Asset Returns
in an Endogenous Growth Model
with Incomplete Markets*

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Abstract

This paper analyzes a class of stochastic endogenous growth models with uninsurable idiosyncratic income risk. The model economy is populated by infinitely-lived households who own and operate their own business, work for a stock company, and participate in stock and bond markets. Households have time- and state-additive log-utility preferences and production functions exhibit constant returns to scale with respect to produced input factors (physical and human capital). This paper shows that if the idiosyncratic component of productivity and depreciation shocks is unpredictable, then there exists an equilibrium in which households choose not to trade bonds. This no-trade result implies that equilibria can be found by solving a one-agent decision problem. The paper also analyzes the asset return implications of a calibrated model economy with an individual income process that displays realistic variations in idiosyncratic income risk. The calibrated model economy generates a sizable mean equity premium (1%) if the volatility of implied stock returns matches the volatility of observed U.S. stock returns.

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I. Introduction

Dynamic general equilibrium models provide a useful framework for the study of business-cycle fluctuations and asset returns. For the most part, the literature on business cycles has developed independently from the asset pricing literature.[1] This dichotomy is unfortunate since movements in both prices (asset returns) and quantities (output, consumption, employment) provide useful information for tests of general equilibrium models. In this paper we assess the business cycle and asset return implications of a tractable incomplete-markets model of economic growth.

The model we analyze in this paper is an incomplete-markets version of the class of convex growth models analyzed, among others, by Alvarez and Stokey (1998), Jones and Manuelli (1990), Jones, Manuelli, and Siu (2000), and Rebelo (1991). In this class of models, production displays constant-returns-to-scale with respect to reproducible input factors, households are infinitely-lived and have homothetic preferences, and markets are competitive. In the particular model analyzed here, there are two input factors, physical and human capital, and households have log-utility preferences. Moreover, we assume that production of the homogeneous good takes place in two sectors. The first sector consists of many ex-ante identical businesses owned and operated by individual households (the "entrepreneurial sector"). Production in this sector is subject to productivity and depreciation shocks, and the idiosyncratic component of these shocks is assumed to be unpredictable.[2] The second sector consists of a large stock company (the "stock-market

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[1] More precisely, the business cycle literature has usually ignored the asset pricing implications, and the asset pricing literature has followed Lucas (1978) and Mehra and Prescott (1985) by confining attention to exchange economies with exogenous aggregate consumption. See Cooley (1995) for a survey of the business cycle literature and Campbell (1999),Constantinides (2002), and Kocherlakota (1996) for recent surveys of the asset pricing literature. For notable exceptions, see Boldrin, Christiano, and Fisher (2001), Cochrane (1991), Danthine and Donaldson (1994), Jermann (1998), Lettau and Uhlig (2000), McGrattan and Prescott (2001), Storesletten, Telmer, and Yaron (2001), and Tallarini (2000).

[2] The introduction of a production sector in addition to the stock-market sector allows us to discuss two sources of idiosyncratic risk: labor income risk and proprietary income risk (entrepreneurial risk). Empirically, uninsured proprietary income risk appears to be an important component of idiosyncratic risk (Heaton and Lucas 1999).

sector") and households have the opportunity to participate in the production process by purchasing equity shares and supplying labor in competitive markets. The market structure is incomplete in the sense that households can trade stocks, bonds, and human capital in frictionless markets, but cannot directly insure against idiosyncratic income shocks.[3]

In this paper, we show that there exists an equilibrium in which households optimally choose not to use bond trading (borrowing and lending) to smooth out idiosyncratic income shocks.[4] This no-trade result extends the work by Constantinides and Duffie (1996) to production economies with not necessarily normally distributed random variables.[5] As in Constantinides and Duffie (1996), the idiosyncratic component of log-income follows (approximately) a random walk, and borrowing and lending is therefore a highly ineffective means to insulate consumption from income shocks. In contrast to Constantinides and Duffie (1996), in this paper the random walk property of log-income is an endogenous outcome, and not all income is consumed (aggregate saving is positive).

