Good Enough*

Salvador Barberà†

Geoffroy de Clippel‡

Alejandro Neme§

Kareen Rozen¶

July 2018

Abstract

A decision maker may not perfectly maximize her preference over the feasible set. She may feel it is good enough to maximize her preference over a sufficiently large consideration set; or just require that her choice is sufficiently well-ranked (e.g., in the top quintile of options); or even endogenously determine a threshold for what is good enough, based on an initial sampling of the options. We introduce and investigate a class of theories, Order-k Rationality, encompassing heuristics such as these.

*The theories of Order-k Rationality studied in this paper were independently proposed under the name of Ordinal Satisficing by Barberà and Neme (2014) and under the name of Minimal Consideration in a supplement to de Clippel and Rozen (2012). The present paper replaces these earlier discussions of this interesting class of theories.

†MOVE, Universitat Autònoma de Barcelona, and Barcelona GSE. 08193, Spain. Departament d'Economia i d'Història Econòmica, Edifici B, 08193 Bellaterra, Spain. E-mail: barbera.salvador@gmail.com

‡Department of Economics, Brown University. E-mail: declippel@brown.edu

§Instituto de Matemática Aplicada de San Luis. Universidad Nacional de San Luis and CONICET. Avenida Italia 1554. 5700, San Luis, Argentina. E-mail: neme.alejandroj@gmail.com

¶Department of Economics, Brown University. E-mail: kareen_rozen@brown.edu

1. Introduction

In its classic expression, an individual is *rational* if she chooses the best available alternative according to a preference ordering. A growing literature has arisen proposing a variety of departures from this simple model, in light of the evidence against its behavioral implications. In the spirit of Simon (1955)'s satisficing, we investigate the behavioral implications of a decision maker (henceforth DM) who picks alternatives that are good enough, but not necessarily best. Alternatively, one can think of a preference-maximizing DM with limited attention, who feels that the number of options she has identified and evaluated is good enough.

As is well known, the empirical content of Rationality is captured by the Strong Axiom of Revealed Preference (SARP), namely that Samuelson's revealed preference is acyclic (Samuelson 1938, Houthakker 1950). Suppose instead, as in our theory of *Order-k Rationality*, that the DM picks from any choice problem S an option that falls in the top $k(S)$ elements according to her preference ordering. Samuelson's revealed preference does not apply when $k(S) > 1$. Even so, witnessing the DM choose x from S does restrict the DM's possible preference: there must be at least $|S| - k(S)$ options in S in the strict lower-contour set of x. Observing the DM pick options from different choice problems induces multiple restrictions of this form, and consistency with the theory implies the existence of an acyclic relation that jointly satisfies all these restrictions. As we will see in Section 3, this extension of SARP – which we name k-SARP – is not only necessary for consistency, but also sufficient.

The restrictions the data imposes on the DM's putative preference are more complex than the direct comparisons arising from Samuelson's revealed preference. Hence k-SARP may at first seem hard to check. Section 3, however, provides a simple iterative procedure to test k-SARP, making it comparable to testing SARP (namely, doable in polynomial time). This is perhaps surprising, since we also show that testing a theory as simple as picking the second best for a preference ordering (as suggested in Sen (1993)) is NP-hard.

As is also true for Rationality, multiple preference orderings can explain the same observed choices under Order-k Rationality.[1] An option x is unambiguously preferred to an alternative y under a theory if x is ranked above y for all preference orderings that are consistent with observed choices. Also, an option x is a valid forecast for an out-of-sample problem S if the augmented dataset, where x is picked out of S on top of observed choices, remains consistent with the theory. Under Rationality, the unambiguous preference relation is simply the transitive closure of Samuelson's revealed preference, and x is a valid forecast for S if and only if no feasible alternative is unambiguously preferred to it. Things get more complicated when relaxing the assumption of preference maximization, as the revealed preference restrictions will typically be more complex than direct comparisons between alternatives. Even so, we explain in Section 4 how our testing procedure can be adapted to tractably tackle the questions of identification and out-of-sample forecasting.

Recent papers have studied limited attention as a culprit for bounded rationality. That literature has considered various restrictions on how attention sets vary with the feasible set; see Manzini and Mariotti (2007, 2012), Masatlioglu et al. (2012), Cherepanov et al. (2013), and Lleras et al. (2017). For instance, Lleras et al. (2017) captures choice overload by assuming that if the DM pays attention to a feasible option in a large set, then she also pays attention to it in smaller sets. A potential issue is that some choice functions are consistent with a theory only because the DM pays attention to very few options in some choice problems – in some case just the choice itself. On the other hand, a possible reason for Order-k Rationality is a DM's insistence on considering a sufficient number of options. We suggest that it might be fruitful to enhance theories of attention by adding a lower bound on the cardinality of the consideration set in different choice problems. In Section 5, we characterize the empirical content of such an enrichment of Lleras et al. (2017), and discuss the tractability of testing.

In Section 6, we suggest and investigate a class of theories that use limited

[1] Of course, multiplicity is more prevalent for more permissive theories. For instance, choices pin down a unique preference ordering under Rationality with full data, but not necessarily when the DM picks options that are good enough instead of optimal.

attention to endogenously define what good enough means. Formally, the DM makes choices from a list, as in Rubinstein and Salant (2006). She starts by reviewing a predetermined number of options in the list (e.g. a fraction of its length), and then picks the first ensuing option that is better than those she reviewed. We characterize the empirical content of this theory when lists are observable, and show that testing is tractable once again.

As in a majority of papers on bounded rationality, we focused so far on single-valued choice functions. Relaxing the assumption of preference maximization opens the possibility of a multi-valued choice correspondence even when the DM has a preference ordering (which is, by definition, a strict relation). Quite naturally, when choosing an option that is good enough, the DM may sometimes pick different options from the same choice problem. It may be, for instance, that the DM entertains the different options sequentially and settles on the first one that is good enough according to this order. Facing options in a different sequence may impact choices. Having more time to search may bring her closer to preference maximization. Choosing in the presence of others, she may settle for the second best, but pick her most-preferred option when choosing alone. These are all examples of framing effects or ancillary conditions (Salant and Rubinstein 2008, Bernheim and Rangel 2009). In many cases, the modeler does not have access to contextual information (e.g. order in which options are presented or considered, the DM's time constraint, or the social setting), and observes only the feasible set and final choices. We extend our analysis to such circumstances in Section 7.

Our approach in this paper is ordinal, in that choices are based on comparisons given a preference order. Intensities play no role: if represented by utilities, choices are invariant to increasing utility transformations. Alternatively, it would be possible to model the notion of 'good enough' in a cardinal way. In Section 8, we consider a DM who is happy to settle on an option, so long as its utility is at least as large as her expected utility from drawing (and consuming) a new feasible option. This theory is cardinal in nature: an increasing transformation of a given Bernoulli utility function can modify resulting choices. When it comes to the testable implications of such a theory, however, the modeler does not know what the utility function is; instead, the modeler must check whether choices are consistent with the theory for some utility function. As it turns out, the empirical content of this cardinal theory coincides with the empirical content of Order k-Rationality, with $k(S) = |S|-1$ for all S. While a thorough study of cardinal theories would be the subject of a different paper, this suggests that incorporating cardinal comparisons does not necessarily create a theory whose empirically content is distinguishable from an ordinal one.

