i}} \lambda_i(m_{-i}) > 0$. If $u_i(m^4_i; \theta) > u_i(z^\theta_i; \theta)$, we define $\hat{m}_i$ as the same as m_i except that $\hat{m}^6_i$ is chosen to be larger than m^6_i. In doing so, agent i decreases the probability that $\underline{z}$ is chosen in Rule 3. So, conditional on Rule 3, we have

$$\sum_{m_{-i} \in M^3_{-i}} \lambda_i(m_{-i}) u_i(g(\hat{m}_i, m_{-i}); \theta) > \sum_{m_{-i} \in M^3_{-i}} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i}); \theta).$$

If $u_i(m^4_i; \theta) \leq u_i(z^\theta_i; \theta)$, we define $\hat{m}_i$ as the same as m_i except that $\hat{m}^4_i \in F(\theta)$ and $\hat{m}^6_i$ is chosen to be larger than m^6_i. Similarly, conditional on Rule 3, we obtain the same inequality.

Now suppose that agent i believes that Rule 2 will be triggered with positive probability, so that $\sum_{m_{-i} \in M^2_{-i}} \lambda_i(m_{-i}) > 0$. We again consider a deviation from m_i to $\hat{m}_i$ and observe that the choice of $\hat{m}^4_i$ does not affect the outcome of the mechanism conditional on Rule 2.

First, assume that $m^1_j = \theta' \neq \theta$ for each $j \neq i$. Suppose $u_i(m^3_i[\theta', 1]; \theta) \geq u_i(z_i(\theta, \theta'); \theta)$. In this case, agent i could change m_i to $\hat{m}_i$ by having $\hat{m}^6_i$ larger than m^6_i and keeping m_i unchanged otherwise. Since $u_i(m^3_i[\theta', 1]; \theta) \geq u_i(z_i(\theta, \theta'); \theta) > u_i(z_i(\theta', \theta'); \theta)$, we have that conditional on Rule 2,

$$\sum_{m_{-i} \in M^2_{-i}} \lambda_i(m_{-i}) u_i(g(\hat{m}_i, m_{-i}); \theta) > \sum_{m_{-i} \in M^2_{-i}} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i}); \theta).$$

Otherwise, suppose that $u_i(m^3_i[\theta', 1]; \theta) < u_i(z_i(\theta, \theta'); \theta)$. In this case, agent i could change m_i to $\hat{m}_i$ by having $\hat{m}^3_i[\theta', 1] = z_i(\theta, \theta')$, $\hat{m}^6_i > m^6_i > 1$, and keeping m_i unchanged otherwise. Since $u_i(z_i(\theta, \theta'); \theta) > u_i(z_i(\theta', \theta'); \theta)$, we have that, conditional on Rule 2,

$$\sum_{m_{-i} \in M^2_{-i}} \lambda_i(m_{-i}) u_i(g(\hat{m}_i, m_{-i}); \theta) > \sum_{m_{-i} \in M^2_{-i}} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i}); \theta).$$

Second, assume that $m^1_j = \theta$ for each $j \neq i$. We choose $t^* \neq i$ and $m^*_{-i} \in \text{supp}(\lambda_i(\cdot))$ such that for each $j \neq i$ and $m_{-i} \in \text{supp}(\lambda_i(\cdot))$,

$$u_i(m^{*2}_{t^*}[\theta]; \theta) \geq u_i(m^2_j[\theta]; \theta).$$

Then, in this case, agent i could change m_i to $\hat{m}_i$ by having $\hat{m}^3_i[\theta, 1] = m^{*2}_{t^*}[\theta]$ and $\hat{m}^6_i > m^6_i > 1$, keeping m_i unchanged otherwise. Since $u_i(m^{*2}_{t^*}[\theta]; \theta) > u_i(z_i(\theta, \theta); \theta)$, we have that, conditional on Rule 2,

$$\sum_{m_{-i} \in M^2_{-i}} \lambda_i(m_{-i}) u_i(g(\hat{m}_i, m_{-i}); \theta) > \sum_{m_{-i} \in M^2_{-i}} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i}); \theta).$$

It follows that, in all cases, these choices of $\hat{m}_i$ strictly improve the expected payoff of agent i if either Rule 2 or Rule 3 is triggered. This implies that m_i is never a best response to any belief λ_i, which contradicts our hypothesis that $m_i \in S_i^{\Gamma(\theta)}$.■

Step 2: For any $\theta \in \Theta$, there exists $m \in S^{\Gamma(\theta)}$ such that $g(m) \in F(\theta)$.

