Order-*k* Rationality∗

Bravo Working Paper # 2020-010

Salvador Barberà†   Geoffroy de Clippel‡

Alejandro Neme§   Kareen Rozen¶

Abstract: A decision maker (DM) may not perfectly maximize her preference over the feasible set. She may feel it is good enough to maximize her preference over a sufficiently large consideration set; or just require that her choice is sufficiently well-ranked (e.g., in the top quintile of options); or even endogenously determine a threshold for what is good enough, based on an initial sampling of the options. Heuristics such as these are all encompassed by a common theory of Order-*k* Rationality, which relaxes perfect optimization by only requiring choices from a set S to fall within the set's top $k(S)$ elements according to the DM's preference ordering. Heuristics aside, this departure from rationality offers a natural way, in the classic 'as if' tradition, to gradually accommodate more choice patterns as *k* increases. We characterize the empirical content of Order-*k* Rationality (and related theories), and provide a tractable testing method which is comparable to the method of checking SARP.

_____

  †MOVE, Universitat Autònoma de Barcelona, and Barcelona GSE. 08193, Spain. De-partament d'Economia i d'Història Econòmica, Edifici B, 08193 Bellaterra, Spain. E-mail: barbera.salvador@gmail.com

  ‡Department of Economics, Brown University. E-mail: declippel@brown.edu

  §Instituto de Matemática Aplicada de San Luis. Universidad Nacional de San Luis and CONICET. Avenida Italia 1554. 5700, San Luis, Argentina. E-mail: neme.alejandroj@gmail.com

  ¶Department of Economics, Brown University. E-mail: kareen_rozen@brown.edu

Order-k Rationality[*]

Salvador Barberà[†]   Geoffroy de Clippel[‡]

Alejandro Neme[§]   Kareen Rozen[¶]

October 2019

Abstract

A decision maker (DM) may not perfectly maximize her preference over the feasible set. She may feel it is good enough to maximize her preference over a sufficiently large consideration set; or just require that her choice is sufficiently well-ranked (e.g., in the top quintile of options); or even endogenously determine a threshold for what is good enough, based on an initial sampling of the options. Heuristics such as these are all encompassed by a common theory of Order-k Rationality, which relaxes perfect optimization by only requiring choices from a set S to fall within the set's top $k(S)$ elements according to the DM's preference ordering. Heuristics aside, this departure from rationality offers a natural way, in the classic 'as if' tradition, to gradually accommodate more choice patterns as k increases. We characterize the empirical content of Order-k Rationality (and related theories), and provide a tractable testing method which is comparable to the method of checking SARP.

[†]MOVE, Universitat Autònoma de Barcelona, and Barcelona GSE. 08193, Spain. Departament d'Economia i d'Història Econòmica, Edifici B, 08193 Bellaterra, Spain. E-mail: barbera.salvador@gmail.com

[‡]Department of Economics, Brown University. E-mail: declippel@brown.edu

[§]Instituto de Matemática Aplicada de San Luis. Universidad Nacional de San Luis and CONICET. Avenida Italia 1554. 5700, San Luis, Argentina. E-mail: neme.alejandroj@gmail.com

[¶]Department of Economics, Brown University. E-mail: kareen_rozen@brown.edu

1. Introduction

In its classic expression, an individual is rational if she systematically chooses the best available alternative according to a preference ordering. Some may think of rationality as a plausible model of the thought process underlying choices. Others may prefer to remain agnostic about how choices crystalize, but are interested in rationality as long as it captures choices reasonably well. This is the classic 'as if' justification.

Advances in behavioral economics provide robust evidence that people's choices may be incompatible with rationality in some circumstances. In such cases, the 'as if' justification fails, and some alternatives to rational choice should be considered. Besides, the binary classification rational/irrational is not very useful in the imperfect world of actual choice data. Instead, one would like to quantify someone's level of rationality.

To address these considerations, while preserving the central role of preference orderings, we simply propose to relax the perfect maximization requirement. To this end, we introduce an index function k that associates to each choice problem S a strictly positive index $k(S)$ that is smaller than, or equal to, the number of elements in S. Then choices from S will be required to fall within the top $k(S)$ options in S according to some given preference ordering. This defines our notion of Order–k Rationality. The special case $k(S) = 1$ for all S corresponds to the rational benchmark that has been much studied and applied. Our goal is to discuss and analyze the more permissive theories arising from less stringent threshold functions.

So far we have presented Order-k Rationality as a simple, structured departure from rationality that gradually accommodates more and more choice patterns as k increases. This approach should be appealing to those who abide by the 'as if' motivation for rationality. That being said, we also find it insightful to describe some circumstances under which order–k rational choices arise. This forces us to provide more detailed models of how choices crystalize, exploring possible underlying thought processes and causes for bounded rationality. In a first heuristic, we envision a preference-maximizing decision maker (henceforth DM) who finds it good enough to identify and evaluate a

certain number of options. This relates directly to recent papers on limited attention in choice theory,[1] but for an important difference. Attention sets are restricted by imposing a minimal bound on the number of elements they contain, while the literature has so far restricted how they vary across choice problems.

The second heuristic we consider is a formalization of Simon's (1955) satisficing. Instead of maximizing a preference ordering, the DM picks the first alternative that is good enough. In our version, what is good enough may vary across choice problems, and is based on an underlying preference ordering. For instance, the DM may find it good enough to pick an option that falls in the top quintile, according to her preference, of the available options. In Section 3.2, we discuss circumstances under which this version of satisficing seems reasonable, and perhaps more so than others.

The third heuristic endogenizes the preference threshold that makes an option good enough: the DM reviews a certain number of elements, and then continues reviewing options until she finds one that is superior to the best one encountered in the sampling phase (she picks that best option if she runs out of options without having found a better one). We show in Section 3 how choices arising from all three heuristics described above are observationally equivalent to Order-k Rationality (see Observations 1–3). That Order-k Rationality is not only an intuitive relaxation of Rationality to organize choices in the 'as if' tradition, but also arises for multiple heuristics, only reinforces its importance.

In Section 4, we characterize the empirical content of Order-k Rationality through a weakening of the classic Strong Axiom of Revealed Preference (SARP) (Samuelson 1938, Houthakker 1950). Choices reveal preference restrictions, and consistency amounts to finding an acyclic relation satisfying them. While these restrictions simply pin down preference comparisons in the case of rationality (e.g., a revealed preferred to b when a is picked in the presence of b), more complex restrictions arise in general (e.g., 'a revealed preferred to both b and c, or to both b and d, or to both c and d'). Nonetheless, Theorem

---

[1]See for instance Manzini and Mariotti (2007, 2012), Masatlioglu et al. (2012), Cherepanov et al. (2013), and Lleras et al. (2017).