In our production economy with physical and human capital accumulation, the intuition for the no-trade result is as follows. In equilibrium, each household faces a standard multi-asset portfolio choice problem of the Merton-type (Merton, 1969,1971): investment in the risk-free bond, investment of physical and human capital in the entrepreneurial sector, and investment of physical and human capital in the stock-market sector.[6] Because household preferences are logarithmic, the

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[3] The assumption that human capital can be sold is problematic, but essential to keep the model tractable. Economic intuition suggests that the introduction of non-negativity constraints on human capital investment is likely to lead to a larger effect of human capital risk on individual consumption and asset prices.

[4] This no-trade result still holds if households can trade an arbitrary number of assets whose payoffs only depend on the aggregate state of the economy.

[5] There is also an extensive computational literature on infinite-horizon, incomplete-market models. See, for example, Huggett (1993) for work on exchange economies and Aiyagari (1994) and Krusell and Smith (1998) for papers dealing with production economies.

[6] Although there is an extensive literature on intertemporal portfolio choice building on Merton's work, this framework has not previously been used to study formally the process of human capital formation. Khan and Ravikumar (2001), Obstfeld (1994), Saito (1998), and many others have used models with only physical capital but no human capital (linear production functions) to analyze the

optimal portfolio shares are wealth independent. Because idiosyncratic shocks are unpredictable, the portfolio shares are also independent of current and past idiosyncratic shocks. Thus, the demand for bonds is the same for all households. Since the bond is in zero net-supply, the only way to achieve market clearing is to have zero individual demand (no bond trading).

The no-trade feature of the model simplifies the computation of equilibria dramatically since the problem of finding an equilibrium is reduced to solving a one-agent decision problem. To illustrate the usefulness of the model for quantitative work, we analyze the business cycle and asset return implications of a calibrated version of the model. Our analysis produces several results. First, the model's implications for aggregate quantity variables are similar to the implications of the representative-agent version of the model, which have been extensively studied by Jones, Manuelli, and Siu (2000). However, in contrast to Jones et al. (2000), in this paper we allow for aggregate depreciation shocks, and this extension significantly improves the model's ability to match both consumption volatility and output volatility. In our incomplete-markets model these aggregate depreciation shocks correspond to changes in the rate of business failure (loss of specific physical capital) and job destruction (loss of specific human capital).[7]

Our second finding is that the presence of uninsurable idiosyncratic income risk substantially increases the mean equity premium. In our model, as in the work by Constantinides and Duffie (1996) and Storesletten, Telmer, and Yaron (2001), this result is driven by two features of the income process: idiosyncratic income shocks are permanent and the amount of idiosyncratic risk is increasing during economic downturns. Recent empirical work has shown (Meghir and Pistaferri, 2001, and Storesletten et al., 2001) that individual income data are well-described by an income process that exhibits a large and counter-cyclical permanent component, and we use the estimates of this empirical literature to calibrate the model economy. In other words, our results are based on

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effects of incomplete risk sharing, and Cox, Ingersoll, and Ross (1985) derive the asset pricing implications of a complete-markets model with linear technologies. Campbell (1996) uses a representative-agent model with physical and human capital to study stock and bond returns.

[7] Aggregate depreciation shocks are also used in Storesletten et al. (2001).

realistic assumptions about the amount and variation of permanent idiosyncratic income risk.[8]

We also find that introducing aggregate depreciation shocks increases the volatility of stock returns significantly. In particular, our model implies a stock return volatility that is much higher than the stock return volatility in the simple exchange economy analyzed by Mehra and Prescott (1985) .[9] This volatility increase provides a second reason for our model's ability to generate a non-negligible equity premium: higher (unpredictable) volatility of stock returns means stock market investment is riskier, which in turn increases the equity premium demanded by risk-averse investors.