Proof of Step 2: Fix $\theta \in \Theta$. For each $i \in N$, we define

$$a_i(\theta) \in \arg\max_{a \in F(\theta)} u_i(a; \theta).$$

Define $\lambda_i \in \Delta(M_{-i})$ as follows: for any $m_{-i} \in M_{-i}$, if $\lambda_i(m_{-i}) > 0$,

$$\begin{aligned} m_j^1 &= \theta; \\ m_j^2 &= \{m_j^2[\tilde{\theta}]\}_{\tilde{\theta} \in \Theta} \text{ where } m_j^2[\tilde{\theta}] = a_j(\tilde{\theta}); \\ m_j^5 &= 1; \text{ and} \\ m_j^6 &= 1, \end{aligned}$$

for all $j \neq i$. Define $m_i = (\theta, m_i^2, m_i^3, m_i^4, i, 1)$ such that $m_i^2 = \{m_i^2[\tilde{\theta}]\}_{\tilde{\theta} \in \Theta}$ where $m_i^2[\tilde{\theta}] = a_i(\tilde{\theta})$. With this belief λ_i, agent i is convinced that Rule 1 is triggered with probability 1, so that i becomes the winner of the modulo game. Thus, m_i is a best response to λ_i so that m_i survives the first round of iterative deletion of never best responses. We next claim that the support of λ_i is rationalizable. Fix $j \neq i$. Let $m_j \in M_j$ be defined such that

$$\begin{aligned} m_j^1 &= \theta; \\ m_j^2 &= \{m_j^2[\tilde{\theta}]\}_{\tilde{\theta} \in \Theta} \text{ where } m_j^2[\tilde{\theta}] = a_j(\tilde{\theta}); \\ m_j^5 &= j; \text{ and} \\ m_j^6 &= 1 \end{aligned}$$

Define $\lambda_j \in \Delta(M_{-j})$ as follows: for any $m_{-j} \in M_{-j}$, if $\lambda_j(m_{-j}) > 0$,

$$\begin{aligned} m_k^1 &= \theta; \\ m_k^2 &= \{m_k^2[\tilde{\theta}]\}_{\tilde{\theta} \in \Theta} \text{ where } m_k^2[\tilde{\theta}] = a_k(\tilde{\theta}); \\ m_k^5 &= 1; \text{ and} \\ m_k^6 &= 1, \end{aligned}$$

for all $k \neq j$. Recall that we define $m_j = (\theta, m_j^2, m_j^3, m_j^4, j, 1)$. With this belief λ_j, agent j is convinced that Rule 1 is triggered with probability 1, so that j becomes the winner of the modulo game. Thus, m_j is a best response to λ_j and each such

m_j survives the first round of iterative deletion of never best responses. We can repeat this argument iteratively so that m_i survives the iterative deletion of never best responses. Thus, a message profile m where each m_i is as constructed above generates the outcome $a_i(\theta)$, which is in $F(\theta)$. This completes the proof.■

Step 3: $m_i \in S_i^{\Gamma(\theta)} \Rightarrow \lambda_i(m_{-i}) = 0$ for any profile (m_i, m_{-i}) under Rules 2 or 3, where $\lambda_i \in \Delta(M_{-i})$ represents the belief held by i to which m_i is a best response.

Proof of Step 3: Suppose $m_i \in S_i^{\Gamma(\theta)}$. By Step 1, m_i has the form of $m_i = (\theta', m_i^2, m_i^3, m_i^4, m_i^5, 1)$ for some $\theta' \in \Theta$, where the θ' announced by different agents might be different. Given the message m_i, we define the set of messages of the remaining agents which trigger Rule 1, 2, or 3. Let M_{-i}^1 be the set of $m_{-i} \in M_{-i}$ such that (m_i, m_{-i}) triggers Rule 1 and $M_{-i}^{2,i}$ be the set of $m_{-i} \in M_{-i}$ such that (m_i, m_{-i}) triggers Rule 2 with agent i as the deviating player.