2 shows how the usual method for checking SARP can be adapted to test its weakening as well, making the testing of Order-k Rationality comparable to the testing of rationality itself, whatever the index function k. These results prove that the weaker rationality assumption still imposes significant testable restrictions on the admissible choice behavior of DMs, provides operational guidance for discussing identification and forecasting exercises, and can be used to test the 'degree of rationality' that is reflected by a DM's choices.

We conclude the paper with a few observations about testing and related theories.[2] First, while the order in which a DM reviews alternatives is often unknown to the modeler (e.g., subjective to the DM), there are situations where this information is available (Rubinstein and Salant, 2006). Clearly, having access to such enriched data may matter when it comes to testing the second and third heuristics. We show in Section 5.1 how our testing result for Order-k Rationality can easily be adjusted to tackle those situations as well. Once again, testing is doable in a way that is comparable to checking SARP.

Second, the ability to test Order-k Rationality in a way comparable to checking SARP (as established in Theorems 1 and 2) is by no means obvious a priori. To put the result in perspective, consider a theory as simple as picking the second best for a preference ordering, as suggested by Sen (1983). We prove in Section 5.2 that testing it can be much more demanding (in a sense to be made precise) than testing rationality and/or order-2 rationality.

Third, we suggest in Section 5.3 that it may be fruitful to enhance earlier theories of limited attention by adding (in the spirit of the first heuristic) a lower bound on the cardinality of consideration sets. Indeed, a potential issue with earlier approaches is that the dataset's consistency with the theory may be possible only when the DM pays attention to very few options in some choice problems – in some cases just the observed choice itself. While the

---

[2]For another related theory, Aleskerov et al. (2007) considers settings where *the entire acceptable set is chosen* (e.g., team building): given utility function $u : X \to \mathbb{R}$ and threshold $\tau(S)$, the DM picks all elements whose utility is above $\tau(S)$. Our DM picks an option at a time; not observing the DM pick x doesn't imply x is unacceptable. Hence Aleskerov's formulation has no testable implication in our setting (e.g., set $\tau(S) = 0$ and u always positive). Related to Aleskerov et al (2007)'s approach but with added structure, Eliaz et al. (2011) study selecting two finalists, and Chambers and Yenmez (2017) the top q options.

DM may suffer from choice overload in large choice problem, assuming that she pays attention to a single option may be unrealistic too. We characterize the testable implications of such an enrichment to Lleras et al. (2017). Once again, testing is doable in a way that is comparable to checking SARP.

Finally, in Section 5.4 we observe that Order-k Rationality may also arise from some collective decision processes, expanding the settings to which our model applies.

2. ORDER-k RATIONALITY

Remember from the introduction that X denotes the finite set of all conceivable alternatives, that a choice problem is any nonempty subset of X, and that the *index function k* associates to each choice problem S a strictly positive index $1 \leq k(S) \leq |S|$. An *observed choice correspondence C_{obs}* associates to each choice problem S a subset $C_{obs}(S)$ of recorded choices from S. Notice that $C_{obs}(S)$ could be empty (no choice data pertaining to S has been recorded) or contain multiple elements (DM picked different options when encountering S on different occasions). For each choice problem S, integer $t \geq 1$, and ordering $\succ$, let $M_\succ^t(S)$ be the set of the top t options in S according to $\succ$:

$$M_\succ^1(S) = \arg\max_\succ S, \text{ and } M_\succ^t(S) = M^{t-1}(S) \cup \arg\max_\succ [S \backslash M^{t-1}(S)], \quad \forall t \geq 2.$$

Definition 1. C_{obs} *is* consistent with Order–k Rationality *if there exists a preference ordering $\succ$ on X such that $C_{obs}(S) \subseteq M_\succ^{k(S)}(S)$, for all S.*

3. THEORIES OF GOOD ENOUGH

We develop three heuristics below that give rise to order–k rational choices. While distinct, they share the common feature of formalizing what it means for choices to be 'good enough.'

3.1 Minimal Consideration

As has been well-documented, decision makers don't always actively consider and compare all their available options. Yet they may still be boundedly

rational, in the sense of maximizing a preference ordering over those options that are considered.

For instance, a DM may find it good enough to review m options in choice problems with at least that many alternatives (and to review all options in smaller choice problems). The process by which options catch the DM's attention may be random, nudged by third parties (e.g. through advertisement), or the result of some targeted search (e.g. focusing on the m cheapest options). Most often, the modeler does not know how consideration sets crystallize. For this reason, the theory we present below restricts only their sizes.

Varying m defines a collection of nested theories, with fewer choice patterns compatible the larger m is. Indeed, a choice that is optimal in a consideration set with m options remains optimal when other options are dropped from consideration. Thus positing instead that the DM considers *at least* m options in all problems containing at least that many (and reviews all options in smaller problems) would be observationally equivalent. With this new interpretation in mind, we could view m as one way to calibrate the level of bounded rationality, with the modeler seeking the largest m allowing to explain the data.

In some cases, the modeler may find other threshold functions worth considering as well. The modeler may conjecture, for instance, that the DM contemplates a certain fraction of alternatives (e.g., a prospective employer reviewing more files the more candidates there are). To cover a wide class of possible theories, we consider a threshold function α that associates to each choice problem S an integer $\alpha(S) \in \{1, \ldots, |S|\}$. It is up to the modeler to decide which theory (that is, which threshold function) is most natural and worth testing.

Finally, we recognize the possibility that the DM uses different consideration sets across different occasions of facing the same choice problem. This would be the case, for instance, if attention is stochastic, or if it is impacted by contextual information (e.g. advertisement, placement, or packaging). This added feature seems reasonable and accommodates situations where choice may vary even when preference is strict.[3] However, our analysis and results

---

[3]As data is oftentimes limited, we think primarily of cases where one has at most a

also apply when the consideration set is uniquely determined by the feasible set. This simply amounts to restricting attention to single-valued choice functions.

Definition 2. C_{obs} *is consistent with the theory of Minimal Consideration with Threshold α if there exists a preference ordering $\succ$ such that, for each S and each $x \in C_{obs}(S)$, there exists an attention set $A(S)$ that contains at least $\alpha(S)$ elements and such that x is $\succ$-maximal over $A(S)$.*

Observation 1. C_{obs} *is consistent with Order-k Rationality if, and only if, it is consistent with the theory of Minimal Consideration with Threshold α, where $\alpha(S) = |S| - k(S) + 1$ for all S.*

3.2 Ordinal Relative Satisficing

Order–k Rationality is reminiscent of Simon (1955)'s satisficing procedures. The general idea is that, instead of maximizing her preference, the DM reviews feasible options sequentially, and picks the first one that she finds acceptable. There are many ways, however, to formalize this. The purpose of this section is to make precise which form of satisficing Order–k rationality encompasses, and under which circumstances it might make sense.

