Frequency of trade and the determinacy of equilibrium paths: logarithmic economies of overlapping generations under certainty [1] [2]

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[2]Gaetano Bloise, Chris Edmond and Stefano Lovo made interesting comments.

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Abstract

Equilibrium paths in economies of overlapping generations depend on the frequency of trade. In a logarithmic example, determinacy obtains as the frequency of trades tends to infinity or trade occurs in continuous time.

If time extends infinitely into the infinite past as well as into the infinite future, in continuous time, all non-stationary equilibrium paths of prices are time-shifts of a single path; in addition, there are two stationary solutions; in discrete time, there is a one dimensional family of non-stationary solutions, up to time-shift, but the indeterminacy vanishes as the frequency of trade tends to infinity.

If, alternatively, time has a finite starting point, in discrete time the degree of indeterminacy increases with the frequency of trade, and, in continuous time, it is infinite; however, these are families of exponentially decreasing oscillations that, although they may exhibit pseudo-chaotic behavior for a while, as time tends to infinity, all get damped, and asymptotic behavior is that of the economy that originates in the infinite past.

This is different from the effect of increases in the life span of individuals.

Key words: frequency of trade, continuous time, overlapping generations, determinacy.

Journal of Economic Literature classification numbers: D50; D90.

1 Introduction

Economies of overlapping generations, introduced by Allais (1947) and Sa - muelson (1958), display equilibrium properties different from the equilibrium properties of economies over a finite horizon or with a finite number of individuals [1]. In particular, the determinacy of equilibrium that obtains for finite economies under standard assumptions, fails for economies of overlapping generations.

The divergence between the equilibrium properties of economies of overlapping generations and economies with finitely many individuals poses a modelling dilemma.

Debreu (1970) first proved the generic determinacy or local uniqueness of competitive equilibria for finite economies; Kehoe and Levine (1985), Shannon (1996) and Shannon and Zame (1999) extended the argument to economies with a finite number of individuals over an infinite horizon, even if genericity and local uniqueness are less evident notions when the commodity space is infinite dimensional.

The focus here is on the indeterminacy of competitive equilibrium paths in economies of overlapping generations. Ever since Samuelson's (1958) remark that "we can try to cut the gordian knot by our special assumption of stationariness" and Gale's (1973) exposition of the problem, it has been recognized that competitive equilibrium paths may be indeterminate, and that the extent of indeterminacy depends on the number of commodities and the number of periods in the life-span of individuals, as well as the presence of aggregate debt.

The degree of indeterminacy is the generically maximal dimension of an open set of distinct equilibrium allocations. The argument in Geanakoplos (1987) and Geanakoplos and Brown (1982, 1985) was simple and convincing: in an economy with 2-period life-span and L commodities per period, the degree of indeterminacy is $2L-1$ if time extends infinitely into the past as well as into the future; it is $L-1$ if time has a finite starting point; in the latter case, the degree of indeterminacy increases to L for competitive equilibria with debt. An argument for determinacy in Burke (1987) and Geanakoplos and Brown (1985) restricted attention to nearly stationary equilibrium paths.

The computation of the degree of indeterminacy follows by considering a finite truncation of the economy and counting the effective degrees of freedom, the excess of endogenous, equilibrating variables, the prices of commodities, over the number of independent equilibrium conditions, the vanishing excess demand for commodities. If the exchange of commodities is restricted to dates in the interval $-T, \ldots, 0, \ldots, T$, the relevant prices are $p_{-(T+1)}, p_{-T}, \ldots, p_0, \ldots, p_T, p_{(T+1)}$, while the excess demands that should vanish along an equilibrium path are $p_{-T}, \ldots, p_0, \ldots, p_T$; the price vectors $p_{-(T+1)}$ and $p_{(T+1)}$ matter, since they affect the demand of individuals whose life span terminates at $-T$ or commences at T, respectively. Since there are L commodities at each date, while relative

[1] Geanakoplos and Polemarchakis (1991) surveyed the properties of competitive equilibria in economies of overlapping generations.

prices suffice to determine the excess demand of individuals, the number of effective equilibrating variables is $L(2T+3)-1$; the number of equilibrium equations is $L(2T+1)$, the number of commodities in the truncated economy — the value of excess demand of individuals whose life span terminates at $-T$ or commences at T need not coincide with the value of their endowments at those dates, and, as a consequence, Walras' law does not apply. It follows that, generically, the set of distinct equilibrium consumption paths contains an open set of maximal dimension $L(2T+3) - 1 - L(2T+1) = 2L - 1$. For economies with a finite starting date, the argument is analogous.