Additional Related Literature

Simon (1955) introduced the idea of satisficing behavior as a simpler, perhaps more realistic alternative to rationality. Instead of identifying and evaluating all feasible options in order to pick the best one, DMs may be happy to pick the first acceptable option they encounter.

There are different ways, however, to formalize what is acceptable to a DM. Most relatedly, Aleskerov et al. (2007) assumes that there is a utility function $u : X \to \mathbb{R}$ and a utility threshold $\tau(S)$ such that the DM picks the entire set of elements whose utility is superior to $\tau(S)$. While written in terms of utility, this is tantamount to requiring that there exist an index function k and a preference ordering such that the DM chooses all the top $k(S)$ elements of S. As a special case, Eliaz et al. (2011) study the method of selecting two finalists, while Chambers and Yenmez (2017) analyze a DM who picks all the top q options according to an ordering. Testing amounts to checking the acyclicity of the revealed preference where chosen elements are revealed preferred to unchosen alternatives. Contrary to these papers, we don't assume that the DM necessarily deems unacceptable an option that we haven't yet seen her choose. In that case, Aleskerov et al.'s model has no testable implications.[2] In light of this, it is important to emphasize that we envision the index function k as fixing a theory; it is something the modeler chooses a priori when selecting a theory to test. Testing gets subtler than

[2]Indeed, any observed choice function is consistent in that sense with the theory, simply using $k(S) = |S|$ for each choice problem S.

checking for acyclicity of a revealed preference, but remains tractable.

The results in Section 3 are similar to those in Barberà and Neme (2014), which is superseded by this current paper, and also appeared in a supplement to de Clippel and Rozen (2012), as one of the applications of the general methodology there. The testing methodology we follow is in line with de Clippel and Rozen (2012) for all the theories analyzed in the present paper (see Sections 3, 5, 6, and 7). We start by identifying restrictions that observed choices place on the DM's preferences should she comply with the theory, and then show that all relevant restrictions have been unearthed by proving that the existence of an acyclic relation satisfying them guarantees consistency. Though more complex than direct comparisons between pairs of options, these restrictions oftentimes pertain to the lower-contour set of an option, or of a set of options. In that case, testing can be done tractably using de Clippel and Rozen's enumeration procedure. Tests in this paper are instances of this procedure. In the presence of more complex restrictions, testing is often NP-hard, as suggested by Proposition 3 of de Clippel and Rozen (2018). That result is used here to establish the NP-hardness of testing Sen (1993)'s theory of choosing the second best.

2. Order-k Rationality

Denote by X the finite set of all conceivable alternatives. A choice problem is any nonempty subset of X. A choice function singles out an element of each choice problem. An *index function* is a mapping k that associates to each choice problem S a strictly positive index $k(S)$ that is smaller than or equal to the number of elements in S. Under *Order-k Rationality*, the DM has a strict preference ordering $\succ$ and picks an option that is either the 1^{st}-best, the 2^{nd}-best, $\ldots$, or the $k(S)^{th}$-best option in S according to $\succ$.

We have thus defined a class of choice theories, one for each index function k. The special case where $k(S) = 1$ for each choice problem S corresponds to the theory of rationality. There are various circumstances where it makes sense to consider larger k's, so as to capture bounded rationality.[3] It is up to the

[3]Notice that any choice function that is consistent with Order-k Rationality is also con-

modeler to conjecture which index function is appropriate for the situation at hand (that is, the modeler must decide on a theory). The modeler must then check whether observed choices are consistent with the theory, for instance by using the efficient testing method we provide in Section 3.

'Good-enough' decision making, leading to Order-k Rationality, arises naturally in different circumstances. As seen in the heuristics below, the notion of 'good enough' may apply to the preference or the consideration set (or even both, as considered in Section 6).

Relative satisficing. The first type of decision making captures a relative form of satisficing:[4] the DM considers an option 'good enough' if it is among her top $k(S)$ alternatives in the choice problem S. For instance, the DM may be willing to settle for second best, but no less. Modeling $k(S)$ as a fixed percentage of S corresponds to meeting quantile criteria, such as being ranked above the median, or in the top quintile. To illustrate, suppose the DM is a new car shopper with a preference ranking over possible vehicle specifications (model trims and color combinations). She knows what specifications she can afford, but not in which dealerships they can be found; and she may choose to stop visiting dealerships once she finds a car that she ranks highly enough. Beyond sequential search, there may be different reasons for choosing something that is only 'good enough,' and different procedures for reaching that preference threshold. Order-k Rationality is meaningful whatever the procedure and motivations the DM follows to identify the good-enough option.

Minimal consideration. Another 'good-enough' heuristic applies to the consid-

sistent with Order-k' Rationality if $k'(S) \geq k(S)$ for all choice problems S.

[4]We use the term *relative* because the simplest incarnation of satisficing uses a single aspiration level, which fixes one benchmark alternative. However, Simon's discussion of satisficing is more nuanced, noting that the aspiration level may vary with the success of one's search: "A vague principle would be that as the individual, in his exploration of alternatives, finds it easy to discover satisfactory alternatives, his aspiration level rises; as he finds it difficult to discover satisfactory alternatives, his aspiration level falls." (Simon 1955, p. 111) If the aspiration level shifts dynamically and the realized sequence of examination is unobserved, then Order-k Rationality would arise, with the index function k given by the worst possible aspiration level for each set.

eration set instead of preference: the DM may feel it is sufficient to contemplate and compare a certain number of alternatives when making a choice. There are many rules of thumb that lead to such behavior. When searching on Google, for instance, the DM may restrict attention to the first page of results. More broadly, Masatlioglu et al. (2012) suggest that the DM may form her consideration set by taking the top-n options according to some auxiliary criterion (e.g. price, quality, order of appearance). In general, the choice problem may affect the number of alternatives considered. For instance, the DM may insist on contemplating a certain proportion of alternatives; or she may be overwhelmed by large choice sets, with the number of alternatives she considers decreasing in S. Whatever it is, the DM's rule for the size of the minimal consideration set has direct meaning for the index function: if at least $n(S)$ alternatives are considered in each set S, then the DM's behavior is described by Order-k rationality with the index function given by $k(S) = |S| - n(S) + 1$ for all S.

Remark 1. *While Order-k Rationality presumes a single preference ordering, it can also provide a useful framework for understanding some testable implications of multi-self and interactive decision making. Consider the real-life problem of arbitrator selection (see e.g. de Clippel et al. (2014) and references therein), which is commonly implemented using an alternate-strikes procedure. Anbarci (1993) showed that backward induction outcomes must fall above the median of each party's preference ordering.*[5] *Thus Order-k Rationality, with the index function given by* $k(S) = \lfloor \frac{|S|}{2} \rfloor$ *for all* S*, is a necessary condition for the backward induction equilibrium.*

3. Testable implications through k-SARP

We now investigate the testable implications of Order-k Rationality. As we cannot observe the DM's thought process, we cannot identify for certain what

[5]More generally, one expects this to be a property of fair bargaining outcomes; the equilibrium of other non-cooperative procedures have this property as well, such as Anbarci's method of voting by alternating offers and vetoes, or de Clippel et. al.'s shortlisting method, a special instance of the rule of k-names (see Barberà and Coelho (2010, 2017, 2018)).

theory she uses. However, we can observe the DM's choices. By understanding the regularities Order-k Rationality imposes on choice behavior, we know when it is consistent with the DM's choices, and when it is refuted.