As should be clear right away, the set of acceptable options for S should be $M_{\succsim}^{k(S)}(S)$ if Order-k Rationality is to be interpreted as a form of satisficing. This set is defined relative to, and hence may vary with, the choice problems. While this is not the case in the simplest incarnations of Simon's satisficing that use a single aspiration level, we think this depature is in fact more natural in many settings. For instance, a prospective employee may be satisfied to accept a lower-wage job in case of economic downturn, but not otherwise. A consumer may be satisfied with a consumption bundle when the budget set is small, but not anymore when her income increases. Dependence of $k(S)$ on the set S is

---

few observations for a limited number of choice problems. However, our approach remains meaningful if one had access to very many observations per choice problem, resulting in a stochastic choice dataset. Our theory then applies to the support of those distributions, while placing no restriction on relative frequencies.

also consistent with the findings of Caplin, Dean and Martin (2011), who test the satisficing model with an innovative experiment generating choice-process data; they estimate that the reservation utility increases with set size, and depends on object complexity as well.[4]

While indeed intuitive, we emphasize that relative satisficing thus presumes the DM knows the set of feasible options when making a decision. Satisficing behavior occurs because the DM is forced to search for them. When hiring, for instance, an employer may know the set of applicant characteristics available, as a function of the posted wage and the market conditions, but has to read through files and conduct interviews to learn each particular applicant's characteristics. When shopping for a new car, one can easily discover the possible vehicle specifications (model trims and color combinations) that are affordable, but effort is required to figure out in which dealerships they can be found. When looking to buy a product or a service, one may be aware of prices to expect based on past observations, but effort and patience is required to identify what each provider charges. Nowadays, there are websites reporting, for instance, the range of prices other customers paid for a car, or for hiring a contractor to do some specific work.

To make a satisficing procedure complete, we must also discuss how the DM encounters alternatives, to determine which is the first acceptable one she faces. When reviewing applicant files, the employer may review them in the order they were uploaded online. Car dealerships can nudge consumers to visit them first through advertisements and promotions (e.g., a free giftcard

---

[4]Caplin et al (2011) introduces such process data because, as they point out, the standard incarnation of satisficing does not have testable implications unless one observes search order (following their approach, Papi (2012)'s theoretical analysis, which exogenously fixes the DM's reference level independently of the set, assumes such choice-process data as well). Caplin et al (2011) further observes that the lack of empirical content means one cannot tell how poor the DM's choices are. By contrast, each of the satisficing theories we propose, indexed by the function k, does have testable implications within a standard choice domain; and our approach allows the modeler to both parametrize irrationality and garner information about preferences. For example, the modeler may believe the DM always picks from the top $x\%$ based on the size of the set, and may find that the smallest x fitting the data is 10%. Combined with the resulting revealed restrictions on preferences (discussed later), the modeler can say quite a bit about the quality of the DM's choices.

for test driving). Sellers can strategize to make their products more visible on the internet. When a traveler familiar with a region decides where to stop for lunch at one of the restaurants she expects to find along the highway, the order in which options are reviewed vary with both her location at lunchtime and the order in which restaurants are lined up along the highway. As these examples show, it is possible that the DM picks different options when facing the same choice problem on different occasions, as randomness or contextual information unknown to the modeler may impact the order in which the DM reviews alternatives. Interpreted as a model of satisficing, Order–k Rationality accommodates, but does not require, such effects; indeed, it may be that a choice problem is observed only once, or that the DM always picks the same option whenever she faces it.[5]

To formalize these ideas, we start by introducing the notion of a *list*, which is simply a sequence $\ell = (x_1, x_2, \ldots, x_K)$ of distinct options in X. The notion of a list thus refines that of a choice problem: one knows *both* the feasible options and the order in which they appear to the DM. Suppose first that we know the lists faced, not just the option sets. Observing the DM's choice from lists in which the *same* options appear in a different order thus provides richer data, with the potential for more stringent tests and better identification (see Rubinstein and Salant (2006) for a first analysis of choice from lists).

Definition 3. *Observed choices from known lists are* consistent with Ordinal Relative Satisficing with Threshold Function k *if there exists a preference ordering* $\succ$ *such that the chosen element for each list* ℓ *is the first option in the list that belongs to* $M_\succ^{k(S)}(S)$.

In Section 5.1, we develop the testable implications of Ordinal Relative Satisficing when observing choices from known lists. However, as discussed before Definition 3, there are many cases where the list is not known to the mod-

---

[5]As data is oftentimes limited, we think primarily of cases where one has at most a few observations for a limited number of choice problems. However, our approach remains meaningful if one had access to very many observations per choice problem, resulting in a stochastic choice dataset. Our theory defined shortly then applies to the support of those distributions, while placing no restriction on relative frequencies.

eler. The next observation then formalizes the straightforward link between Order-k Rationality and ordinal relative satisficing with threshold function k.

Observation 2. *C_{obs} is consistent with Order-k Rationality if, and only if, for each S and each $x \in C_{obs}(S)$, there exists a list $\ell_{S,x}$ such that the modified choice data where x is picked from the list $\ell_{S,x}$, for each S and $x \in C_{obs}(S)$, is consistent with ordinal relative satisficing with threshold function k.*

Like for the theories of minimal consideration, the DM ends up maximizing her preference over a smaller set of options. In contrast to those earlier theories, however, the set here can be very small (sometimes down to a single alternative) or much larger, depending on how soon the DM encounters an acceptable option.

3.3 A Hybrid: Minimal Attention to Determine the Satisficing Threshold

In this section, we endogenize the acceptance threshold. We envision the DM as first reviewing a given fraction of the list to set up a threshold, and picking the first option that surpasses that threshold according to her preference. For each list ℓ, let $n(\ell)$ be a number between 1 and the length of the list. The DM reviews the first $n(\ell)$ options in the list, and sets up her satisficing threshold based on this sample. Like the index function k, the modeler chooses the function n and sets out to test the resulting theory. A natural threshold would be to set n according to a fixed fraction of the list's length, but more complex functions can be accommodated without complication.

Definition 4. *Observed choices from lists are* consistent with Satisficing after n-Sampling *if there exists a preference ordering $\succ$ such that the chosen element for each observed list ℓ is the first option in the list that is $\succ$-superior to $x^*(\ell)$, the best option in the first $n(\ell)$ elements of the list. If no such option exists, then the chosen element is $x^*(\ell)$ itself.*[6]

---

[6]This bears similarity to the classic secretary problem, but departs from it in the ability to choose an element, $x^*(\ell)$, from earlier in the list.

This procedure thus combines features of both relative satisficing and limited attention, as the threshold for each list is set by paying attention to a limited number of options. The higher is $n(\ell)$, the more stringent is the DM's bar for what is good enough. As discussed earlier, observing choices from lists is a form of enriched data, where the modeler knows not only the set of available options, but also the sequence in which the DM assessed them. If the modeler does not have access to that information (e.g., the order in which options are presented is not recorded, or the order is subjective), then we are back to the notion of Order–k Rationality.