Under certainty, an economy which is individually finite: every individual is endowed with and consumes a finite number of commodities, reduces to an economy with life-span of two periods; Balasko and Shell (1981) and Geanakoplos and Polemarchakis (1991) developed the construction. In particular, an economy with one commodity at each date and life-span of n dates reduces to an economy with $L = n - 1$ commodities at each date and life-span of two dates: it suffices to observe that, then, no individual consumes at more than two, consecutive dates. It follows that the degree of indeterminacy is $2n - 3$ if time extends infinitely into the past as well as into the future and $n - 1$ if time has a finite starting date and there is aggregate debt.

Indeterminacy for a truncation of the economy is a first step but not a conclusive argument for indeterminacy over an infinite horizon. Two issues remain to be checked: (1) that the positivity constraints on consumption and prices do not interfere with the extension of equilibrium paths to infinite time; (2) that, for economies that extend into the infinite past as well as the infinite future, distinct solutions are not simply translations over time.

Concerning the impact of positivity constraints, Geanakoplos and Polemarchakis (1986) showed that indeterminacy of degree 1 is indeed generic in the very special case of one commodity at each date and life-span of two dates; the argument is geometric, and it exploits the non-linearity of equilibrium paths, but does not generalize. For economies with multiple commodities, Kehoe and Levine (1984,b) and Santos and Bonna (1989) constructed robust examples of high degrees of indeterminacy; they exploited the stability of steady state equilibria, which circumvents the non-linearity of the equilibrium path, and they restricted their attention to economies with a finite starting date. Alternatively, Kehoe, Levine, Mas-Colell and Zame (1989) developed the argument for indeterminacy in an abstract framework that allows for negative prices; the interpretation of the model is not clear.

Intertemporally separable economies, in which the utilities functions of individuals are additively separable over the two dates in their life-span, are conducive to uniqueness of the equilibrium. If time has a finite starting point, there is no debt, and generations with life span of two periods aggregate into a representative individual, Geanakoplos and Polemarchakis (1984) and Kehoe and Levine (1984,a) showed that equilibrium is generically unique; nevertheless, the restriction to a representative individual disables the equivalence between economies with multiple commodities and economies with long life span. More pertinently, Kehoe, Levine,Mas-Colell and Woodford (1991) considered economies that display gross-substitute condition at each date, and showed the uniqueness of equilibrium without debt; indetetrminacy arises with debt, but all equilibrium paths display the turnpike property of convergence to a steady state — the case of an economy with logarithmic preferences in Balasko and Shell (1981) is an instance of gross-substitutability and uniqueness.

Aiyagari (1988, 1989) and Reichlin (1992) studied the effects on equilibrium paths of increases in the life span of individuals; appropriate but not general specifications could yield convergence to the characteristics of equilibrium paths with infinitely lived individuals.

Though a conclusive demonstration is lacking, the sense of the results in the literature is that the degree of indeterminacy increases with the number of commodities at each date or, equivalently, the length of the life-span of individuals. This is particularly bothersome for policy analysis, where empirically relevant models involve life-span equal to the number of years in the economic life of an individual.

It is this intuition that is put to the test here by considering economies with increasing frequency of trade and, at the limit, economies over continuous time.

Overlapping generations over continuous time were considered by Burke (1995), but under the simplifying assumption that, at each instant, only a finite number of individuals is active; the focus was on the existence of equilibrium paths. Here, a continuum of individuals is active at each instant; but the economy is stationary and equilibrium paths, steady state paths, in particular, exist a fortiori.

High frequency of trade differs from a large number of dates in the life span: the frequency of trade can vary while the life span remains fixed.

Continuous time also serves to examine whether distinct equilibrium paths are simply translations in time; discrete time makes translations harder to identify, even define.

The argument is explicitly computational; which is the reason for restricting attention to logarithmic economies. But the results reverse the standard intuition concerning indeterminacy in economies with long life-span; and there is good evidence that they extend to an open set of utility functions.

With continuous time, equilibrium paths of prices are smooth; this, even for endowments and discount factors of individuals that do need not depend continuously on time.

With discrete time, as the frequency of trade increases, equilibrium price paths converge to the continuous time solutions.

If time extends infinitely into the infinite past as well as into the infinite future, in continuous time, all equilibrium paths of prices are smooth and are time shifts of one another, except for the two exceptional cases of stationary solutions; in discrete time there is a one dimensional family of solutions, up to time shift; however the indeterminacy should be considered an artifact due to the discretization, since vanishes as the frequency of trade tends to infinity [2].