We consider a given belief λ_i of agent i. If $\sum_{m_{-i} \in M_{-i}^1} \lambda_i(m_{-i}) = 0$, then Rule 2 or 3 will be triggered with probability one. Although Rule 2 can now be triggered with a "deviating agent" being different from i, it is easily checked that a similar argument to that in Step 1 applies so that the message m_i cannot be a best reply to λ_i. So, suppose that

$$0 < \sum_{m_{-i} \in M_{-i}^1} \lambda_i(m_{-i}) < 1.$$

For each $\tilde{\theta} \in \Theta$, define

$$\hat{m}_i^3(\tilde{\theta}) = \begin{cases} (m_{j^*}^2[\theta'], m_i^3[\theta', 2]) & \text{if } \tilde{\theta} = \theta' \\ m_i^3[\tilde{\theta}] & \text{otherwise,} \end{cases}$$

where $j^* = \arg\max_{j \in N} u_i(m_j^2[\theta']; \theta)$. Define $\hat{m}_i^4 = \arg\max_{y \in \Delta(A)} u_i(y; \theta)$. We set $\hat{m}_i^6$ to be an integer sufficiently large. Define $\hat{m}_i = (\theta', m_i^2, \hat{m}_i^3, \hat{m}_i^4, m_i^5, \hat{m}_i^6)$ as i's alternative message in which we keep $m_i^1 = \theta', m_i^2$ and m_i^5 unchanged. Then, as $\hat{m}_i^6$ tends to infinity, agent i's expected utility from choosing $\hat{m}_i$ is approximately at least as high as

$$\sum_{m_{-i} \in M_{-i}^1 \cup M_{-i}^{2,i}} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i}); \theta) + \sum_{m_{-i} \notin M_{-i}^1 \cup M_{-i}^{2,i}} \lambda_i(m_{-i}) u_i(\hat{m}_i^4; \theta),$$

which is strictly larger than i's expected payoff from choosing m_i. Hence, by choosing $\hat{m}_i^6$ large enough, $\hat{m}_i$ is a better response to λ_i (in words, the loss in Rule 2 can always be offset by a bigger gain in Rule 3). This is a contradiction.

So, if $m_i = (\theta', m_i^2, m_i^3, m_i^4, m_i^5, 1) \in S_i^{\Gamma(\theta)}$, it follows that agent i must be convinced that each $j \neq i$ is choosing a message of the form $(\theta', m_j^2, m_j^3, m_j^4, m_j^5, 1)$ and hence $\sum_{m_{-i} \in M_{-i}^1} \lambda_i(m_{-i}) = 1$.■

Step 3 implies that one can partition the set of rationalizable message profiles into separate components, $\theta, \theta^{'}, \theta^{''}, \ldots$. For instance, in the $\theta^{'}$ component, this is the choice of state that each agent makes in the first item of their messages, which also determines the event to which each of them assigns probability 1. That is, in that component, each agent i believes that all the others are using strategies of the form $(\theta^{'}, \cdot, \cdot, \cdot, \cdot, 1)$ with probability 1.

We introduce an additional piece of notation. For any $\theta, \theta^{'} \in \Theta$ and $i \in N$, define

$$S_i^{\Gamma(\theta)}[\theta^{'}] = \left\{ m_i \in S_i^{\Gamma(\theta)} \middle| \; m_i^1 = \theta^{'} \text{ and } m_i^6 = 1 \right\}.$$

Let $S^{\Gamma(\theta)}[\theta^{'}] = \times_{i \in N} S_i^{\Gamma(\theta)}[\theta^{'}]$

Step 4: $m \in S^{\Gamma(\theta)} \Rightarrow g(m) \in F(\theta)$.

Proof of Step 4: By Step 3, we know that if $m_i \in S_i^{\Gamma(\theta)}$, there exists $\theta^{'} \in \Theta$ such that agent i both is using and is convinced that every agent j is using a message of the form $m_j = (\theta^{'}, m_j^2, m_j^3, m_j^4, m_j^5, 1)$. If $\theta^{'} = \theta$, by Rule 1 and the construction of the mechanism, $g(m) \in F(\theta)$. So, in what follows, we assume that $\theta^{'} \neq \theta$. Suppose by way of contradiction that there exists $m^* \in S^{\Gamma(\theta)}[\theta^{'}]$ such that $g(m^*) \notin F(\theta)$. By strong set-monotonicity, we know that there exist $i \in N$, $a^* \in F(\theta^{'})$, and $z^* \in \Delta(A)$ such that $u_i(a^*; \theta^{'}) \geq u_i(z^*; \theta^{'})$; and $u_i(a^*; \theta) < u_i(z^*; \theta)$. Let $a^* = g(m^*)$.