Observation 3. C_{obs} *is consistent with Order-k Rationality if, and only if, for each S and each $x \in C_{obs}(S)$ there exists a list $\ell_{S,x}$ such that the modified choice data where x is picked from the list $\ell_{S,x}$, for each S and $x \in C_{obs}(S)$, is consistent with Satisficing after n-sampling, where $n(S) = |S| - k(S)$.*

4. Testing, Identification and Forecasting

Suppose one is interested in Order–k Rationality, either because it offers a natural departure from Rationality as an 'as if' story, or because of the heuristics-based motivations offered in Observations 1–3. How does one test whether observed choices are consistent with Order–k Rationality?

As is well-known, a DM's observed choices are consistent with Rationality if, and only if, the Strong Axiom of Revealed Preference (SARP) holds. We propose to extend this classic result to Order–k Rationality by pursuing de Clippel and Rozen (2019)'s general methodology to test bounded rationality theories. The first step is to identify an exhaustive collection of revealed preference restrictions, by proving that consistency is equivalent to checking whether the set of restrictions is acyclically satisfiable, that is, whether there exists an acyclic relation satisfying them. The second step is to look into the type of preference restrictions needed for such characterizations. If they are all lower contour set (LCS) restrictions (e.g., 'a preferred to b or c' restricts the lower contour set of a), then acyclic satisfiability can be checked essentially in the same way SARP is checked.

Suppose the DM is consistent with Order–k Rationality. Observed choices then impose the following restrictions on what the DM's preference may look like: if $x \in C_{obs}(S)$, then there exists $|S| - k(S)$ alternatives in S that are all inferior to x. Let $\mathcal{R}_k(C_{obs})$ be the set of all such restrictions. Consistency with Order–k rationality thus implies that $\mathcal{R}_k(C_{obs})$ is acyclically satisfiable, that is, there exists an acyclic relation sarisfying them. Could it be that by thinking harder we would identify other revealed preference restrictions that are critical for testing? No, as the following result shows.

Theorem 1. *C_{obs} is consistent with Order-k Rationality if, and only if, $\mathcal{R}_k(C_{obs})$ is acyclically satisfiable.*

Proof. Necessity was proved in the discussion above. As for sufficiency, let P be an acyclic relation satisfying the restrictions in $\mathcal{R}_k(C_{obs})$, and let $\succ$ be a completion of P into an ordering. For each each $x \in C_{obs}(S)$, the corresponding restriction in $\mathcal{R}_k(C_{obs})$ tells us that one can find at least $|S| - k(S)$ alternatives in S that are P-inferior to it. Since P-comparisons remain valid under $\succ$, x is among the top $k(S)$ options in S according to $\succ$, and C_{obs} is consistent with Order-k Rationality. ∎

One might be concerned that checking acyclic satisfiability for $\mathcal{R}_k$ is much more challenging than checking SARP, as the revealed preference restrictions do not pin down comparisons between pairs of alternatives. Instead, one must check whether one can *find* an acyclic relation satisfying restrictions of the form 'a must be preferred to both b and c, or a must be preferred to both b and d, or a must be preferred to both b and d' (the revealed preference restriction arising from a being picked from $S = \{a, b, c, d\}$ with $k(S) = 2$). Yet, checking whether $\mathcal{R}_k$ is acyclically satisfiable can be done in a way comparable to checking SARP.[7]

Let X_1 be the image of C_{obs}, and let $Y_1 = X \setminus X_1$. Intuitively, Y_1, the set of elements never chosen, contains obvious candidates for the DM's worst

---

[7]The procedure we defined to prove this appeared in Barberà and Neme (2014), as well as in a supplement to de Clippel and Rozen (2012) as a special instance of their more general enumeration procedure, which applies here because $\mathcal{R}_k$ contains only LCS restrictions.

elements, with those in X_1 ranked somewhere above those in Y_1. The data may suggest further information, however, on how elements in X_1 rank in relation to each other. For each $x \in X_1$, let $C_{obs}^{-1}(x)$ be the set of choice problems S such that $x \in C_{obs}(S)$. Consider the set

$$Y_2 = \{x \in X_1 \mid |S \cap Y_1| \geq |S| - k(S), \text{ for all } S \in C_{obs}^{-1}(x)\},$$

which is the set of $x \in X_1$ such that for every set S in which x is chosen, x already belongs to the top $k(S)$ elements, either because there are sufficiently many worse-ranked elements (i.e., those in Y_1), or because the size of the set is at most $k(S)$ (e.g., it is a two-element set and the theory is that the DM chooses from the top two). As elements in Y_2 are not forced by the theory to be better than any other element in X_1, the set Y_2 contains obvious candidates for the DM's worst elements in X_1. After taking out the elements of Y_2 and ranking these somewhere above the elements of Y_1, we would next want to investigate the set of remaining elements, $X_2 = X_1 \setminus Y_2$. Building on this idea, we can define by induction two sequences of sets:

$$\begin{aligned} Y_{\ell+1} &= \{x \in X_\ell \mid |S \cap [\cup_{i=1}^{\ell} Y_i]| \geq |S| - k(S), \text{ for all } S \in C_{obs}^{-1}(x)\}, \\ X_{\ell+1} &= X_\ell \setminus Y_{\ell+1}. \end{aligned} \quad (1)$$

In each step ℓ, the set $\cup_{i=1}^{\ell} Y_i$ represents the set of elements that have already been ranked below the remaining elements (i.e., those in X_ℓ). Then $Y_{\ell+1}$ contains the candidates for worst elements in X_ℓ: each $x \in Y_{\ell+1}$ has enough elements ranked below it that it is already belongs to the top $k(S)$ for every S where it is chosen. The set $Y_{\ell+1}$ is then removed, and ranked somewhere above those previously removed, to generate the next set of remaining elements, $X_{\ell+1}$.

Clearly, the set of remaining elements weakly shrinks in each step: $X_{\ell+1} \subseteq X_\ell$ for each ℓ. Given that X_1 has at most $|\mathcal{D}|$ elements, where $\mathcal{D}$ is the set of choice problems S such that $C_{obs}(S) \neq \emptyset$, the sequence $(X_\ell)_{\ell \geq 1}$ becomes constant in at most that many steps. Let X^* be this limit set, that is, $X^* = X_\ell$ where ℓ is the lowest index such that $X_\ell = X_{\ell+1}$. The next result shows that $\mathcal{R}_k(c_{obs})$ is acyclically satisfiable holds (and thus C_{obs} is consistent with our

theories by Theorem 1) if and only if $X^* = \emptyset$.

Theorem 2. $\mathcal{R}_k(C_{obs})$ *is acyclically satisfiable holds if, and only if, $X^* = \emptyset$.*

Proof. (*Sufficiency*) Suppose that $X^* = \emptyset$. Let ℓ be the smallest index such that $X_\ell = X^*$. Consider the partition $\{Y_1, \ldots, Y_L\}$ of X, and the (strict) relation P defined by $x \succ y$ if the atom of the partition to which x belongs has a larger index than the atom to which y belongs. Clearly P is acyclic, and acyclic satisfiability holds using P.