Suppose we observe the DM making a choice for each of several choice problems. Formally, this can be captured by an *observed choice function* $c_{obs} : \mathcal{D} \to X$, where the *dataset* $\mathcal{D}$ is the class of choice problems for which observations have been collected. The observed choice function c_{obs} *is consistent with* Order-k Rationality if there is a strict preference $\succ$ such that $c_{obs}(S)$ is one of the top $k(S)$ options in S, for each $S \in \mathcal{D}$. Otherwise, the observed choices refute the theory. Our goal is to design a simple test to inform us, for each observed choice function, whether it is consistent with the theory or refutes it.

One can test the theory of rationality (the special case $k(S) = 1$ for all S) by checking whether Samuelson's revealed preference is acyclic; this is known as the Strong Axiom of Revealed Preference, or SARP. Under rationality, x is revealed preferred to y if x is picked in the presence of y, that is, there exists $S \in \mathcal{D}$ containing y and such that $c_{obs}(S) = x$.

Such an inference is invalid if $k(S) \geq 2$, since there may exist alternatives in S that are preferred to the choice. However, observed choices do impose restrictions on what the DM's preference may look like: there must exist $|S| - k(S)$ alternatives in S that are all inferior to x. We now show that observed choices are consistent with Order-k Rationality if and only if there exists an acyclic (strict) relation satisfying such restrictions. This condition corresponds to a generalization of SARP:

k**-SARP** *There exists an acyclic (strict) relation P such that, for each $S \in \mathcal{D}$, there is a set $S' \subset S$ of $|S| - k(S)$ alternatives with $c_{obs}(S)Py$, for all $y \in S'$.*

After proving the result, we will propose a practical way of checking k-SARP, illustrating how it is comparable in complexity to testing SARP.

Theorem 1. *The observed choice function c_{obs} is consistent with Order-k Rationality if and only if k-SARP is satisfied.*

Proof. Necessity follows from the discussion above. As for sufficiency, let P

be such an acyclic relation, and let $\succ$ be a completion of P into an ordering. Since P-comparisons remain valid under $\succ$, $c_{obs}(S) \succ y$ for all $y \in S'$, given any $S \in \mathcal{D}$. Thus $c_{obs}(S)$ is among the top $k(S)$ options in S according to $\succ$, and c_{obs} is consistent with Order-k Rationality. $\square$

One might conjecture that checking k-SARP is difficult, as it requires checking for the existence of an acyclic relation satisfying certain restrictions (as opposed to SARP, which requires checking that a given relation is acyclic). Fortunately, that is not the case: testing is of comparable complexity to testing Rationality. We now explain how the testing is implemented.

Let X_1 be the image of c_{obs}, and let $Y_1 = X \setminus X_1$. Intuitively, Y_1, the set of elements never chosen, contains obvious candidates for the DM's worst elements, with those in X_1 ranked somewhere above those in Y_1. The data may suggest further information, however, on how elements in X_1 rank in relation to each other. For each $x \in X_1$, let $c_{obs}^{-1}(x)$ be the set of choice problems $S \in \mathcal{D}$ such that $c_{obs}(S) = x$. Consider the set

$$Y_2 = \{x \in X_1 \mid |S \cap Y_1| \geq |S| - k(S), \text{ for all } S \in c_{obs}^{-1}(x)\},$$

which is the set of $x \in X_1$ such that for every set S in which x is chosen, x already belongs to the top $k(S)$ elements, either because there are sufficiently many worse-ranked elements (i.e., those in Y_1), or because the size of the set is at most $k(S)$ (e.g., it is a two-element set and the theory is that the DM chooses from the top two). As elements in Y_2 are not forced by the theory to be better than any other element in X_1, the set Y_2 contains obvious candidates for the DM's worst elements in X_1. After taking out the elements of Y_2 and ranking these somewhere above the elements of Y_1, we would next want to investigate the set of remaining elements, $X_2 = X_1 \setminus Y_2$. Building on this idea, we can define by induction two sequences of sets:

$$\begin{aligned} Y_{\ell+1} &= \{x \in X_\ell \mid |S \cap [\cup_{i=1}^{\ell} Y_i]| \geq |S| - k(S), \text{ for all } S \in c_{obs}^{-1}(x)\}, \\ X_{\ell+1} &= X_\ell \setminus Y_{\ell+1}. \end{aligned} \tag{1}$$

In each step ℓ, the set $\cup_{i=1}^{\ell} Y_i$ represents the set of elements that have already been ranked below the remaining elements (those in X_ℓ). Similarly, $Y_{\ell+1}$ looks for candidates for worst elements in X_ℓ, based on checking which chosen elements already have enough worse-ranked elements that they automatically belong to the top $k(S)$ whenever chosen in some S. The set $Y_{\ell+1}$ is then removed, and ranked somewhere above those previously removed (i.e., $\cup_{i=1}^{\ell} Y_i$), to generate the next set of remaining elements, $X_{\ell+1}$. Clearly, $X_{\ell+1} \subseteq X_\ell$ for each ℓ. Given that X_1 has at most $|\mathcal{D}|$ elements, the sequence $(X_\ell)_{\ell \geq 1}$ becomes constant in at most that many steps. Let X^* be this limit set, that is, $X^* = X_\ell$ where ℓ is the lowest index such that $X_\ell = X_{\ell+1}$. The next result shows that k-SARP holds (and thus c_{obs} is consistent with Order-k Rationality by Theorem 1) if and only if $X^* = \emptyset$. Thus k-SARP can be checked in polynomial time (in the cardinalities of X and $\mathcal{D}$).

Theorem 2. *k-SARP holds if and only if $X^* = \emptyset$.*

Proof. (*Sufficiency*) Suppose that $X^* = \emptyset$. Let ℓ be the smallest index such that $X_\ell = X^*$. Consider the partition $\{Y_1, \ldots, Y_L\}$ of X, and the (strict) relation P defined by $x \succ y$ if the atom of the partition to which x belongs has a larger index than the atom to which y belongs. Clearly P is acyclic, and k-SARP is satisfied using P.