If, alternatively, time has a finite starting point, in discrete time the degree

[2] Chris Edmond emphasized this.

of indeterminacy increases with the frequency of trade and is that predicted by Geanakoplos (1987), while, in continuous time, it is infinite; however these are families of exponentially decreasing oscillations that, as time tends to infinity, all get damped, and asymptotic behavior is that of the economy that originates in the infinite past, which confirms the turnpike property in Kehoe, Levine, Mas-Colell and Woodford (1991).

Certainty is an important assumption: in examples, with stochastic shocks, equilibrium paths are irregular, even brownian-like.

2 The economy

Alternative specifications consider time discrete or continuous; it is instructive to consider both and contrast the results.

The economy is stationary: the distribution of the fundamentals does not vary with calendar time.

One commodity is available at each date; there is no storage or production.

2.1 Discrete time

Discrete time extends into the infinite future as well as the infinite past:

$$\ldots, -(t/n), \ldots, -(1/n), 0, (1/n), \ldots, (t/n), \ldots;$$

alternatively, it has a finite starting point:

$$0, (1/n), \ldots, (t/n), \ldots.$$

The indexation of time by t/n, where t is an integer, allows for comparisons of equilibrium paths for different values of n, the reciprocal of the length of a date of time.

At each date, an individual $\tau = t + 1$ is born, and his life span extends until date $t + n$. At date $s = \tau - 1, \ldots \tau + n - 2$, the consumption of the individual is $x_{\tau,s}$, and his endowment is $e_{s-\tau+2}$, a non -negative amount; across the life-span of the individual,

$$\sum_{s=\tau-1}^{\tau+n-2} e_{s-\tau+2} = 1,$$

and the intertemporal utility function is

$$u = \sum_{s=\tau-1}^{\tau+n-2} k_{s-\tau+2} \ln(x_{\tau,s}),$$

where $k_{s-\tau+2}$ is the discount factor, a non-negative coefficient, with

$$\sum_{s=\tau-1}^{\tau+n-2} k_{s-\tau+2} = 1.$$

It is convenient to define

$$e_s = k_s = 0, \quad s \neq 1, \ldots, n.$$

The price of the commodity at date t is p_t. The wealth of individual τ is

$$w_\tau = \sum_{s=\tau-1}^{\tau+n-2} p_s e_{s-\tau+2};$$

since the individual maximizes his intertemporal utility subject to the intertemporal budget constraint

$$\sum_{s=\tau-1}^{\tau+n-2} p_s x_{\tau,s-\tau+2} \leq w_\tau,$$

his consumption demand is

$$x_{\tau,s} = \frac{k_{s-\tau+2}}{p_s} w_\tau.$$

Individuals active at date t are $\tau = t+2-n, \ldots, t+1$; since, at equilibrium, aggregate demand must coincide with the aggregate endowment,

$$\overline{x}_t = \sum_{\tau=t+2-n}^{t+1} e_{\tau,t} = 1,$$

it is necessary and sufficient that

$$\sum_{\tau=t+2-n}^{t+1} \frac{k_{t-\tau+2}}{p_t} w_\tau = 1;$$

substituting for w_τ yields the equilibrium equation for prices

$$p_t = \sum_{\tau=t+2-n}^{t+1} \sum_{s=\tau-1}^{\tau+n-2} k_{t-\tau+2} p_s e_{s-\tau+2};$$

changing the order of summation,

$$p_t = \sum_{s=t-n+1}^{t+n-1} p_s \sum_{\tau=t+2-n}^{\min\{t+1,s+1\}} k_{t-\tau+2} e_{s-\tau+2},$$

and, by a sequence of changes of variables, of $s+t$ for s and of r for $t+\tau-2$,

$$p_t = \sum_{s=1-n}^{n-1} p_{s+t} \sum_{r=-n}^{r=\min\{-1,s-1\}} k_r e_{s+r};$$

equilibrium paths of prices satisfy

$$p_t = \sum_{s=1-n}^{n-1} c_s p_{t+s},$$

where the coefficients

$$c_s = \sum_{r=-n}^{r=\min\{-1,s-1\}} k_r e_{s+r} \geq 0$$

satisfy

$$\sum_{s=1-n}^{n-1} c_s = 1.$$

To avoid degeneracies, none of the c_s vanishes; for this it is enough to require that all endowments and all discount factors are strictly positive. The equation is a linear difference equation, whose positive solutions characterize equilibrium paths.