(*Necessity*) Suppose that the limit set X^* is nonempty. By construction, X^* is contained in the image of C_{obs}. Let $Y^* = X \setminus X^*$. If $\mathcal{R}_k$ is acyclically satisfiable using P, then the restriction of P to X^* must be such that for all $x \in X^*$ and for all $S \in C_{obs}^{-1}(x)$, $|\{y \in S \mid xPy\}| \geq |S| - k(S)$. Decomposing the lower-contour set into two components,

$$|\{y \in S \cap X^* \mid xPy\}| + |\{y \in S \cap Y^* \mid xPy\}| \geq |S| - k(S).$$

Thus $|\{y \in S \cap X^* | xPy\}| + |S \cap Y^*| \geq |S| - k(S)$ for all $x \in X^*$ and $S \in C_{obs}^{-1}(x)$, since in the most permissive scenario, all elements in $S \cap Y^*$ are P-inferior to x. Acyclicity implies that one can find $x^* \in X^*$ for which there is no $y \in X^*$ such that $x^* P y$. But then applying the above bound to x^*, for all $S \in C_{obs}^{-1}(x^*)$ it must be that $0 \geq |S| - k(S) - |S \cap Y^*|$, which cannot be if X^* is the limit set. Thus X^* must be empty if $\mathcal{R}_k$ is acyclically satisfiable, as desired. ∎

This testing procedure is of a similar spirit, and similar complexity,[8] to procedures used to check SARP (the special case where $k(S) = 1$ for all S). The relative simplicity of testing may be surprising, as many other theories of bounded rationality are NP-hard to test. Consider, for instance Sen's (1993)'s theory of choosing the second best according to a preference ordering (as opposed to choosing one of the top two elements, as permitted by Order-k Rationality with $k(S) = 2$ for all S). Under full data (when the entire choice function is observed), Baigent and Gaertner (1996) characterize Sen's theory

---

[8]A number of steps which is polynomial in the size of the data.

using simple and easy-to-check axioms.[9]

They do not capture, however, the empirical content of this theory when data is limited. In this general case, Sen's rather simple theory turns out to be NP-hard to test, as we show in Theorem 4 of Section 5.2. This provides an interesting contrast with our results above. Determining whether choices are second-best for some preference ordering can be hard, but checking whether choices are top best for some ordering (Rationality), or among the top two elements for some ordering (2-Rationality) is tractably tested using a method comparable to checking SARP.

What becomes clear from the testing procedure above is that it is possible for different preference orderings to generate the same observed choices under Order-k Rationality. Contrary to Rationality, this is true even for a complete dataset. If the testing procedure yields the sequence of sets $Y_1, \ldots, Y_L$, then there are at least $\prod_{\ell=1}^{L} |Y_\ell|!$ possible preference orderings. The rationalizing preference constructed has all the elements of Y_ℓ ranked below all the elements of $Y_{\ell+1}$, for each ℓ; but within each Y_ℓ, it does not matter how the alternatives are ranked.

Nonetheless, there are choice configurations that completely pin down the preference between some alternatives.[10] Formally, x is *unambiguously revealed preferred to* y if xPy for any preference ordering P that generates C_{obs} under Order–k Rationality. Identifying whether this is so amounts to ruling out the possibility of generating observed choices with a preference P such that yPx.

---

[9]For single-value choice functions, the two axioms are simply: (i) if x is picked both from $\{x, y\}$ and $\{x, z\}$, then x is not picked from problems containing x, y, and z, and (ii) each problem S has an element $y \neq c(S)$ such that $c(\{y, z\}) = z$ for all $z \in S$ different from y.

[10]Suppose $X = \{x_1, x_2, x_3, x_4, x_5\}$ and the modeler posits $k(S) = 2$ for all S. Observed choices $c_{obs}(\{x_1, x_2, x_3, x_5\}) = x_1$ and $c_{obs}(\{x_1, x_2, x_3, x_4\}) = x_2$ are consistent with Order-k Rationality using various preferences, including $x_1 P x_2 P x_3 P x_4 P x_5$, $x_5 P x_1 P x_2 P x_3 P x_4$, and $x_4 P x_2 P x_1 P x_3 P x_5$, among others. In *all* rationalizing preferences, both x_1 and x_2 must be ranked above x_3; that is, x_1 and x_2 are revealed preferred to x_3. Intuitively, x_1, x_2, x_3 all appear in two choice problems, where two of these are observed to be chosen; as $k(S) = 2$, this leaves no space for x_3 to be ranked above either. The first choice reveals that at most one element among x_2, x_3 and x_5 is preferred to x_1. The second choice reveals that at most one element among x_1, x_3 and x_4 is preferred to x_2. But if x_3 is the one alternative preferred to x_2 (x_1), and if at most one of these can be preferred to x_1 (x_2), then its availability in both observed problems renders it impossible to explain the choices.

14

We may use Theorems 1 and 2 to address this problem, by augmenting the collection of restriction $\mathcal{R}_k$ with the restriction yPx. Clearly, x is unambiguously revealed preferred to y if, and only if, acyclic satisfiability of the augmented restrictions fails. Modifying the definition of $Y_{\ell+1}$ to remove y for any ℓ with $x \in X_\ell$ then provides a simple test as in Theorem 2.

Theorem 2 is also useful for out-of-sample predictions. Given C_{obs}, option $x \in S$ is a *valid prediction* for S under Order–k Rationality if there exists $\succ$ generating C_{obs} under the theory such that x falls in the top $k(S)$ element of S according to $\succ$. To check this, simply append the choice x from S to C_{obs}, and apply the method suggested in Theorem 2 to check whether this extension of observed choices is consistent with Order–k Rationality. Consistency holds if, and only if, x is a valid forecast for S.

5. COMMENTS ON TESTING AND RELATED THEORIES

In Section 5.1, we develop the empirical content of Ordinal Relative Satisficing and Satisficing after Sampling for rich datasets where choice from lists are observed. Section 5.2 illustrates that the ability to test Order-k Rationality in a matter similar to SARP is highly nontrivial, as doing so is impossible even for Sen (1993)'s related theory of choosing the second best. The ensuing subsections develop interesting extensions of our approach. Section 5.3 considers a hybrid model, whereby a common theory of consideration sets is enriched with our minimal-consideration requirement (the first heuristic). In Section 5.4 we show Order-k Rationality may also arise from collective decision-making processes, expanding the types of datasets where the model may be of use.

5.1 Enriched Data: Choice from Lists

In Sections 3.2 and 3.3, we introduced the theories of Ordinal Relative Satisficing, and of Satisficing after Sampling, noting that their testable implications amount to Order-k rationality when the DM's choices are observed knowing only the set of options faced, but not their order. We now explore the testable implications when the modeler sees these lists.