(*Necessity*) Suppose that the limit set X^* is nonempty. Let $Y^* = X \setminus X^*$. If k-SARP is satisfied using P, then the restriction of P to X^* must be such that $|\{y \in S \mid c_{obs}(S)Py\}| \geq |S| - k(S)$ for all $S \in \mathcal{D}$ such that $c_{obs}(S) \in X^*$. Decomposing the lower-contour set into two components,

$$|\{y \in S \cap X^* \mid c_{obs}(S)Py\}| + |\{y \in S \cap Y^* \mid c_{obs}(S)Py\}| \geq |S| - k(S)$$

and thus $|\{y \in S \cap X^* | c_{obs}(S)Py\}| + |S \cap Y^*| \geq |S| - k(S)$ for all $S \in \mathcal{D}$ such that $c_{obs}(S) \in X^*$, since in the most permissive scenario, all elements in $S \cap Y^*$ are P-inferior to $c_{obs}(S)$. Acyclicity implies that one can find $x \in X^*$ for which there is no $y \in X^*$ such that xPy. For all $S \in c_{obs}^{-1}(x)$, it must be that $0 \geq |S| - k(S) - |S \cap Y^*|$, which cannot be if X^* is the limit set. Thus X^* must be empty if k-SARP holds, as desired. $\square$

Theorems 1 and 2 are similar to those derived in Barberà and Neme (2014)[6], and are special instances of the general methodology suggested and pursued for a variety of theories in de Clippel and Rozen (2012, 2018). According to that methodology, testable implications are first expressed in terms of the existence of an acyclic relation satisfying some restrictions (as in k-SARP). Second, provided that the restrictions all correspond to 'lower-contour sets' (as is the case for k-SARP), testing can be done using a simple enumeration procedure (similar to the iterative process used to define X^*) that is comparable in complexity to checking whether a given relation is acyclic (as is case when testing Rationality through SARP). de Clippel and Rozen show how this methodology can be followed successfully for different theories of bounded rationality, but by far not all. The present paper establishes that Order-k Rationality provides another interesting class of theories where the methodology can be followed successfully.

Remark 2. *Not all theories of bounded rationality can be tested easily: some are computationally complex to test, namely NP-hard. The simplicity of testing Order-k Rationality may be surprising. Indeed, consider Sen's (1993)'s theory of choosing the second best according to a preference ordering (as opposed to choosing one of the top two elements, as permitted by Order-k Rationality with $k(S) = 2$ for all S). Under full data (when the entire choice function is observed), Baigent and Gaertner (1996) characterize Sen's theory using simple and easy-to-check axioms.*[7] *They do not capture, however, the empirical content of this theory when data is limited. In this general case, Sen's rather simple theory turns out to be NP-hard to test, as we prove in Theorem 9 of the Appendix. The result follows as a corollary of de Clippel and Rozen (2018)'s Proposition 3, by showing that their NP-hard problem of testing acyclic satisfiability of a 'mixed set of binary restrictions' is reducible in polynomial time to testing Sen's theory. Hence, determining whether choices are second-best*

[6]This paper supersedes Barberà and Neme (2014).

[7]For single-value choice functions, the two axioms to consider are simply: (i) if x is picked both from $\{x, y\}$ and $\{x, z\}$, then x is not picked from problems containing x, y, and z, and (ii) each problem S admits an element $y \neq c(S)$ such that $c(\{y, z\}) = z$ for all $z \in S$ different from y.

for some preference ordering can be hard, but checking whether it is top best for some ordering (Rationality), or testing whether it is among the top two elements for some ordering (2-Rationality) is relatively easy. This is because k-SARP is a natural and tractable generalization of SARP.

4. Forecasting and Preference Identification

A modeler may have further questions upon determining that observed choices are consistent with a theory. For instance, what forecasts can be made regarding potential choices in unobserved choice problems? Can the DM's preference over any pairs of alternatives be unambiguously identified? We address each of these questions in turn.

4.1 Forecasting

Having observed the DM's selections for choice problems in $\mathcal{D}$, the modeler would like to guess the DM's choice in an out-of-sample problem S. The possible guesses consistent with the theory comprise each $x \in S$ such that there is a complete choice function c under Order-k Rationality that extends c_{obs} and that has $c(S) = x$. Hence, the problem of forecasting is intimately related to the problem of testing consistency. Indeed, determining whether $x \in S$ is a possible choice is tantamount to testing whether the extended observed choice function $\bar{c}_{obs} : \mathcal{D} \cup \{S\} \to \mathcal{P}(X)$, given by $\bar{c}_{obs}(T) = c_{obs}(T)$ for $T \in \mathcal{D}$ and $\bar{c}_{obs}(S) = x$, is consistent with Order-k Rationality. As we know how to tractably test consistency, we can easily generate forecasts for S.

4.2 Identification

Even with a complete dataset ($\mathcal{D} = \mathcal{P}(X)$), it is sometimes possible for different preference orderings to generate the same observed choices under Order-k Rationality. Recalling the construction from Section 3, if the sequence of sets $Y_1, \ldots, Y_L$ is built according to (1) and partitions X, then it is easy to see that there are at least $\prod_{\ell=1}^{L} |Y_\ell|!$ possible preference orderings. Nonetheless, there are choice configurations that pin down the preference between some alternatives. Formally, suppose c_{obs} has already been deemed consistent with Order-k Rationality. Then x is *revealed preferred to y* if xPy for any preference ordering P that generates c_{obs}. Identifying whether x is revealed preferred to y is therefore equivalent to ruling out the possibility of a preference P with yPx generating observed choices.

To illustrate, suppose X has five elements. The modeler posits that $k(S) = 2$ for all S. Observed choices $c_{obs}(\{x_1, x_2, x_3, x_5\}) = x_1$ and $c_{obs}(\{x_1, x_2, x_3, x_4\}) = x_2$ are consistent with Order-k Rationality using various preferences, including for instance $x_1Px_2Px_3Px_4Px_5$, $x_5Px_1Px_2Px_3Px_4$, and $x_4Px_2Px_1Px_3Px_5$, among others. In *all* rationalizing preferences, however, both x_1 and x_2 must be ranked above x_3; that is, x_1 and x_2 are both revealed preferred to x_3. Intuitively, x_1, x_2, x_3 all appear in two choice problems, where two of these are observed to be chosen; as $k(S) = 2$, this leaves no space for x_3 to be ranked above either.[8]

In general, we may use the testing technique of Theorems 1 and 2 to address the problem of identification. Suppose one wonders whether x is revealed preferred to y. Then it would be impossible to find a preference ordering ranking y above x and generating the observed choices under Order k-Rationality. In other words, x is revealed preferred to y if and only if k-SARP, augmented by adding the restriction yPx on the acyclic relation P, fails. Modifying the definition of $Y_{\ell+1}$ by dropping y for all ℓ such that $x \in X_\ell$ then provides a simple test as in Theorem 2.

5. Enhancing Existing Theories

Order k-Rationality can be interpreted as capturing a DM whose attention may be limited, but who is known nevertheless to be minimally attentive. This condition on attention differs from earlier approaches, which typically restrict how attention sets vary across choice problems. For instance, Lleras et al. (2017) captures choice overload by requiring the DM's attention correspondence to satisfy IIA: any option paid attention to in large problems is

[8]Formally, the first observed choice reveals that at most one element among x_2, x_3 and x_5 is preferred to x_1. The second observed choice reveals that at most one element among x_1, x_3 and x_4 is preferred to x_2. But if x_3 is the one alternative preferred to x_2 (x_1), and if at most one of these can be preferred to x_1 (x_2), then its availability in both observed problems renders it impossible to explain the choices.

also paid attention to in smaller problem that contains it. This property also describes the DM's attention in theories suggested by Manzini and Mariotti (2012) and Cherepanov et al. (2013).