By our hypothesis, there exists $m_i^* = (\theta^{'}, m_i^{*2}, m_i^{*3}, m_i^{*4}, m_i^{*5}, 1) \in S_i^{\Gamma(\theta)}$. Since $m_i^* \in S_i^{\Gamma(\theta)}[\theta^{'}]$, there exists $\lambda_i \in \Delta(M_{-i})$ such that (i) $\lambda_i(m_{-i}) > 0 \Rightarrow m_j = (\theta^{'}, m_j^2, m_j^3, m_j^4, m_j^5, 1)$ for any $j \neq i$ and (ii) $\sum_{m_{-i}} \lambda_i(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta) \geq \sum_{m_{-i}} \lambda_i(m_{-i}) u_i(g(\tilde{m}_i, m_{-i}); \theta)$ for all $\tilde{m}_i \in M_i$. Define

$$\hat{m}_{-i}(m_i^*) \in \arg \max_{(m_i^*, m_{-i}) \in S^{\Gamma(\theta)}[\theta^{'}]} u_i(g(m_i^*, m_{-i}); \theta).$$

Note that we have $g(m_i^*, \hat{m}_{-i}(m_i^*)) \in \arg\max_{a \in F(\theta^{'})} u_i(a; \theta)$, i.e., $g(m_i^*, \hat{m}_{-i}(m_i^*))$ induces one of i's best outcomes under Rule 1 in which all agents unanimously announce $\theta^{'}$ in state θ. Without loss of generality, assume that the winner of the modulo game that gives this great outcome to agent i is actually not agent i himself.

Then, we define $\tilde{\lambda}_i \in \Delta(M_{-i})$ as follows: $\tilde{\lambda}_i(m_{-i}) = 0$ if and only if $m_{-i} \neq \hat{m}_{-i}(m_i^*)$. By construction of $\tilde{\lambda}_i$, we have that m_i^* must be a best response to the redefined belief $\tilde{\lambda}_i$. Since m_i^* is a best response to $\tilde{\lambda}_i$ and m_i^* triggers Rule 1 with probability one under $\tilde{\lambda}_i$, agent i should not in particular have an incentive to induce Rule 2. This implies that we must have $u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) \geq u_i(g(m_i^{'}, \hat{m}_{-i}(m_i^*)); \theta)$ for any $m_i^{'}$.

We can organize the argument in two cases:

Case 1. Assume $g(m_i^*, \hat{m}_{-i}(m_i^*)) = a^*$. Then, we define $\hat{m}_i^3$ as follows: for any $\tilde{\theta} \in \Theta$,

$$\hat{m}_i^3[\tilde{\theta}, 1] = \begin{cases} z^* & \text{if } \tilde{\theta} = \theta' \\ m_i^{*3}[\tilde{\theta}, 1] & \text{otherwise,} \end{cases}$$

and

$$\hat{m}_i^3[\tilde{\theta}, 2] = \begin{cases} a^* & \text{if } \tilde{\theta} = \theta' \\ m_i^{*3}[\tilde{\theta}, 2] & \text{otherwise.} \end{cases}$$

Define $\hat{m}_i = (\theta', m_i^{*2}, \hat{m}_i^3, m_i^{*4}, m_i^{*5}, \hat{m}_i^6)$ where we only change the third and sixth components of m_i^*, i.e., $\hat{m}_i^3$ and $\hat{m}_i^6$. With this choice of strategy, agent i changes the outcome with respect to using m_i^* only when the outcome under m_i^* was a^*. By choosing $\hat{m}_i^6$ sufficiently large, we conclude that $\hat{m}_i$ is an even better response than m_i^* to $\tilde{\lambda}_i$. This contradicts the hypothesis that m_i^* is a best response to $\tilde{\lambda}_i$.

Case 2. Assume, on the other hand, that $g(m_i^*, \hat{m}_{-i}(m_i^*)) \neq a^*$. We shall show that this case is impossible. In this case, relying on the strategy $\hat{m}_i$ as defined in Case 1, note that $g(\hat{m}_i, \hat{m}_{-i}(m_i^*)) = g(m_i^*, \hat{m}_{-i}(m_i^*))$ with probability $\hat{m}_i^6/(\hat{m}_i^6 + 1)$, as the only change happened upon a^* being the outcome.