Proposition 1. *If time extends into the infinite past, up to normalization, equilibrium prices satisfy*

$$p_t = a + bq_0^t,$$

where $0 \leq q_0$, *and* a *and* b *are non-negative real numbers, not both equal to 0; if time has a finite starting point, equilibrium prices satisfy*

$$p_t = a + bq_0^t + \sum_{k=1}^{m} q_k^t (a_k \cos(\omega_k t) + b_k \sin(\omega_k t)) + \sum_{i=1}^{s} d_i q_i^t (-1)^t,$$

where $0 \leq q_0$, a_k, b_k *and* d_i *are real numbers chosen such that price remain positive,* $2m + s = n - 2$, *and* $q_k(\cos(\omega_k) + i\sin(\omega_k))$ *and* q_i *are the* $n-2$ *complex or negative zeroes of the function* $\sum_{s=1-n}^{n-1} c_s q^s (\cos(\omega s) + i \sin(\omega s)) - 1$, *with* $0 < q_k, q_i \leq \min\{1, q_0\}$.

By homogeneity, distinct equilibrium paths are associated with distinct paths of the real rate of interest or, equivalently, of the rate of inflation,

$$\pi_t = \frac{p_{t+1}}{p_t} - 1.$$

The dimension of the set of equilibrium paths is equal to $n-1$, when time has a finite starting point; however, positivity constraints reduce the dimension to 1, as opposed to $2n-3$ when time extends to the infinite past. Furthermore, in a sense that can be made exact in continuous time, the single dimension of indeterminacy amounts to translations in time.

2.2 Continuous time

Time extends continuously into the infinite future as well as the infinite past:

$$-\infty < t < +\infty;$$

alternatively, it has a finite starting point:

$$0 \leq t < +\infty.$$

At each date, an individual, $\tau = t$ is born, and his life span extends to $\tau + 1$. At date $\tau \leq s \leq \tau + 1$, the consumption of the individual is $x_\tau(s)$, and his endowment is $e(s-\tau)$, where e is an integrable [3], positive function on the interval $[0,1]$; over the life-span of the individual,

$$\int_\tau^{\tau+1} e(s-\tau)ds = 1,$$

and the intertemporal utility function of the individual is

$$u = \int_\tau^{\tau+1} k(s-\tau)\ln x_\tau(s)ds,$$

where $k(s-\tau)$, is the discount factor is and k is an integrable, positive function on the interval $[0,1]$, which is absolutely continuous except for finitely many jumps and such that

$$\int_\tau^{\tau+1} k(s-\tau)ds = 1.$$

It is convenient to define

$$e(s) = k(s) = 0, \quad s \notin [0,1].$$

The price of the commodity at date t is $p(t)$, where p is a locally integrable, positive function, such that $p(t) \leq Ke^{N|t|}$, for some large, real numbers K and N; indeed, it will follow that, at equilibrium, the price function is smooth [4].

The wealth of individual τ is

$$w_\tau = \int_\tau^{\tau+1} e(s-\tau)p(s)ds;$$

since he maximizes his intertemporal utility subject to the intertemporal budget constraint

$$\int_\tau^{\tau+1} x_\tau(s-\tau)p(s)ds \leq w_\tau,$$

his demand is

$$x_\tau(s) = \frac{k(s-t)}{p(s)}w_\tau.$$

[3] L^1.
[4] C^∞.

At date t, individuals $t-1 \leq \tau \leq t$ are active; aggregate demand is

$$x(t) = \int_{t-1}^{t} x_\tau(t) d\tau;$$

since, at equilibrium, aggregate demand must coincide with the aggregate endowment,

$$\overline{x}(t) = \int_{t-1}^{t} e(t-\tau) d\tau = 1,$$

it is necessary and sufficient that

$$\int_{t-1}^{t} \frac{k(t-\tau)}{p(t)} w_\tau d\tau = 1$$

or

$$p(t) = \int_{t-1}^{t} k(t-\tau) (\int_{\tau}^{\tau+1} e(s-\tau) p(s) ds) d\tau.$$

The function g, defined by

$$g(t-s) = \int_{-\infty}^{+\infty} k(t-\tau) e(s-\tau) d\tau,$$

is continuous function, with generalized derivative, g', which is integrable; moreover $g(t-s) = 0$ if $|t-s| > 1$. The equilibrium equation is

$$p(t) = \int_{t-1}^{t+1} g(t-s) p(s) ds,$$

which writes in compact form as the convolution equation

$$p = g * p.$$

If p is locally integrable and satisfies the equilibrium equation, then it must be a smooth function; and the argument extends to non-logarithmic preferences.

The solutions of the equilibrium equation can be found by using a combination of Laplace and Fourier transforms and their form depends on the zeroes of the function

$$F(\lambda) = (\int_{-\infty}^{+\infty} e^{\lambda t} g(t) dt) - 1.$$

As a real function, it is a convex function of λ, and, in the generic case, it has two distinct real zeroes: 0 and another, λ_0 — they coincide in non-generic cases.