While it is both intuitive and well-documented that a DM may be overloaded when facing too many options, it may be unrealistic that she would pay attention to only a single option in a problem. This, however, is a feature of certain choice behaviors that some theories of attention explain. To illustrate, consider the following observed choice function: $c_{obs}(X) = x$ and $c_{obs}(\{x, y\}) = y$, for all $y \neq x$. These choices are consistent with the theory of Lleras et al., but only if x is the *only* option that the DM considers when facing X, no matter how large or small X is. This is because IIA requires the DM to consider x in each pair containing it, and hence the DM must prefer all alternatives to x.

One may desire to enhance theories of attention by adding a lower bound on the number of alternatives considered. Doing so retains their appealing features, while restraining over-permissiveness. Here, we consider this approach for the theory of Choice Overload. For each choice problem S, let $\alpha(S)$ be the modeler's lower bound on the number of options that the DM considers when facing S. Under *Choice Overload with α-Minimal Attention*, the DM has a preference ordering P and an attention correspondence $A : \mathcal{P}(X) \to \mathcal{P}(X)$ such that:

(1) A satisfies IIA,

(2) $A(S)$ is a non-empty subset of S, for each S,

(3) $A(S)$ contains at least $\alpha(S)$ elements, for each S,

(4) The choice from S is the P-maximal element in $A(S)$, for each S.

To understand the testable implications of this new theory, we follow the methodology suggested by de Clippel and Rozen (2012), by looking for restrictions that observed choices place on the DM's preference should her choices be consistent with the theory. One type of restriction arises from the IIA property: it implies that xPy whenever $x = c_{obs}(S)$, $y = c_{obs}(T)$, and $y \in S \subset T$. Denote by $\mathcal{R}_{IIA}$ the set of all such restrictions. This revealed preference was identified by Manzini and Mariotti (2012), Cherepanov et al. (2013) and Lleras et al. (2017). Another type of restriction arises from the minimal consideration property (3), which requires the DM's choices to be in the top $k(S) = |S| - \alpha(S) + 1$ options according to her preference. Thus, as in Section 3, one must add a restriction that at least $\alpha(S) - 1$ elements of S are P-inferior to $c_{obs}(S)$. Let $\mathcal{R}_\alpha$ be the set of all such restrictions.

To summarize, consistency with the theory requires the existence of an acyclic (strict) relation satisfying the restrictions listed in $\mathcal{R}_{IIA} \cup \mathcal{R}_\alpha$. This is a more restrictive condition than satisfying either $\mathcal{R}_{IIA}$ or $\mathcal{R}_\alpha$ alone, as it is possible to satisfy one but not the other. Does the enhanced theory entail any restrictions beyond these two types? The next theorem confirms that $\mathcal{R}_{IIA} \cup \mathcal{R}_\alpha$ fully captures the empirical content of Choice Overload with α-Minimal Attention. This conclusion is nontrivial: as we will demonstrate below for another major theory, an enhanced theory need not always be characterized by the union of the restrictions of the original theory with $\mathcal{R}_\alpha$.

Theorem 3. *An observed choice function c_{obs} is consistent with Choice Overload with α-Minimal Attention if and only if there is an (strict) acyclic relation P satisfying the restrictions in $\mathcal{R}_{IIA} \cup \mathcal{R}_\alpha$.*

Proof. Necessity follows from the discussion above. As for sufficiency, let P be an acyclic relation satisfying $\mathcal{R}_{IIA} \cup \mathcal{R}_\alpha$, and let $\succ$ be a completion of P into an ordering. Clearly, $\succ$ also satisfies $\mathcal{R}_{IIA} \cup \mathcal{R}_\alpha$. For each choice problem S, let $B(S)$ be the bottom $\alpha(S)$ elements of S according to $\succ$, let

$$A^*(S) = S \cap \{c_{obs}(T) \mid T \in \mathcal{D},\ S \subseteq T\},$$

and let $A(S) = A^*(S) \cup B(S)$. We conclude the proof by showing that $c_{obs}(S)$ is the $\succ$-maximal element of $A(S)$ for each $S \in \mathcal{D}$. By definition, $c_{obs}(S) \in A^*(S) \subseteq A(S)$. Suppose, by contradiction, that there exists $x \in A^*(S)$ such that $x \succ c_{obs}(S)$. If $x \in A^*(S)$, then $c_{obs}(S) \succ x$ since $\succ$ satisfies the restrictions in $\mathcal{R}_{IIA}$. This cannot be, so $x \in B(S)$. Then $c_{obs}(S)$ has at most $\alpha(S) - 2$ that are $\succ$-inferior to it in S, since x has at most $\alpha(S) - 1$ below it ($x \in B(S)$)

and c_{obs} is $\succ$-inferior to x. This contradicts $\succ$ satisfying the restrictions in $\mathcal{R}_\alpha$. Hence no such x exists, and $c_{obs}(S)$ is the $\succ$-maximal element of $A(S)$. $\square$

The tractability of testing Order-k Rationality extends to Choice Overload with α-Minimal Attention. Indeed, testing can be implemented using a similar approach. Let $\tilde{X}_1$ be the image of c_{obs}, and let $\tilde{Y}_1 = X \setminus \tilde{X}_1$. Letting $k(S) = |S| - \alpha(S) + 1$, define by induction, for each $\ell \geq 1$:

$$\begin{aligned}\tilde{Y}_{\ell+1} =& \{x \in \tilde{X}_\ell \mid |S \cap [\cup_{i=1}^{\ell} \tilde{Y}_i]| \geq |S| - k(S), \text{ for all } S \in c_{obs}^{-1}(x),\\ &\text{and } \nexists y \in \tilde{X}_\ell \setminus \{x\} \text{ s.t. } x = c_{obs}(T),\ y = c_{obs}(T') \text{ for } x, y \in T \subset T'\}\\ \tilde{X}_{\ell+1} =& \tilde{X}_\ell \setminus \tilde{Y}_{\ell+1}.\end{aligned}$$

The sequence of $\tilde{Y}_\ell's$ differs from its Section 3-counterpart in that excludes alternatives which are ranked above another remaining one according to the IIA-based revealed preference. The sequence $\tilde{X}_\ell$ is decreasing, and becomes constant in at most $|\mathcal{D}|$ steps. Letting $\tilde{X}^*$ be this limit set, it is easy to see that c_{obs} is consistent with Choice Overload with α-Minimal Attention if and only if $\tilde{X}^* = \emptyset$. Thus the theory can be checked in polynomial time (in the cardinality of $\mathcal{D}$). Moreover, preference identification can be performed as in Section 4.

Theorem 4. *c_{obs} is consistent with Choice Overload with α-Minimal Attention if and only if $\tilde{X}^* = \emptyset$.*

Proof. (*Sufficiency*) Suppose $\tilde{X}^* = \emptyset$. Let ℓ be the smallest index such that $\tilde{X}_\ell = \tilde{X}^*$. Consider the partition $\{\tilde{Y}_1, \ldots, \tilde{Y}_L\}$ of X, and the (strict) relation P defined by $x \succ y$ if $x \in \tilde{Y}_j$ and $y \in \tilde{Y}_k$, with $j > k$. Since P is acyclic and satisfies $\mathcal{R}_{IIA} \cup \mathcal{R}_\alpha$, Theorem 3 applies.