Although our incomplete-markets model with production generates values for the equity premium and the volatility of stock returns that are substantially higher than the ones found by Mehra and Prescott (1985), these values are still far below the observed values for the U.S. stock market. In our production economy, the main reason for the model's inability to generate realistic variations in stock returns is the assumption that there are no market frictions in addition to market incompleteness (no capital adjustment costs). Thus, stock returns are equal to the marginal product of capital net depreciation, and stock returns therefore inherit the relatively low volatility of the time series of the marginal product of capital and aggregate depreciation. In order to investigate the model's equity premium once this volatility problem has been overcome, we also analyze a version of the model in which there is only an entrepreneurial sector so that we can match the observed stock return volatility without generating unrealistically large variation in output and consumption. That is, we consider an economy without a stock market sector and back out stock returns by finding that stock return process which induces households not to hold any stocks. For this version of the model economy the implied equity premium is quite large (1%). In other words, once the volatility problem has been overcome, the present incomplete-markets model is capable of generating a

[8] Heaton and Lucas (1996) use an econometric specification that does not allow for income shocks of different degrees of persistence, and estimate an overall autocorrelation coefficient around .50.

[9] Notice that in the simple exchange model studied by Mehra and Prescott (1985) one important reason for the low stock return volatility is the low volatility of dividends, which in turn is a consequence of the assumption that dividends are equal to output. This problem has been noticed by several authors. See, for example, McGrattan and Prescott (2001).

substantial equity premium. An important topic for future research is to extend the current model so that even with a sizable stock market sector it can match both the volatility of stock returns and the volatility of aggregate consumption and output.[10]

II. The Model

a) The Economy

We consider a discrete-time, infinite-horizon production economy populated by infinitely-lived households. Time is indexed by t and individual households are indexed by i. To avoid mathematical technicalities, we assume that the number of households, I, is finite.

Information and uncertainty are modeled as follows. A complete description of the state of the economy in period t is given by a vector $(s_{1t},\ldots,s_{It},S_t)$, where we interpret s_{it} as a household-specific (idiosyncratic) shock and S_t as an economy-wide (aggregate) shock. We assume that S_t is an element of a time-independent set, $\boldsymbol{S}$, and that s_{it} is an element of a time- and household-independent set, $\boldsymbol{s}$. The formal arguments assume that the two sets $\boldsymbol{s}$ and $\boldsymbol{S}$ are finite. We denote the vector of idiosyncratic shocks in period t by $s_t = (s_{1t},\ldots,s_{It})$. A (partial) history of idiosyncratic and aggregate shocks is denoted by $s^t = (s_0,\ldots,s_t)$ and $S^t = (S_0,\ldots,S_t)$, respectively. Clearly, the (ordered) set of all histories defines an event tree with date-events (nodes) (s^t,S^t). Throughout the analysis, we fix this event tree. We assume that all households observe (s^t,S^t) at time t , but all results still hold if agent i only observes (s_i^t,S^t).

Households have common prior beliefs so that the probability of the date-event (s^t,S^t), denoted by $\pi(s^t,S^t)$, is the same for all households. For simplicity, we assume $\pi(s^t,S^t) > 0$ for all date-events (s^t,S^t). We make two further assumptions on these probabilities. First, past idiosyncratic shocks have no predictive power: $\pi(s_{t+1},\ldots,s_{t+n},S_{t+1},\ldots,S_{t+n} \mid s^t,S^t) = \pi(s_{t+1},\ldots,s_{t+n},S_{t+1},\ldots,S_{t+n} \mid S^t)$, where the symbol $\pi(A \mid B)$ stands for the probability that A given B. This assumption implies $\pi(s_{t+1},\ldots,s_{t+n} \mid S^{t+n}) =$

[10] Jermann (1998) and Lettau and Uhlig (2000) introduce a quadratic adjustment cost to break the tight link between stock returns and the net marginal product of capital, and Boldrin et al. (2001) use a two-sector model with limited inter-sectoral factor mobility.