Proposition 2. *If time extends into the infinite past, equilibrium prices satisfy*

$$p(t) = a + b e^{\lambda_0 t},$$

where a and b are non-negative real numbers, not both equal to 0; if time has a finite starting point, equilibrium prices satisfy

$$p(t) = a + be^{\lambda_0 t} + \sum_{\lambda_k \leq \min\{0,\lambda_0\}} e^{\lambda_k t}(a_k \cos \mu_k t + b_k \sin \mu_k t),$$

where the a_k and b_k are real numbers chosen such that the sum converge and the price remain positive, and $\lambda_k + i\mu_k$ are the complex zeroes of the function

$$F(\lambda + i\mu) = \int_{-\infty}^{+\infty} e^{(\lambda+i\mu)t} g(t)dt - 1.$$

Since what matters are relative prices, one studies the inflation rate

$$\pi(t) = \frac{1}{p(t)} \frac{dp(t)}{dt} = \frac{\lambda_0 e^{\lambda_0 t}}{\frac{a}{b} + e^{\lambda_0 t}}.$$

To fix ideas, $\lambda_0 > 0$.

If time extends to the infinite past, there are three types of solutions: the non-stationary ones, for both a and b different from zero, that start from a value close to a constant, for t near $-\infty$, and increase exponentially with rate q as t goes to $+\infty$; and two stationary ones, when one of a or b is equal to zero, that give the two steady states.

Along non-stationary solutions, the rate of inflation converge to 0, the steady state of constant prices, as $t \to -\infty$, and to λ_0, the autarkic stationary state, as $t \to +\infty$.

Proposition 3. *If time extends to the infinite past, the non-stationary solutions are all equivalent up to time shift.*

If one is interested in solutions for positive time only, it follows from the form of the equations that one has convergence of $\pi_{a,b}(t)$ to the largest of 0 and λ_0, with exponentially decreasing fluctuations.

2.3 From discrete to continuous time

In order to compare discrete time with n dates and continuous time, one assumes that the coefficients e_j and k_j are approximations of the corresponding functions in continuous time in the sense that

$$e_j = \int_{(j-1)/n}^{j/n} e(t)dt, \quad \text{and} \quad k_j = \int_{(j-1)/n}^{j/n} k(t)dt,$$

and writes the solution of the discretized equation as

$$p_n(t) = 1 + bq_n^{[nt]},$$

where $[nt]$ is the integer part of a real number; this is a piece-wise constant function corresponding to prices of the n-date model. In the same way one defines the rate of inflation $\pi_{a,b,n}(t)$.

Proposition 4. *As $n \to \infty$, $\pi_{a,b,n}(t)$ converges uniformly to $\pi_{a,b}(t)$.*

This proposition also justifies the growth condition on prices assumed for continuous time.

3 Proofs

The proofs use similar ideas in continuous and discrete time; the latter is technically simpler. In both cases, one first looks for real valued solutions and then imposes the positivity constraint. To solve the equation in continuous time, the essential tool is the Laplace transform, which transforms convolutions to products, easier to analyze. For solutions that are bounded, or bounded by a polynomial, the standard tools of Fourier analysis yield that, generically, the only solutions are the constants. For solutions that are exponential, the argument requires a refinement, as follows: first one splits the solution, $p(t)$ into two parts, $p_-(t)$ and $p_+(t)$, that have support in a half line and satisfy the convolution equation up to an error term, $f(t)$, with compact support. To these, one applies the Laplace transform and uses estimates to prove that $p(t)$ is a sum of exponentials, exact solutions of the convolution equation, and a bounded error term, r(t), that has to be a solution as well. To the bounded term, the standard argument in Fourier analysis applies to show that it is necessarily constant.

Proof of Proposition 1 Complex solutions of the equilibrium equation depend on the roots of [5]

$$P(z) = \sum_{s=1-n}^{n-1} c_s q^s (\cos(\omega s) + i \sin(\omega s)) = 1,$$

where $z = q(\cos(\omega) + i\sin(\omega))$ is a complex number. For simplicity, all are simple zeroes — the case of multiple zeroes can be easily worked.

Lemma 1. *As a function on the positive real line, that is for $\omega = 0$, the function $P(q)$ is convex, it tends to infinity as q tends to infinity or zero, and it is such that*

1. *$P(1) = 1$, and there is another root, $P(q_0) = 1$, where, $q_0 \neq 1$ in the generic case, while the two roots coincide in the non-generic case;*

2. *if $\min\{1, q_0\} < q < \max\{1, q_0\}$, then $P(q) < 1$;*

3. *other than 1 and q_0, there are no other real or complex roots of $P(z) = 1$ with $\min\{1, q_0\} \leq \|z\| \leq \max\{1, q_0\}$; in particular, there are no solutions with $\|z\| = 1$;*

4. *there are $2(n-2)$ other roots, all complex or negative, of which $n-2$ with $\|z\| < \min\{1, q_0\}$ and $n-2$ with $\|z\| > \max\{1, q_0\}$.*

[5] Goldberg (1958).