(*Necessity*) Suppose that the limit set $\tilde{X}^*$ is nonempty yet there exists a preference ordering P from which the choices arise under Choice Overload with α-Minimal Attention. Following the same reasoning as in the proof of Theorem 2, acyclicity of P implies that one can find $x \in X^*$ for which there is no $y \in \tilde{X}^*$ such that xPy, yet for all $S \in c_{obs}^{-1}(x)$, it must be that $0 \geq |S| - k(S) - |S \cap \tilde{Y}^*|$, where $\tilde{Y}^* = X \setminus \tilde{X}^*$. Since $\tilde{X}^*$ is the limit set, this is possible only if there exists $y \in \tilde{X}^* \setminus \{x\}$ such that $x = c_{obs}(T)$ and $y = c_{obs}(T')$ for $x, y \in T \subset T'$. As P must satisfy $\mathcal{R}_{IIA}$, this means xPy. Thus x is not P-minimal in $\tilde{X}^*$, a contradiction. Hence $\tilde{X}^*$ must be empty, as desired. □

Theorem 3 showed that the testable implications of Choice Overload with α-Minimal Attention boil down to testing whether there is a (strict) acylic relation satisfying the union of $\mathcal{R}_\alpha$ with the theory's original restrictions. When enhancing other theories with α-Minimal Attention, however, further restrictions could potentially be generated. Recall Masatlioglu et al. (2012)'s theory, which posits that the DM has a preference ordering and a consideration set mapping that is unchanged when unconsidered alternatives are taken away; that is, they require that $A(S) = A(T)$ whenever $A(T) \subseteq S \subset T$. Suppose we see the following dataset when $X = \{a, b, x, y, z\}$:

S	$\{a,b,x,z\}$	$\{a,b,x\}$	$\{a,b,y,z\}$	$\{a,b,y\}$	$\{x,y,z\}$
$c_{obs}(S)$	x	a	y	a	z

These observed choices are consistent with α-Minimal Attention for $\alpha(S) = 2$ for all S. To see this, take the preference ordering P given by $zPxPyPaPb$. These observed choices are also consistent with Masatlioglu et al. (2012)'s theory. To see this, take the preference ordering P given by $xPyPzPaPb$ and the consideration set mapping A defined by $A(\{x,y,z\}) = A(\{x,z\}) = A(\{y,z\}) = z$, $A(\{a,b,x\}) = A(\{a,b,y\}) = \{a,b\}$ and $A(S) = S$ otherwise. Thus there exists an acyclic relation satisfying the union of the restrictions for these two theories.[9] However, this data is inconsistent with an enhancement of Masatlioglu et al. (2012)'s theory in which we require the DM to pay attention to at least two elements in every choice problem. To see this, suppose by contradiction that the choices are consistent with this new theory. Observe that z must be considered in $\{a,b,x,z\}$ (since the choice switches to a when z is dropped), so x is revealed preferred to z. Similarly, z must be considered in $\{a,b,y,z\}$, so y is revealed preferred to z. This yields a contradiction: because

[9]See de Clippel and Rozen (2012) for the limited-data restrictions for Masatlioglu et al. (2012)'s theory.

at least two options must be considered, z cannot be chosen from $\{x, y, z\}$.

6. Determining 'good enough' endogenously

Encountering options sequentially may impact a DM's choices. In such cases, a 'good-enough' choice heuristic may be particularly salient due to the inclination to stop looking after having reviewed sufficiently many options. Following Rubinstein and Salant (2006), we formalize a *list* as a sequence $(x_k)_{k=1}^{K}$ of distinct options in X. The notion of a list refines that of a choice problem: one knows *both* the feasible set of options and the order in which they appear to the DM. Observing a DM's choices over lists in which the same options appear in a different order thus provides richer data, with the potential for more stringent tests and better identification.

In this section, we envision the DM as first reviewing a given fraction of the list to set up a threshold, and picking the first option that surpasses that threshold according to her preference. For each list ℓ, let $n(\ell)$ be a number between 1 and the length of the list. The DM is assumed to set up her threshold by reviewing the first $n(\ell)$ options in the list. Like the index function k, the modeler chooses the function n and sets out to test the resulting theory. A natural threshold would be to set n according to a fixed fraction of the list's length, but more complex functions can be accommodated without complication. Let $\tau(\ell)$ be the best option in the first $n(\ell)$ elements of the list ℓ, according to the DM's (unknown) preference ordering $\succ$. The DM's choice is then the first option among the remaining ones in the list that is $\succ$-superior to $\tau(\ell)$, if any, and is otherwise $\tau(\ell)$.[10] This procedure thus combines features of both relative satisficing and limited attention, as the threshold for each list is set by paying attention to a limited number of options.

In this framework, complete data would mean observing the DM's choice from every list. Testing is rather straightforward in that case. For each set $S \subseteq X$, one looks for a list of S where the DM picks its first element, and

[10] This bears similarity to the classic secretary problem, but departs from it in the ability to choose an element, $\tau(\ell)$, from earlier in the list.

denotes that element $c^*(S)$.[11] Consistency occurs if and only if (a) the function c^* satisfies IIA (is rational), and (b) observed choices can be derived by applying the choice procedure to the preference ordering revealed by c^*, i.e., x is preferred to y if and only if $c^*(\{x, y\}) = x$.

Of course, having access to such rich data is implausible. However, the general testing methodology of de Clippel and Rozen (2012) can be applied to obtain a tractable test of this theory for any dataset. One starts by identifying restrictions that observed choices place on the DM's preference, should her choices conform to the theory. First, if she chooses one of the first $n(\ell)$ options in a list, then the DM must prefer it over all alternatives in the list. Second, the element the DM chooses from a list must be preferred to any alternative that precedes it. Third, any element appearing between the $n(\ell) - th$ element of the list and before the chosen element must be inferior to at least one of the first $n(\ell)$ options in the list. Consistency with the theory thus requires the existence of an acyclic relation satisfying all these restrictions implied by observed choices. Do observed choices reveal any other essential restrictions on the DM's putative preference? No, as the next theorem shows.

Theorem 5. *Observed choices are consistent with the theory if and only if there exists an (strict) acyclic relation P such that, for each observed list ℓ,*

(a) If the observed choice x is one of the first $n(\ell)$ elements, then xPy for all $y \neq x$ in the list;

(b) If x is the observed choice, then xPy for all y preceding x;

(c) For any y following the first $n(\ell)$ elements but preceding the observed choice, there exists x among the first $n(\ell)$ elements such that xPy.