For each $\varepsilon > 0$ and m_{-i}, we define

$$\lambda_i^\varepsilon(m_{-i}) = \begin{cases} 1 - \varepsilon & \text{if } m_{-i} = \hat{m}_{-i}(m_i^*) \\ \varepsilon & \text{for one } \tilde{m}_{-i} : \tilde{m}_{-i}^1 = \theta', \tilde{m}_{-i}^2 = a^*, \tilde{m}_{-i}^5 = \hat{m}_{-i}^5(m_i^*), \tilde{m}_{-i}^6 = 1, \end{cases}$$

where we denote $\tilde{m}_j^1 = \theta'$ for all $j \neq i$ by $\tilde{m}_{-i}^1 = \theta'$ and the same notation applies to $\tilde{m}_{-i}^2, \tilde{m}_{-i}^5$, and $\tilde{m}_{-i}^6$. Since agent i, by our hypothesis, is not the winner of the modulo game under $(m_i^*, \hat{m}_{-i}(m_i^*))$, by construction of $\tilde{m}_{-i}$, we have $g(m_i^*, \tilde{m}_{-i}) = a^*$. Moreover, each $\tilde{m}_j$ is rationalizable because agent j can believe that agent i chooses j's best outcome in the place of $m_i^2[\theta']$ and i becomes the winner of the modulo game.

Consider again the strategy $\hat{m}_i$ as in Case 1. As $\hat{m}_i^6$ tends to infinity, we obtain

$$\begin{aligned} & \sum_{m_{-i}} \lambda_i^\varepsilon(m_{-i}) u_i(g(\hat{m}_i, m_{-i}); \theta) \\ \approx\ & (1-\varepsilon) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + \varepsilon u_i(z^*; \theta) \\ >\ & (1-\varepsilon) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + \varepsilon u_i(a^*; \theta) \ \ (\because\ u_i(z^*; \theta) > u_i(a^*; \theta)) \\ =\ & \sum_{m_{-i}} \lambda_i^\varepsilon(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta), \end{aligned}$$

where the last equality follows from the fact that agent i is not the winner of the modulo game when the others are using the specified strategy with probability ε.

Therefore, $\hat{m}_i$ is an even better response than m_i^* to λ_i^ε. We conclude that m_i^* is *not* a best response to λ_i^ε for any $\varepsilon > 0$.

Finally, to show impossibility, we claim that m_i^* is a best response to λ_i^ε as long as we choose $\varepsilon > 0$ sufficiently small. Consider an alternative message $m_i^{'}$ that induces Rule 2. Fix any such alternative message $m_i^{'}$. No matter how large $m_i^{'6}$ can be, one can choose $\varepsilon > 0$ small enough so that

$$\begin{aligned} & \frac{1}{m_i^{'6}+1}\left[u_i(g(m_i^*, \tilde{m}_{-i}); \theta) - u_i(z_i(\theta^{'}, \theta^{'}); \theta)\right] \\ > \; & \varepsilon\left[u_i(g(m_i^{'}, \tilde{m}_{-i}); \theta) - u_i(g(m_i^*, \tilde{m}_{-i}); \theta)\right]. \end{aligned}$$

Indeed, this is so because $g(m_i^*, \tilde{m}_{-i}) = a^* \in F(\theta^{'})$, and hence, $u_i(g(m_i^*, \tilde{m}_{-i}); \theta) > u_i(z_i(\theta^{'}, \theta^{'}); \theta)$ by SNWA. Moreover, given agent i's belief λ_i^ε, m_i^* results in the best outcome with probability $1-\varepsilon$ so that $u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) \geq u_i(g(m_i^{'}, \hat{m}_{-i}(m_i^*)); \theta)$. Therefore, once we choose $\varepsilon > 0$ small enough, m_i^* is made an even better response to λ_i^ε than $m_i^{'}$. Since this argument applies to any such alternative message $m_i^{'}$, agent i has no incentive to trigger Rule 2 himself. This establishes that m_i^* is a best response to λ_i^ε.■