Proof (1) The function $P(q)$ is convex on the positive real line, since it is the sum of convex functions, integer powers of q, with positive coefficients. Since $\sum_{s=1-n}^{n-1} c_s = 1$, $P(1) = 1$. If the first derivative $P'(1) < 0 (> 0)$, there is another root, $P(q_0) = 1$, with $q > 1 (< 1)$; in the non-generic case of $P' = 0$, there is a double root at 1.

(2) follows immediately, since P is convex.

(3) Since $P(z) < 1$, for z real, with $1 < z < q_0$, and all the coefficients in $P(z)$ are non-negative, if $\min\{1, q_0\} < \|z\| < \max\{1, q_0\}$, then $\|P(z)\| < P(\|z\|) < 1$, while, if $\|z\| = 1$ or $\|z\| = q_0$, and z is not real and positive, still $\|P(z)\| < 1$, with strict inequality, since, in the sum that defines $P(z)$, not all terms can be positive.

(4) The family of polynomials $P(z)$ as the coefficients c_s vary is parametrized by the open simplex $\Delta^{2n-2} = \{c > 0 : \sum_{s=1-n}^{n-1} c_s = 1\}$, and, thus it is connected. Given any two polynomials in it, $P_0(z)$ and $P_1(z)$, the path joining them is $[P_\tau(z) : 0 \leq \tau \leq 1]$. From (3), no root of $P_\tau(z)$, other than 1 or q_0 can cross the annulus $\min\{1, q_0\} \leq \|z\| \leq \max\{1, q_0\}$; it follows that the number of solutions, with multiplicity, inside the disc $\|z\| \leq min\{1, q_0\}$ is constant along the path, as is the number outside it. If $P_0(z) = P_1(z^{-1})$, both polynomials are in the family, and every solution of $P_0(z) = 1$ with $\|z\| < 1$ yields a solution of $P_1(z^{-1}) = 1$ with $\|z\| > 1$. Since the two polynomials can be joined by a path along which the number of solutions inside or outside the disc is constant, it follows that they are both equal to $n - 2$. □

The formula for the equilibrium price, p_t, implies that, as t tends to infinity, the argument of p_t is determined by the argument of the term $a_k z_k^t$, where z_k is the root with largest absolute value. It follows that this root must be real and equal to the largest of 1 and q; and that all other z_k with $a_k \neq 0$ must lie inside the unit circle.

This gives the formula for the equilibrium paths when time has a finite starting point: $t \geq 0$. If positivity of prices is required for $t \leq 0$, and, in particular as $t \to -\infty$, this eliminates also the z_k with $\|z_k\| < 1$, and the only solutions are $p_t = a + bq_0^t$. □

Proof of Proposition 2 In continuous time the equilibrium equation is

$$p(t) = \int_{t-1}^{t+1} g(t-s)p(s)ds$$

or, in compact form,

$$p = g * p,$$

where g is a nonnegative, absolutely continuous function with support on the closed interval $[-1, +1]$, such that $\int_{-1}^{+1} g(t)dt = 1$.

One seeks the solution in the space of locally integrable functions that satisfy the growth condition $p(t) < Ke^{N|t|}$ almost everywhere, for some K and N.

Lemma 2. *The function $p(t)$ is smooth.*

Proof The derivative, g', in the sense of distributions, of g is an integrable function. If p' is the distributional derivative of p, then $p' = g' * p$, and since the convolution of two integrable functions is an integrable function, p' is integrable and p is absolutely continuous; in this manner, all derivatives of p are continuous. □

Next one characterizes complex solutions of the equilibrium equation.

Lemma 3. *If p solves the equilibrium equation and g is generic in the sense to be specified below, then p is of the form*

$$p(t) = \sum_{-N \leq \lambda_k \leq N} c_k e^{(\lambda_k + i\mu_k)t}.$$

Proof If $\phi(t)$ is a smooth function such that $\phi(t) \geq 0$, $\phi(t) = 0$, for $t < 0$, and $\phi(t) = 1$, for $t > 1$, and if $p_+(t) = \phi(t)p(t)$, and $p_-(t) = (1 - \phi(t))p(t)$, then it follows from the equilibrium equation that p_+ satisfies the equation

$$p_+(t) - p_+(t) * g(t) = f_+(t),$$

where $f_+(t)$ is a smooth function with support in $[-1, 2]$.