Things are rather straightforward in the case of Ordinal Relative Satisficing with Threshold Function k. If option x is the observed choice from a list ℓ, then we learn that (i) x falls within $M_\succ^{k(S)}(S)$, where S is the set of options appearing in ℓ; and (ii) the set of elements preceding x in the list, which we denote $P_\ell(x)$, does not belong to $M_\succ^{k(S)}(S)$. Thus observing the list allows us to refine the revealed-preference restrictions listed in $\mathcal{R}_k(C_{obs})$: (a) x is revealed preferred to each option in $P_\ell(x)$, and (b) one must find $|S| - k(S) - |P_\ell(x)| \geq 0$ options in $S \setminus P_\ell(x)$ that are ranked below x. Similarly to Theorem 1, it is not difficult to check that consistency with the theory is equivalent to acyclic satisfiability of the set of revealed preference restrictions derived this way from C_{obs}. Since all restrictions pertain to lower-contour sets of options in X, one can adapt the procedure given in Theorem 2 to tractably test consistency in a way comparable to checking SARP.

We next consider Satisficing after n-Sampling when lists are known. Some necessary conditions on the DM's preference are easy to develop. First, if she chooses one of the first $n(\ell)$ options in a list, then the DM must prefer it over all alternatives in the list. Second, the element the DM chooses from a list must be preferred to any alternative that precedes it. Third, any element appearing between the $n(\ell) - th$ element of the list and before the chosen element must be inferior to at least one of the first $n(\ell)$ options in the list. The next theorem shows that observed choices do not reveal any other essential restrictions on the DM's putative preference.

Theorem 3. *Observed choices from known lists are consistent with Satisficing after $n-$Sampling if and only if the following set of restrictions is acyclically satisfiable: for each subset of alternatives S there exists a list ℓ_S such that*

(a) *If the observed choice x is one of the first $n(\ell_S)$ elements, then $x \succ y$ for all $y \neq x$ in the list;*

(b) *If x is the observed choice, then $x \succ y$ for all y preceding x;*

(c) *For any y following the first $n(\ell_S)$ elements but preceding the observed choice, there exists x among the first $n(\ell_S)$ elements such that $x \succ y$.*

Since all restrictions pertain to lower-contour sets of options in X, one can adapt the procedure given in Theorem 2 to tractably test consistency in a manner comparable to checking SARP.

5.2 Sen's (1993) Theory of Choosing Second-Best Alternatives

Order-k Rationality, which allows for the DM to choose *any* of her top-$k(S)$ elements from S, may be contrasted with Sen (1993)'s theory that the DM chooses *exactly* her second-best alternative according to her preference ordering. While Order-k Rationality can be tested in a way that generalizes the way we test SARP, it turns out that Sen (1993)'s more precise theory cannot. More specifically, if one *could* find a way to test Sen's theory in a manner akin to SARP, then one could find a tractable way to test any NP-hard problem, overturning the conventional wisdom that no such method exists.

Theorem 4. *Testing consistency with Sen (1993)'s theory of choosing the second best according to a preference ordering is NP-hard.*

The proof builds on de Clippel and Rozen (2018)'s Proposition 3, which says that it is NP-hard to test whether there is an acyclic relation satisfying 'mixed sets' of binary restrictions, where some are of the form *x is worse than y or z,* while others are of the form *x' is better than y' or z'* By contrast, in k-SARP, there are only lower-contour set restrictions. The connection of such mixed restrictions with Sen's theory may not be entirely obvious. Seeing that x is chosen from $\{x, y, z\}$ does mean that *x is worse than y or z*, and that *x is better than y or z*, but determining which alternative is better than x also determines which is worse. To see how the 'better than' and 'worse than' restrictions can be disentangled, suppose we observe that x is chosen from $\{b, x\}$ and that b is chosen from $\{b, y, z\}$. Then, we know that y or z is preferred to b. As b is preferred to x, we know at least one of y or z is preferred to x, and there is no implication that the other belongs to the lower-contour set. Formally, the proof proceeds by showing that for any collection of mixed binary restrictions, one can construct (in polynomial time) a dataset

where testing Sen's theory is *equivalent* to testing whether there is an acyclic relation satisfying the restrictions.

5.3 Choice Overload with Minimal Consideration

The theory of Minimal Consideration puts a lower bound on attention *within* any choice problem. This differs from earlier approaches to attention, which typically restrict how consideration sets vary *across* choice problems. For instance, Lleras et al. (2017) captures choice overload by requiring the DM's attention correspondence to satisfy an 'IIA' property: if x is considered when facing a choice problem S, then x must also be considered when facing any subset of S in which it is contained. This same property arises in Cherepanov et al. (2013)'s theory of rationalization. Both theories presume a single-valued choice function is observed (implicitly, each choice problem is associated with a unique consideration set), and we will thus assume the same in this subsection. To avoid confusion, observed choices will now be denoted as a function $c_{obs} : \mathcal{D} \to X$ with $c_{obs}(S) \in S$ for each $S \in \mathcal{D}$, where $\mathcal{D}$ is the set of choice problems for which a choice has been recorded.

It is intuitive and well-documented that a DM may be overloaded when facing too many options. However, going to the extreme that she pays attention to a single option may be unrealistic. Say that $c_{obs}(X) = x$ and $c_{obs}(\{x, y\}) = y$, for all $y \neq x$. These choices are consistent with the above theories, but only if x is the *only* option the DM considers when facing X (regardless of X's size). This is because IIA requires the DM to consider x in any pair containing it, and hence the DM must prefer all alternatives to x.

One may desire to enhance theories of attention by adding a lower bound on the number of alternatives considered. Doing so retains their appealing features, while restraining over-permissiveness. Here, we consider this approach for the theory of Choice Overload.

Definition 4 A single-valued choice function c_{obs} is compatible with *Choice Overload with α-Minimal Consideration*, if there exists a strict preference ordering $\succ$ over X, and a consideration-set mapping A that satisfies IIA and for each S, contains at least $\alpha(S)$ elements, with $c_{obs}(S) = M^1(A(S))$.

18

If a DM's choices are consistent with this hybrid theory, then certain choice patterns reveal information about her preference. First, as observed for full datasets by Cherepanov et al. (2013) and Lleras et al. (2017), xPy when $x = c_{obs}(S)$, $y = c_{obs}(T)$, and $y \in S \subset T$. This follows at once from the consideration set mapping satisfying IIA. Denote by $\mathcal{R}_{IIA}(c_{obs})$ the set of all such restrictions. Second, the restrictions $\mathcal{R}_k$ from Section 4, with $k(S) = |S| - \alpha(S) + 1$, must also hold. The next theorem shows that no further restrictions arise beyond these two types.