Steps 1 through 4 together imply that for each $\theta \in \Theta$ and $m \in M$, (i) $S^{\Gamma(\theta)} \neq \emptyset$ and (ii) $m \in S^{\Gamma(\theta)} \Rightarrow \exists \theta^{'} \in \Theta$ such that $m_i^1 = \theta^{'}$; $m_i^6 = 1$; and $g(m) \in F(\theta)$. Thus, for each $\theta \in \Theta$ and $m \in M$, $S^{\Gamma(\theta)} \neq \emptyset$ and $m \in S^{\Gamma(\theta)} \Rightarrow g(m) \in F(\theta)$. This completes the proof of the theorem.■

When focusing only on SCFs, we obtain the following result as a corollary of Theorem 2.

Corollary 1 *Suppose that there are at least three agents ($n \geq 3$). If an SCF f satisfies Maskin monotonicity and NWA, it is fully implementable in rationalizable strategies.*

Proof: We know that set-monotonicity and SNWA are reduced to Maskin monotonicity and NWA (the SCF-version used by BMT (2011)), respectively, as long as the social choice rule is single-valued. Note also that the difference between weak and full implementation is inconsequential in the case of SCFs. The rest of the proof is completed as that of Theorem 2.■

This is a logical strenghthening of Proposition 2 of BMT (2011) because we completely dispense with the responsiveness of SCFs, which is assumed there. We say that an SCF f is *responsive* if, for any $\theta, \theta^{'} \in \Theta$, whenever $\theta \neq \theta^{'}$, $f(\theta) \neq f(\theta^{'})$.

7 Sufficient Conditions for Full Implementation in Rationalizable Strategies

Next, we state the general sufficiency result for full implementation in rationalizable strategies.

Theorem 3 *Suppose that there are at least three agents ($n \geq 3$). If an SCC F satisfies set-monotonicity and SNWA, it is* ***fully*** *implementable in rationalizable strategies.*

Proof: Our proof is based upon the canonical mechanism proposed in the proof of Theorem 2. Recall that Steps 1 through 4 of Theorem 2 together imply that for each $\theta \in \Theta$ and $m \in M$, $S^{\Gamma(\theta)} \neq \emptyset$ and $m \in S^{\Gamma(\theta)} \Rightarrow g(m) \in F(\theta)$. Hence, it only remains to establish the following property of the mechanism:

Step 5: For any $\theta \in \Theta$ and $a \in F(\theta)$, there exists $m^* \in S^{\Gamma(\theta)}$ such that $g(m^*) = a$.

Proof of Step 5: Fix $\theta \in \Theta$ as the true state, and fix $a \in F(\theta)$. Define $m_1^* = (\theta, m_1^{*2}, m_1^{*3}, m_1^{*4}, 1, 1)$, where $m_1^{*2}[\theta] = a$. For each $j \in \{2, \ldots, n\}$, define $m_j^* = (\theta, m_j^{*2}, m_j^{*3}, m_j^{*4}, 1, 1)$, where $m_j^{*2}[\theta] = a_{j-1}(\theta)$, which denotes one of the maximizers of $u_{j-1}(\cdot; \theta)$ within all the outcomes in $F(\theta)$. Then, the constructed message profile m^* induces Rule 1 and agent 1 becomes the winner of the modulo game. We thus have $g(m^*) = a$ by construction. What remains to show is that $m^* \in S^{\Gamma(\theta)}$.

By construction of the mechanism, Rule 3 cannot be triggered by any unilateral deviation from Rule 1. So, the specification of m_i^{*4} does not affect our argument. Moreover, also by construction of the mechanism, no agent has an incentive to induce Rule 2 with a unilateral deviation from a truthful profile under Rule 1. So, effectively, the specification of m_i^{*3} does not affect our argument either.

We first show that m_1^* can be made a best response to some belief. Define $\lambda_1^* \in \Delta(M_{-1})$ as follows: for any $m_{-1} \in M_{-1}$, if $\lambda_1^*(m_{-1}) > 0$,

$$\begin{aligned} m_j^1 &= \theta; \\ m_j^2[\theta] &= a_{j-1}(\theta); \\ m_j^5 &= \begin{cases} 2 & \text{if } j = 2 \\ 1 & \text{otherwise} \end{cases} \\ m_j^6 &= 1. \end{aligned}$$

for all $j \in \{2, \ldots, n\}$. Given this belief λ_1^* and m_1^*, agent 2 becomes the winner of the modulo game so that the outcome $a_1(\theta)$, which is the best one for agent 1, is

generated. Therefore, m_1^* is a best response to λ_1^* so that it survives the first round of deletion of never best responses.