Proof. Necessity was proved above. As for sufficiency, we can assume without loss of generality that P is an ordering, as any acyclic relation can be completed into an ordering and the completion will still satisfy the listed restrictions. We conclude the proof by checking that observed choices can be derived from the

[11] Consistency with the theory requires the existence of such a list, e.g. any list that has the DM's most-preferred element of S in first position.

theory by using P as the DM's preference. Let ℓ be an observed list. Suppose that the observed choice x is one of the first $n(\ell)$ options. By (a), x is P-maximal in the set of elements that appear in ℓ, as desired. Suppose now x is not one of the first $n(\ell)$ options. By (b), it is P-superior to all preceding options. By (c), x is the first option in the list that succeeds the initial $n(\ell)$ options, and is P-superior to all of them. Thus indeed, the theory applied to P selects x out of ℓ, as desired. $\square$

7. Contextual effects and Multi-valued Choice

Multiple options are satisfactory to the DM when $k(S) \geq 2$. Our theory does not describe how the DM settles on a specific one. It might be, for instance, that she reviews options in some order, and settles on the first that falls among the top $k(S)$. The modeler may not know what this order is, either because it is subjective or unrecorded. More generally, encountering the same choice problem S on different occasions may lead to different choices among the top $k(S)$ for reasons the modeler cannot discern. The data collected then takes the form of an observed choice correspondence, that associates to each $S \in \mathcal{D}$ a nonempty subset $C_{obs}(S) \subseteq S$ of options that the DM has picked when facing S on various occasions. The observed choice correspondence C_{obs} is consistent with Order-k Rationality if there is a preference ordering $\succ$ for which all options in $C_{obs}(S)$ fall within the $\succ$-top $k(S)$ options in S, for each $S \in \mathcal{D}$.

Of course, k-SARP must hold with respect to each selection of C_{obs}. However, consistency is more demanding than this condition. The reason is that the *same* preference ordering must explain all observed choices, including when multiple ones have been observed for the same problem. If observed choices were consistent with the theory, then the following restrictions on the DM's putative preference could be inferred: for each $S \in \mathcal{D}$, there must exist $|S| - k(S)$ alternatives in S, *each* of which is inferior to *all* elements of $C_{obs}(S)$.

Multi-Valued k-SARP *There exists an acyclic (strict) relation P such that, for each $S \in \mathcal{D}$, there is a set $S' \subset S$ of $|S| - k(S)$ alternatives such that xPy,* *for all $x \in C_{obs}(S)$ and all $y \in S'$.*

Both Theorems 1 and 2 extend using Multi-Valued k-SARP, thereby providing a tractable test of the theory for correspondences as well. The proofs are almost identical to those before, and left to the reader.

Theorem 6. *C_{obs} is consistent with Order-k Rationality if and only if Multi-Valued k-SARP is satisfied.*

Theorem 7. *Multi-Valued k-SARP holds if and only if $X^* = \emptyset$, where X^* is the limit of decreasing sequence of sets $(X_\ell)_{\ell \geq 1}$ as in Theorem 2 after replacing c_{obs} by C_{obs}.*[12]

8. A Cardinal Approach to 'Good Enough'

Our notion of 'good enough' is defined in relative terms, rather than through utilities. A more general framework would allow cardinal information to matter. Here, we introduce and study an intuitive model of search where the notion of good enough is defined cardinally. The perhaps surprising conclusion is that its empirical content is indistinguishable from an ordinal theory (more specifically, one of the theories in the Order-k Rationality class). We leave the study of other cardinal models to future work.

Consider a DM with unit demand searching for a product, say a car for concreteness. A car is described by an array of characteristics (e.g., colors, seating materials, packages), and is captured by the variable x. Let X be the finite set of possible car types. Prices of different car types and the DM's budget determine her feasible set S. The DM has a strict Bernoulli utility function $u : X \to \mathbb{R}$ over car types. The DM knows the measure $p(x) > 0$ of cars of any type $x \in X$, but not where to locate such a car. As such, she must search sequentially through different car lots. The DM considers a car of type x 'good enough,' and concludes her search, if and only if its utility $u(x)$ is at least as high as her expected utility $\sum_{x \in S} p(x|S)u(x)$ from continuing, where

[12] The image of C_{obs} is the set of options in X that belong to $C_{obs}(S)$ for some $S \in \mathcal{D}$, and $C_{obs}^{-1}(x)$ is the union of all problems $S \in \mathcal{D}$ such that $x \in C_{obs}(S)$.

$p(x|S) = p(x)/\sum_{x' \in S} p(x')$. The DM's choices are consistent with this model if there exists a strict Bernoulli u such that the DM's choice in each observed choice problem S qualifies as good enough in this sense.

The above model has a rich structure, and one might expect its testable implications to go beyond those of Order-k Rationality. Taking an increasing transformation of the DM's Bernoulli utility function could certainly change her choices. However, the modeler does not know the DM's utility function and has the flexibility of choosing it when checking consistency of observed choices with the theory. As it turns out, the testable implications boil down to those of an ordinal theory.

Theorem 8. *Observed choices are consistent with this cardinal model if and only if they are consistent with Order-k Rationality, for $k(S) = |S| - 1 \ \forall\ S$.*

In other words, we can only conclude that the DM does not choose her worst element from any set. Theorem 8 takes as input any fixed, full-support probability distribution p. A fortiori, the theorem also holds when both the distribution p and the utility function u are taken to be subjective.

There are other cardinal notions of 'good enough' that are more obviously captured by Order-k Rationality, such as falling above some fixed quantile of the DM's utility in the set. In general, however, it is possible for a cardinal model of 'good enough' to have testable implications that cannot be captured by an ordinal one.

References

Aleskerov, F., D. Bouyssou, and B. Monjardet (2007), Utility Maximization, Choice and Preference, Springer Verlag.

Anbarci, N. (1993), Noncooperative Foundations of the Area Monotonic Solution, *The Quarterly Journal of Economics*, 108, 245–258.

Apesteguia, J. and M. Ballester (2015), A Measure of Rationality and Welfare, *Journal of Political Economy*, 123(6), 1278–1310.

Baigent, N. and W. Gaertner (1996), Never Choose the Uniquely Largest: a Characterization, *Economic Theory*, 8, 239–249.

Barberà, S. and D. Coelho (2010), On the Rule of k Names, *Games and Economic Behavior*, 70(1), 44–61.

Barberà, S. and D. Coelho (2017), Balancing the Power to Appoint Officers, *Games and Economic Behavior*, 101, 189–203.

Barberà, S. and D. Coelho (2018), On the Advantages and Disadvantages of Being the First Mover under Rules of k Names, *International Journal of Economic Theory*, 14, 51–60.

Barberà, S. and A. Neme (2014), Ordinal Relative Satisficing Behavior, *Barcelona GSE Working Paper no. 790.*

Bernheim, D. and A. Rangel (2009), Beyond Revealed Preference: Choice-Theoretic Foundations for Behavioral Welfare Economics, *The Quarterly Journal of Economics*, 124(1), 51–104.

Chambers, C., and B. Yenmez (2018), A Simple Characterization of Responsive Choice, *mimeo.*

Cherepanov, V., T. Feddersen and A. Sandroni (2013), Rationalization, *Theoretical Economics*, 8, 775–800.

de Clippel, G., K. Eliaz and B. Knight (2014), On the Selection of Arbitrators, *American Economic Review*, 104, 3434–3458.

de Clippel, G., and K. Rozen (2012), Bounded Rationality and Limited Datasets, *Cowles Discussion Paper 1853.*

de Clippel, G., and K. Rozen (2018), Bounded Rationality and Limited Datasets, *mimeo.*

Eliaz, K., M. Richter and A. Rubinstein (2011), Choosing the Two Finalists, *Economic Theory*, 46, 211–219.