Theorem 5. *An observed choice function c_{obs} is consistent with Choice Overload with α-Minimal Consideration if and only if $\mathcal{R}_{IIA}(c_{obs}) \cup \mathcal{R}_k(c_{obs})$ is acyclically satisfiable, where $k(S) = |S| - \alpha(S) + 1$ for each S.*

Acyclic satisfiability of $\mathcal{R}_{IIA}(c_{obs}) \cup \mathcal{R}_k(c_{obs})$ is more demanding than having $\mathcal{R}_{IIA}(c_{obs})$ and $\mathcal{R}_k(c_{obs})$ both acyclically satisfiable. To put it differently, just because data is consistent with both the theories of Choice Overload and α-Minimal Consideration does not guarantee consistency with the hybrid theory; we include such an example dataset in the Appendix. The reason is that the *same* acyclic relation must satisfy both sets of restrictions simultaneously to be compatible with the hybrid theory. The tractability of testing Order-k Rationality, and the ability to perform preference identification, extend to Choice Overload with α-Minimal Consideration. Indeed, testing can be implemented using a very similar approach. Details are provided in the Appendix; loosely speaking, one constructs the sequence of sets for Theorem 2, with the only difference being that one must also exclude from Y_ℓ alternatives which are ranked above another remaining one according to $\mathcal{R}_{IIA}(c_{obs})$.

5.4 Rationalizing Collective Decisions

A classical issue in the theory of preference aggregation is whether the choices generated through voting or other methods of collective choice can be viewed 'as if' the group was acting with some degree of rationality. Indeed, we can identify joint decision processes whose outcomes would satisfy our notion of good enough decision-making. One is the case when two agents participate

in choosing from a set of n candidates through the use of rules of r names. Under these rules, one of the agents proposes a set of r candidates to the other, who then selects one from that set. The subgame-perfect equilibrium of the induced extensive-form game is the best alternative for the proposer out of the $(n - r + 1)$ that are best for the chooser (Barberà and Coelho 2010, 2017, 2018). Hence, the collective choice involved will satisfy the criterion of $(n - r + 1)-$rationality, provided the agents involved play the game's subgame-perfect equilibrium. de Clippel, Eliaz and Knight (2014) experimentally examine the real-life problem of arbitrator selection, which is commonly implemented using an alternate strikes procedure, corresponding to $r = \lfloor (n + 1)/2 \rfloor$. That rule has the attractive fairness property of guaranteeing that the joint choice will be at least as good for both players as the median choices; see also Anbarci (1993). In general, one expects the equilibria of other bargaining procedures to exhibit some degree of 'good enough' rationality as well.

References

ALESKEROV, F., D. BOUYSSOU, AND B. MONJARDET (2007), Utility Maximization, Choice and Preference, Springer Verlag.

ANBARCI, N. (1993), Noncooperative Foundations of the Area Monotonic Solution, *The Quarterly Journal of Economics*, 108, 245–258.

BAIGENT, N. AND W. GAERTNER (1996), Never Choose the Uniquely Largest: a Characterization, *Economic Theory*, 8, 239–249.

BARBERÀ, S. AND D. COELHO (2010), On the Rule of k Names, *Games and Economic Behavior*, 70(1), 44–61.

BARBERÀ, S. AND D. COELHO (2017), Balancing the Power to Appoint Officers, *Games and Economic Behavior*, 101, 189–203.

BARBERÀ, S. AND D. COELHO (2018), On the Advantages and Disadvantages of Being the First Mover under Rules of k Names, *International Journal of Economic Theory*, 14, 51–60.

BARBERÀ, S. AND A. NEME (2014), Ordinal Relative Satisficing Behavior, *Barcelona GSE Working Paper no. 790*.

CHAMBERS, C., AND B. YENMEZ (2018), A Simple Characterization of Responsive Choice, *Games and Economic Behavior*, 111, 217–221.

CHEREPANOV, V., T. FEDDERSEN AND A. SANDRONI (2013), Rationalization, *Theoretical Economics*, 8, 775–800.

DE CLIPPEL, G., K. ELIAZ AND B. KNIGHT (2014), On the Selection of Arbitrators, *American Economic Review*, 104, 3434–3458.

DE CLIPPEL, G., AND K. ROZEN (2012), Bounded Rationality and Limited Datasets, *Cowles Discussion Paper 1853*.

DE CLIPPEL, G., AND K. ROZEN (2018), Bounded Rationality and Limited Datasets, *mimeo*.

ELIAZ, K., M. RICHTER AND A. RUBINSTEIN (2011), Choosing the Two Finalists, *Economic Theory*, 46, 211–219.

HOUTHAKKER, H.S. (1950), Revealed Preference and the Utility Function, *Economica*, 17(66), 159–174.

LLERAS, J., Y. MASATLIOGLU, D. NAKAJIMA, AND E. OZBAY (2017), When More is Less: Choice by Limited Consideration, *Journal of Economic Theory*, 170, 70–85.

MANZINI, P., AND M. MARIOTTI (2007), Sequentially Rationalizable Choice, *American Economic Review*, 97(5), 1824–1839.

MANZINI, P., AND M. MARIOTTI (2012), Categorize Then Choose: Boundedly Rational Choice and Welfare, *Journal of the European Economics Association*, 10(5), 1141–1165.

MASATLIOGLU, Y., D. NAKAJIMA AND E. OZBAY (2012), Revealed Attention, *American Economic Review*, 102(5), 2183–2205.

PAPI, M. (2012), Satisficing Choice Procedures, *Journal of Economic Behavior and Organization*, 84(1), 451–462.

RUBINSTEIN, A. AND Y. SALANT (2006), A Model of Choice from Lists, *Theoretical Economics*, 1, 3–17.

SAMUELSON, P. (1938), A Note on the Pure Theory of Consumer's Behavior., *Economica*, 5(17), 61–71.

SEN, A. (1993), Maximization and the Act of Choice, *Econometrica*, 65, 745–779.

SIMON, H. (1955), A Behavioral Model of Rational Choice, *Quarterly Journal of Economics*, 69, 99–118.

A   Appendix: Details for Section 5

A.1   Proof of Theorem 3

Necessity was proved above. As for sufficiency, we can assume without loss of generality that P is an ordering, as any acyclic relation can be completed into an ordering and the completion will still satisfy the listed restrictions. We conclude the proof by checking that observed choices can be derived from the theory by using P as the DM's preference. Let ℓ be an observed list. Suppose that the observed choice x is one of the first $n(\ell)$ options. By (a), x is P-maximal in the set of elements that appear in ℓ, as desired. Suppose now x is not one of the first $n(\ell)$ options. By (b), it is P-superior to all preceding options. By (c), x is the first option in the list that succeeds the initial $n(\ell)$ options, and is P-superior to all of them. Thus indeed, the theory applied to P selects x out of ℓ, as desired. ∎

A.2   Proof of Theorem 4

Fix a mixed set $\mathcal{R}$ of binary restrictions defined on a set X. For each restriction r, let x_r be the option whose contour set is being restricted and let y_r, z_r be the two options, one of which must belong to the relevant contour set of x_r. We say r is an upper-contour set (UCS) restriction when either y_r or z_r is better than x_r; and similarly, say r is a lower-contour set (LCS) restriction when x_r is better than either y_r or z_r. We assume wlog that $y_r \neq z_r$. Consider the set of options X' that contains all options in X, plus a new option a_r for each LCS restriction r, a new option b_r for each UCS restriction r, and the

following observed choices:

S	$\{a_r, x_r\}$	$\{a_r, y_r, z_r\}$
$c_{obs}(S)$	a_r	a_r

for each LCS restriction r, and

S	$\{b_r, x_r\}$	$\{b_r, y_r, z_r\}$
$c_{obs}(S)$	x_r	b_r

for each UCS restriction r. We conclude the proof by showing that there exists an acyclic relation satisfying the restrictions listed in $\mathcal{R}$ if, and only if, c_{obs} is consistent with picking the second best for some preference ordering.