We next show that the support of λ_1^* is rationalizable. Assume $j \neq 1$. Define

$$\bar{m}_j = \begin{cases} (\theta, \bar{m}_2^2, \bar{m}_2^3, \bar{m}_2^4, 2, 1) & \text{if } j = 2 \\ (\theta, \bar{m}_j^2, \bar{m}_j^3, \bar{m}_j^4, 1, 1) & \text{otherwise} \end{cases}$$

where $\bar{m}_j^2[\theta] = a_{j-1}(\theta)$. Define $\bar{\lambda}_2 \in \Delta(M_{-2})$ as follows: for any $m_{-2} \in M_{-2}$, if $\bar{\lambda}_2(m_{-2}) > 0$,

$$\begin{aligned} m_k^1 &= \theta; \\ m_k^2[\theta] &= a_{k-1}(\theta); \\ m_k^5 &= \begin{cases} 2 & \text{if } k = 1 \\ 1 & \text{otherwise} \end{cases} \\ m_k^6 &= 1. \end{aligned}$$

for all $k \neq 2$. Then, given this belief $\bar{\lambda}_2$ and $\bar{m}_2$, agent 3 becomes the winner of the modulo game so that the outcome $a_2(\theta)$, which is the best one for agent 2, is realized. Therefore, $\bar{m}_2$ is a best response to $\bar{\lambda}_2$ so that it survives the first round of deletion of never best responses. Assume $j \in N \backslash \{1, 2\}$. Define $\bar{\lambda}_j \in \Delta(M_{-j})$ as follows: for any $m_{-j} \in M_{-j}$, if $\bar{\lambda}_j(m_{-j}) > 0$,

$$\begin{aligned} m_k^1 &= \theta; \\ m_k^2[\theta] &= \begin{cases} a_n(\theta) & \text{if } k = 1 \\ a_{k-1}(\theta) & \text{otherwise;} \end{cases} \\ m_k^5 &= \begin{cases} j+1 & \text{if } k = 1 \\ 1 & \text{otherwise;} \end{cases} \\ m_k^6 &= 1, \end{aligned}$$

for all $k \neq 2$. Assume $j < n$. Then, given the belief $\bar{\lambda}_j$ and $\bar{m}_j$, agent $j+1$ becomes the winner of the modulo game so that the outcome $a_j(\theta)$, which is the best one for agent j, is realized. Assume, on the other hand, that $j = n$. Then, given the belief $\bar{\lambda}_j$ and $\bar{m}_j$, agent 1 becomes the winner of the modulo game so that the outcome $a_n(\theta)$, which is the best one for agent n, is realized. Therefore, $\bar{m}_j$ is a best response to $\bar{\lambda}_j$ so that it survives the first round of deletion of never best responses. We can repeat this argument iteratively so that m_1^* survives the iterative deletion of never best responses. Hence, $m_1^* \in S_i^{\Gamma(\theta)}$.

Third, we shall show that, for each $j \neq 1$, m_j^* can be made a best response to some belief. For each $j \in \{2, \ldots, n\}$, define $\lambda_j^* \in \Delta(M_{-j})$ as follows: for any

$m_{-j} \in M_{-j}$, if $\lambda_j^*(m_{-j}) > 0$,

$$\begin{aligned} m_k^1 &= \theta; \\ m_k^2[\theta] &= \begin{cases} a_n(\theta) & \text{if } k = 1 \\ a_{k-1}(\theta) & \text{otherwise;} \end{cases} \\ m_k^5 &= \begin{cases} j+1 & \text{if } k = 1 \\ 1 & \text{otherwise;} \end{cases} \\ m_k^6 &= 1, \end{aligned}$$

for all $k \neq j$. Given this belief λ_j^* and m_j^*, agent $j+1$ becomes the winner of the modulo game so that the outcome $a_j(\theta)$, which is the best one for agent j, is realized. Therefore, for each $j \neq 1$, m_j^* is a best response to λ_j^* so that it survives the first round of deletion of never best responses.