Houthakker, H.S. (1950), Revealed Preference and the Utility Function, *Economica*, 17(66), 159–174.

Lleras, J., Y. Masatlioglu, D. Nakajima, and E. Ozbay (2017), When More is Less: Choice by Limited Consideration, *Journal of Economic Theory*, 170, 70–85.

Manzini, P., and M. Mariotti (2007), Sequentially Rationalizable Choice, *American Economic Review*, 97(5), 1824–1839.

Manzini, P., and M. Mariotti (2012), Categorize Then Choose: Boundedly Rational Choice and Welfare, *Journal of the European Economics Association*, 10(5), 1141–1165.

Masatlioglu, Y., D. Nakajima and E. Ozbay (2012), Revealed Attention, *American Economic Review*, 102(5), 2183–2205.

Rubinstein, A. and Y. Salant (2006), A Model of Choice from Lists, *Theoretical Economics*, 1, 3–17.

Salant, Y. and A. Rubinstein (2008), (A, f): Choice with Frames, *Review of Economics Studies*, 75, 1287–1296.

Samuelson, P. (1938), A Note on the Pure Theory of Consumer's Behavior., *Economica*, 5(17), 61–71.

Sen, A. (1993), Maximization and the Act of Choice, *Econometrica*, 65, 745–779.

Simon, H. (1955), A Behavioral Model of Rational Choice, *Quarterly Journal of Economics*, 69, 99–118.

Appendix

Proof of Theorem 8.

Fix any full-support probability distribution p over X. For any S, let $p(S) = \sum_{x \in S} p(x)$. We prove the result by showing that there exists a utility function $u : X \to \mathbb{R}$ such that for any nonsingleton set S,

$$u(x) \geq \sum_{x' \in S} \frac{p(x')}{p(S)} u(x')$$

for all $x \in S$ except the u-minimal element in S.

Fix an enumeration of X and the ranking where x_i is preferred to x_j when $i < j$. For any x_j, define $\mathcal{S}(x_j)$ to be the set of subsets $S \subseteq X$ such that x_j is the second-worst element in S. For any $S \subset X$, let x_S be the worst element in S. Recursively define $u(x_n) = 0$, $u(x_1) = 1$, and for $j \in \{2, \ldots, n-1\}$,

$$u(x_j) = \max\{\max_{S \in \mathcal{S}(x_j)} \sum_{x \in S \setminus \{x_S\}} \frac{p(x)}{p(S)} + \frac{p(x_S)}{p(S)} u(x_S), \frac{1 + u(x_{j+1})}{2}\}.$$

This yields a strictly monotone sequence with $u(x_1) = 1 > u(x_2) > \cdots > u(x_{n-1}) > u(x_n) = 0$. To see this, first observe $0 = u(x_n) < u(x_{n-1}) = \max\{\max_{S \supseteq \{x_{n-1}, x_n\}} \frac{p(S) - p(x_n)}{p(S)}, \frac{1}{2}\} < 1$ since $p(x_n) \in (0,1)$. Suppose by induction that $0 = u(x_n) < \cdots < u(x_{j+1}) < u(x_j) < 1$ for some $j > 2$. Due to the assumption of full support on p and the induction hypothesis, we have both $1 > \frac{1+u(x_j)}{2} > u(x_j)$ and $\sum_{x \in S \setminus \{x_S\}} \frac{p(x)}{p(S)} + \frac{p(x_S)}{p(S)} u(x_S) < 1$ for any $S \in \mathcal{S}(x_{j-1})$. Hence $u(x_1) = 1 > u(x_{j-1}) > u(x_j)$ as desired.

It remains to show that for any $S \subseteq X$ and $x \in S \setminus \{x_S\}$,

$$u(x) \geq \sum_{x' \in S} \frac{p(x')}{p(S)} u(x'),$$

since clearly $u(x_S)$ must be strictly smaller than this weighted average. We show this inequality for the $\hat{x}$ which is second-worst in S, as a higher-utility element will then satisfy the inequality a fortiori. Thus $S \in \mathcal{S}(\hat{x})$. Note that the RHS increases if we replace $u(x')$ with the value 1 for all $x' \neq x_S$ on the righthand side. But this is precisely the inner maximand, evaluated at the set S, in the definition of $u(\hat{x})$. Hence the inequality follows by construction. $\square$

Theorem 9. *Testing consistency with Sen (1993)'s theory of choosing the second best according to a preference ordering is NP-hard.*

Proof. Fix a mixed set $\mathcal{R}$ of binary restrictions defined on a set X, as in Proposition 3 of de Clippel and Rozen (2018). For each restriction r, let x_r be the option whose contour set is being restricted, and let y_r, z_r be the two options potentially included in the upper (or lower) contour set of x_r if r is an UCS (or LCS) restriction. We assume wlog that $y_r \neq z_r$. Consider the set of options X' that contains all options in X, plus a new option a_r for each LCS constraint r, a new option b_r for each UCS restriction r, and the following observed choices:

S	$\{a_r, x_r\}$	$\{a_r, y_r, z_r\}$
$c_{obs}(S)$	a_r	a_r

for each LCS restriction r, and

S	$\{b_r, x_r\}$	$\{b_r, y_r, z_r\}$
$c_{obs}(S)$	x_r	b_r

for each UCS restriction r. We conclude the proof by showing that there exists an acyclic relation satisfying the restrictions listed in $\mathcal{R}$ if, and only if, c_{obs} is consistent with picking the second best for some preference ordering.

If $\mathcal{R}$ is acyclically satisfiable, then let P be a strict acyclic relation on X satisfying the restrictions in $\mathcal{R}$. We can assume without loss of generality that P is complete, that is, an ordering. Extend this relation into a preference ordering on X' by ranking a_r above the P-smallest element of $\{y_r, z_r\}$ and below any other element of X that is P-superior to it, for each LCS restriction r; and ranking b_r below the P-largest element of $\{y_r, z_r\}$ and above any other element of X that is P-inferior to it, for each UCS restriction r. It is easy to check that c_{obs} coincides with the second-best element according to this preference, for each $S \in \mathcal{D}$.

Conversely, suppose that P' is a preference ordering on X' with the property that $c_{obs}(S)$ is the second-best element of S under P', for each $S \in \mathcal{D}$. Consider now a LCS restriction r. Since a_r is picked out of $\{a_r, y_r, z_r\}$, it must be that a_r is ranked in between y_r and z_r, that is, $y_r P' a_r P' z_r$ or $z_r P' a_r P' y_r$. Given that a_r is P'-inferior to x_r (for a_r to be picked from $\{a_r, x_r\}$), it must be that $x_r P' y_r$ or $x_r P' z_r$. Finally, consider an UCS restriction r. Since b_r is picked out of $\{b_r, y_r, z_r\}$, it must be that b_r is ranked in between y_r and z_r, that is, $y_r P' b_r P' z_r$ or $z_r P' b_r P' y_r$. Given that x_r is P'-inferior to b_r (for x_r to be picked from $\{b_r, x_r\}$), it must be that $y_r P' x_r$ or $z_r P' x_r$. $\square$