If $\mathcal{R}$ is acyclically satisfiable, then let P be a strict acyclic relation on X satisfying the restrictions in $\mathcal{R}$. We can assume without loss of generality that P is complete, that is, an ordering. Extend this relation into a preference ordering on X' by ranking a_r above the P-smallest element of $\{y_r, z_r\}$ and below any other element of X that is P-superior to it, for each LCS restriction r; and ranking b_r below the P-largest element of $\{y_r, z_r\}$ and above any other element of X that is P-inferior to it, for each UCS restriction r. It is easy to check that c_{obs} coincides with the second-best element according to this preference, for each $S \in \mathcal{D}$.

Conversely, suppose that P' is a preference ordering on X' with the property that $c_{obs}(S)$ is the second-best element of S under P', for each $S \in \mathcal{D}$. Consider now a LCS restriction r. Since a_r is picked out of $\{a_r, y_r, z_r\}$, it must be that a_r is ranked in between y_r and z_r, that is, $y_r P' a_r P' z_r$ or $z_r P' a_r P' y_r$. Given that a_r is P'-inferior to x_r (for a_r to be picked from $\{a_r, x_r\}$), it must be that $x_r P' y_r$ or $x_r P' z_r$. Finally, consider an UCS restriction r. Since b_r is picked out of $\{b_r, y_r, z_r\}$, it must be that b_r is ranked in between y_r and z_r, that is, $y_r P' b_r P' z_r$ or $z_r P' b_r P' y_r$. Given that x_r is P'-inferior to b_r (for x_r to be picked from $\{b_r, x_r\}$), it must be that $y_r P' x_r$ or $z_r P' x_r$. ∎

A.3 On Choice Overload with α-Minimal Consideration

Proof of Theorem 5

Necessity follows from the discussion above. As for sufficiency, let P be an acyclic relation satisfying $\mathcal{R}_{IIA}(c_{obs}) \cup \mathcal{R}_k(c_{obs})$, and let $\succ$ be a completion of P into an ordering. Clearly, $\succ$ also satisfies $\mathcal{R}_{IIA}(c_{obs}) \cup \mathcal{R}_k(c_{obs})$. For each choice problem S, let $B(S)$ be the bottom $\alpha(S)$ elements of S according to $\succ$, let

$$A^*(S) = S \cap \{c_{obs}(T) \mid T \in \mathcal{D}, \ S \subseteq T\},$$

and let $A(S) = A^*(S) \cup B(S)$. We conclude the proof by showing that $c_{obs}(S)$ is the $\succ$-maximal element of $A(S)$ for each $S \in \mathcal{D}$. By definition, $c_{obs}(S) \in A^*(S) \subseteq A(S)$. Suppose, by contradiction, that there exists $x \in A^*(S)$ such that $x \succ c_{obs}(S)$. If $x \in A^*(S)$, then $c_{obs}(S) \succ x$ since $\succ$ satisfies the restrictions in $\mathcal{R}_{IIA}(c_{obs})$. This cannot be, so $x \in B(S)$. Then $c_{obs}(S)$ has at most $\alpha(S) - 2$ that are $\succ$-inferior to it in S, since x has at most $\alpha(S) - 1$ below it ($x \in B(S)$) and c_{obs} is $\succ$-inferior to x. This contradicts $\succ$ satisfying the restrictions in $\mathcal{R}_k(c_{obs})$. Hence no such x exists, and $c_{obs}(S)$ is the $\succ$-maximal element of $A(S)$. ∎

Consistency with both theories $\not\Longrightarrow$ consistency with hybrid theory

Consider the following data:

S	$\{a, b, x, y\}$	$\{b, x, y\}$	$\{d, e, y, z\}$	$\{e, y, z\}$	$\{w, x, y, z\}$
$c_{obs}(S)$	y	x	z	y	z

The restrictions in $\mathcal{R}_{IIA}(c_{obs})$ are simply xPy and yPz, which is acyclically satisfiable. Taking $\alpha(S) = |S| - 1$ (overlooking at most one option) or $k(S) = 2$ for all S, the restrictions in $\mathcal{R}_k(c_{obs})$ are satisfied by the ordering $zPxPyPaPbPdPe$. Yet $\mathcal{R}_{IIA}(c_{obs}) \cup \mathcal{R}_k(c_{obs})$ is not acyclically satisfiable. Indeed, the last data point reveals that at least two elements in $\{w, x, y\}$ must be P-inferior to z, which cannot be satisfied acyclically once combined with the restrictions xPy and yPz from $\mathcal{R}_{IIA}(c_{obs})$.

Implementing testing

The tractability of testing Order-k Rationality extends to Choice Overload with α-Minimal Consideration. Indeed, testing can be implemented using a similar approach. Let $\tilde{X}_1$ be the image of c_{obs}, and let $\tilde{Y}_1 = X \setminus \tilde{X}_1$. Letting $k(S) = |S| - \alpha(S) + 1$, define by induction, for each $\ell \geq 1$:

$$\tilde{Y}_{\ell+1} = \{x \in \tilde{X}_\ell \mid |S \cap [\cup_{i=1}^{\ell} \tilde{Y}_i]| \geq |S| - k(S), \text{ for all } S \in c_{obs}^{-1}(x),$$
$$\text{and } \nexists y \in \tilde{X}_\ell \setminus \{x\} \text{ s.t. } x = c_{obs}(T), \ y = c_{obs}(T') \text{ for } x, y \in T \subset T'\}$$
$$\tilde{X}_{\ell+1} = \tilde{X}_\ell \setminus \tilde{Y}_{\ell+1}.$$

The sequence of $\tilde{Y}_\ell' s$ differs from its Section 3-counterpart in that it excludes alternatives which are ranked above another remaining one under the IIA-based revealed preference. The sequence $\tilde{X}_\ell$ is decreasing, and becomes constant in at most $|\mathcal{D}|$ steps. Letting $\tilde{X}^*$ be this limit set, it is easy to see that c_{obs} is consistent with Choice Overload with α-Minimal Attention if and only if $\tilde{X}^* = \emptyset$ (details left to the reader). In particular, checking consistency with the theory can be done in polynomial time. Moreover, preference identification can be performed as before.