Fourth, we will show that the support of λ_j^* is rationalizable. Consider $\bar{m}_1 = (\theta, \bar{m}_1^2, \bar{m}_1^3, \bar{m}_1^4, j+1, 1)$, where $\bar{m}_1^2[\theta] = a_n(\theta)$. Define $\bar{\lambda}_1 \in \Delta(M_{-1})$ as follows: for any $m_{-1} \in M_{-1}$, if $\bar{\lambda}_1(m_{-1}) > 0$,

$$\begin{aligned} m_k^1 &= \theta; \\ m_k^2[\theta] &= a_{k-1}(\theta); \\ m_k^5 &= \begin{cases} n+2-j & \text{if } k = 2 \\ 1 & \text{otherwise;} \end{cases} \\ m_k^6 &= 1, \end{aligned}$$

for all $k \neq 1$. Given this belief $\bar{\lambda}_1$ and $\bar{m}_1$, agent 2 becomes the winner of the modulo game so that the outcome $a_1(\theta)$, which is the best one for agent 1, is realized. Therefore, $\bar{m}_1$ is a best response to $\bar{\lambda}_1$ so that it survives the first round of deletion of never best responses.

Consider agent $k \in N \backslash \{1, j\}$. We first assume $k < n$. Define $\bar{m}_k = (\theta, \bar{m}_k^2, \bar{m}_k^3, \bar{m}_k^4, 1, 1)$, where $\bar{m}_k^2[\theta] = a_{k-1}(\theta)$. Define $\bar{\lambda}_k \in \Delta(M_{-k})$ as follows: for any $m_{-k} \in M_{-k}$, if $\bar{\lambda}_k(m_{-k}) > 0$,

$$\begin{aligned} m_i^1 &= \theta; \\ m_i^2[\theta] &= a_{i-1}(\theta); \\ m_i^6 &= 1, \end{aligned}$$

for all $i \neq k$ and $\sum_{i \neq k} m_i^5 = n + k - 1$. Given this belief $\bar{\lambda}_k$ and $\bar{m}_k$, agent $k+1$ becomes the winner of the modulo game so that the outcome $a_k(\theta)$, which is the best one for agent k, is realized. Therefore, $\bar{m}_k$ is a best response to $\bar{\lambda}_k$ so that it survives the first round of deletion of never best responses.

Assume $n \neq j$. We define $\bar{m}_n = (\theta, \bar{m}_n^2, \bar{m}_n^3, \bar{m}_n^4, 1, 1)$ and $\bar{\lambda}_n \in \Delta(M_{-n})$ as follows: for any $m_{-n} \in M_{-n}$, if $\bar{\lambda}_n(m_{-n}) > 0$,

$$\begin{aligned} m_i^1 &= \theta; \\ m_i^2 &= \begin{cases} a_n(\theta) & \text{if } i = 1 \\ a_{i-1}(\theta) & \text{otherwise;} \end{cases} \\ m_i^5 &= 1; \\ m_i^6 &= 1, \end{aligned}$$

for all $i \neq n$. Given this belief $\bar{\lambda}_n$ and $\bar{m}_n$, agent 1 becomes the winner of the modulo game so that the outcome $a_n(\theta)$, which is the best for agent n, is realized. Therefore, $\bar{m}_n$ is a best response to $\bar{\lambda}_n$ so that it survives the first round of deletion of never best responses.

We conclude that the support of λ_j^* is rationalizable. So, we can repeat this argument iteratively so that for each $j \neq 1$, m_j^* survives the iterative deletion of never best responses. Therefore, $m_j^* \in S_j^{\Gamma(\theta)}$ for each $j \neq 1$. Since $m_1^* \in S_1^{\Gamma(\theta)}$, we obtain $m^* \in S^{\Gamma(\theta)}$. This completes the proof of Step 5.■

Combining Step 5 and Steps 1 through 4 in the proof of the previous theorem, the proof is complete.■

8 Concluding Remarks

By relying on a setwise condition requiring the nestedness of lower contour sets, a condition that we term set-monotonicity, we have shown that rationalizable implementation of correspondences leads to a significantly more permissive theory than its counterpart using Nash equilibrium. The two-agent general sufficiency argument is likely handled by adding the usual requirement of nonempty intersections of lower contour sets; we chose instead to focus on a simple finite mechanism for a useful example. Finally, the extension to incomplete information environments should be our natural next step.

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