If

$$\overline{p}_+(z) = \int_{-\infty}^{+\infty} e^{-zt} p_+(t) dt$$

is the Laplace transform [6] of $p(t)$, where $z = \lambda + i\mu$ is a complex number, then $\overline{p}_+(z)$ exists and is an analytical function on the semi-plane $\lambda \geq N$; similarly, one defines $\overline{f}_+(z)$ and $\overline{g}(z)$, which are defined and analytic on the whole plane, because f and g have compact support and so the integral converges for every z.

Lemma 4. *For all l and K, there exist $C_{l,K}$, such that*

$$|\overline{f}_+(\lambda + i\mu)| < \frac{C_{l,K}}{|\mu|^l}, \quad |\mu| \to \infty$$

is an estimate of the decay of f, while

$$|\overline{g}(\lambda + i\mu)| \leq \frac{o_K(\mu)}{|\mu|}, \quad |\mu| \to \infty.$$

is an estimate the decay of g.

Here, $C_{l,K}$ is a constant that does not depend on λ, as long as $\lambda \in [-K, K]$; $o_K(\mu)$ is a function with the same properties that tends to zero as $|\mu|$ tends to infinity.

[6] Smith (1966).

Proof This is standard, given that $f_+(t)$ is smooth and g is absolutely continuous [7].

In the case of g, integration by parts yields

$$|\overline{g}(\lambda + i\mu)| = |\int_{-\infty}^{+\infty} e^{-(\lambda+i\mu)t} g(t)dt| = \frac{1}{|\lambda + i\mu|} \left| \int_{-\infty}^{+\infty} e^{-(\lambda+i\mu)t} g'(t)dt \right|;$$

the fact that g' is an integrable function, with support $[-1, +1]$ yields the bound

$$\tfrac{1}{|\lambda+i\mu|} \left| \int_{-\infty}^{+\infty} e^{-(\lambda+i\mu)t} g'(t)dt \right| \le \tfrac{e^{\lambda}}{|\lambda+i\mu|} \left| \int_{-\infty}^{+\infty} e^{-(\lambda+i\mu)t} g'(t)dt \right|$$

$$\le \tfrac{e^{k}}{|\mu|} \left| \int_{-\infty}^{+\infty} e^{-(\lambda+i\mu)t} g'(t)dt \right|;$$

the integral is the Fourier transform of $g'(t)$ and the Lebesgue theorem [8] yields that the Fourier transform of an integrable function is continuous and tends to 0 at infinity.

A similar argument, iterating the integration by parts l times yields the result for f. □

Corollary 1. *For any $K > 0$, the function $1 - \overline{g}(\lambda + i\mu)$ has finitely many zeroes in the strip $|\lambda| \le K$, for any K; it has at most two real zeroes, $\lambda = 0$ and $\lambda = \lambda_0$, possibly coinciding in the non-generic case; there are no other complex zeroes in the closed annulus $\{\lambda + i\mu : \min\{0, \lambda_0\} \le \lambda \le \max\{0, \lambda_0\}\}$.*

Proof The assertion about finitely many zeroes follows immediately from the decay of $\overline{g}$. For the rest, one notes that $\overline{g}(\lambda)$ is convex on the real line, since, by differentiation with respect to λ twice under the integral sign,

$$\frac{d^2}{d\lambda^2}\overline{g}(\lambda) = \int_{-\infty}^{+\infty} t^2 e^{-\lambda t} g(t)dt > 0,$$

and tends to ∞ as $|\lambda| \to \infty$. Moreover, one has $\overline{g}(0) = 1$, and, thus, the result about the real zeroes of $\overline{g}$ follows. That there are no complex zeroes in the closed strip follows from the strict inequality

$$|\overline{g}(\lambda + i\mu)| = |\int_{-\infty}^{+\infty} e^{-(\lambda+i\mu)t} g(t)dt| < \int_{-\infty}^{+\infty} |e^{-\lambda t} dt| = \overline{g}(\lambda),$$

which holds when $\mu \ne 0$, since $e^{-\lambda t} g(t)$ is always positive, while the convexity of $\overline{g}(\lambda)$ implies that

$$\overline{g}(\lambda) \le 1, \quad 0 \le \lambda \le \lambda_0.$$

From now on, one assumes that these zeroes are simple; this is true in the generic case, and it simplifies some of the arguments — the extension to the general case is left to the reader.

[7] Rudin (1977).
[8] Rudin(1977).

A standard property of the Laplace transform says that, if p_+ satisfies the auxiliary convolution equation, then

$$\overline{p}_+(z) - \overline{g}(z) * \overline{p}_+(z) = \overline{f}_+(z),$$

so that

$$\overline{p}_+(z) = \frac{\overline{f}_+(z)}{1 - \overline{g}(z)};$$

this means that $\overline{p}_+(z)$ extends as a meromorphic function to the whole complex plane, whose poles are contained in the zeroes of $1 - \overline{g}(z)$; moreover the decay estimates and corollary 1 imply that $\bar{p}_+(\lambda + i\mu))$ has only finitely many poles in the strip $|\lambda| \leq K$, for any K, and that $|\overline{p}_+(\lambda + i\mu)| \leq (C/|\mu|^k)$, as $|\mu|$ tends to infinity. By the assumption on growth, $\overline{p}_+(\lambda + i\mu)$ has no poles in the plane $\lambda \geq N$.

According to the Laplace inversion formula [9], one can recover $p_+(t)$ as the integral

$$p_+(t) = \int_{-\infty}^{+\infty} e^{(\lambda + i\mu)t} \overline{p}_+(\lambda + i\mu) d\mu,$$

for any $\lambda \geq N$.

If one denotes the last integral $I(\lambda, t)$, the decay estimates together with Cauchy formula imply that $I(\lambda, t) = I(\lambda', t)$ if there are no poles of $\overline{p}_+(t)$ in the strip between λ and λ', $\lambda' \leq \lambda$. On the other side, if $\{z_1; \ldots; z_k\}$ is the finite set of poles of $\overline{p}_+(t)$ in this strip, one has

$$I(\lambda, t) = I(\lambda', t) + \sum_k c_k e^{i\mu_k} e^{z_k t},$$

where $c_k e^{i\mu_k}$ are the residues of $\overline{p}_+(z)$ at z_k. Moreover, up to a constant, $|I(\lambda', t)| \leq e^{\lambda' t}$ for $t \to +\infty$, by the decay estimates.

It follows that

$$p_+(t) = (\sum_{0 \leq \lambda_k \leq N} c_k e^{(\lambda_k + i\mu_k)t}) + r_+(t),$$

where $r_+(t)$ is a bounded function for t going to plus infinity by the discussion above and is bounded for t going to minus infinity because $p_+(t)$ is zero in this case and the sum is bounded.

One proves an analogous inequality for $p_-(t)$ and putting the two together one gets

$$p(t) = \sum_{-N \leq \lambda_k \leq N} c_k e^{(\lambda_k + i\mu_k)t} + r(t),$$

where $r(t)$ is a bounded function on the real line.

Now both $p(t)$ and the expression in the sum are solutions of the convolution equation, and it follows that the bounded function $r(t)$ is as well; an easy

[9] Smith (1966).

argument, using the Fourier transform, shows that the only bounded solutions of the convolution equation, in the generic situation of simple zeros of $1-\overline{g}(\lambda+i\mu)$, are the constants. □

Proof of proposition 3 If s is such that $e^{\lambda_0 s} = a/b$,

$$\pi_{a,b}(t+s) = \frac{\lambda_0 (b/a) e^{\lambda_0(t+s)}}{1+\frac{b}{a}e^{\lambda_0(t+s)}} = \frac{\lambda_0 e^{-\lambda_0 s} e^{\lambda_0(t+s)}}{1+e^{-\lambda_0 s}e^{\lambda_0(t+s)}} = \frac{e^{\lambda_0 t}}{1+e^{\lambda_0 t}} = \pi_{1,1}(t).$$

□

Proof of Proposition 4 One considers only the generic case $q \neq 1$.

Given n, the price is $p_n(t) = a + bq_n^{[nt]}$.

If e_j, k_j and c_j are related to $e(t), k(t)$ and $g(t)$, as before, then $q_n^n \to q$, as $n \to \infty$, where q is the solution in continuous time:

In fact by the proof of proposition 1, q_n is the real root of the equation

$$P_n(z) = \sum_{r=1-n}^{n-1} c_r z^r = 1,$$

different from 1, while λ_0 is the root of

$$F(\lambda) = \int_{-1}^{1} e^{-\lambda t} g(t) dt = 1,$$

with the same properties. Setting $z = e^{-\lambda/n}$, one sees that $P_n(e^{-\lambda/n})$ is the Riemann sum of the integral that defines $f(\lambda)$. It follows easily that the sequence of functions $(P_n(e^{\lambda/n}) : n = 2, \ldots)$ of λ converges uniformly with its derivatives on compact subsets to $F(\lambda)$; this, in turn, implies that, if $P_n(e^{\lambda_n/n}) = 1$, then $\lambda_n \to \lambda_0$. Since $q_n = e^{\lambda_n/n}$ and $q = e^{\lambda_0}$, it follows that $q_n^n \to q$. □

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