$\pi(s_{t+1} \mid S^{t+n})\, \pi(s_{t+2} \mid S^{t+n}) \ldots \pi(s_{t+n} \mid S^{t+n})$.[11] Second, the conditional probability distribution of idiosyncratic shocks is symmetric with respect to households: $\pi(\ldots, s_{i,t+m}, \ldots, s_{j,t+m}, \ldots \mid S^{t+n}) = \pi(\ldots, s_{j,t+m}, \ldots, s_{i,t+m}, \ldots \mid S^{t+n})$ for all $m \leq n$ and i, j. The last assumption implies that the marginal distributions, $\pi_i(s_{i,t+m} \mid S^{t+n}) \doteq \sum_{s_{-i,t+m}} \pi(s_{t+m} \mid S^{t+n})$, are the same for all households $i = 1, \ldots, I$. In the special case in which the state process is Markov with transition probabilities $\pi(s_{t+1}, S_{t+1} | s_t, S_t)$, the two assumptions read $\pi(s_{t+1}, S_{t+1} | s_t, S_t) = \pi(s_{t+1}, S_{t+1} | S_t)$ and $\pi(\ldots, s_{i,t+1}, \ldots, s_{j,t+1}, \ldots, S_{t+1} | S_t) = \pi(\ldots, s_{j,t+1}, \ldots, s_{i,t+1} \ldots, S_{t+1} | S_t)$.

Economic variables at time t are defined by functions of the following type: $\{f_t\}_{t=0}^{\infty}, f_t = f_t(s^t, S^t)$ or $\{F_t\}_{t=0}^{\infty}, F_t = F_t(S^t)$.[12] Any function f_t, or F_t, defines a random variable in the canonical way. For this random variable, we denote the unconditional expectations by $E[f_t] = \sum_{(s^t, S^t)} \pi(s^t, S^t) f_t(s^t, S^t)$ and the conditional expectations by $E[f_{t+n} | s^t, S^t] = \sum_{(s^{t+n}, S^{t+n}) \in \boldsymbol{D}_n(s^t, S^t)} \pi(s^{t+n}, S^{t+n} | s^t, S^t) f_{t+n}(s^{t+n}, S^{t+n})$, where $\boldsymbol{D}_n(s^t, S^t)$ is the set of all nodes (s^{t+n}, S^{t+n}) succeeding (s^t, S^t).

There is one good which is non-perishable and can be used for either consumption or production purposes. There are two sectors producing the same "all purpose" good. In the first sector, individual households act as entrepreneurs and invest in their private production opportunities. More specifically, household i produces output, y_{1it}, using physical capital, k_{1it}, raw labor, l_{1it}, and human capital, h_{1it}, according to the production function $y_{1it} = A_{1it} f(k_{1it}, l_{1it} h_{1it})$.[13] Here $l_{1it} h_{1it}$ stands for effective labor supplied by household i, A_{1it} denotes a household-specific stochastic productivity shock, and f is a standard neoclassical production function (in particular, it

[11] Consider the case $n = 2$. We have $\pi(s_{t+1}, s_{t+2} \mid S^{t+2}) = \pi(s_{t+2} \mid s_{t+1}, S^{t+2})\, \pi(s_{t+1} \mid S^{t+2}) = \pi(s_{t+2} \mid S^{t+2})\, \pi(s_{t+1} \mid S^{t+2})$. The proof for the case $n > 2$ uses induction and the preceding argument.

[12] The notation $F_t = F_t(S^t)$ is shorthand for $F_t : \boldsymbol{S}^t \rightarrow \mathbb{R}^n$, $F_t = F_t(S^t)$.

[13] The one-good (sector) model is of course equivalent to a three-good model in which each "good" (consumption good, physical capital, human capital) is produced in different sectors of the economy using the same production function. Given the one-good (identical production function) assumption, one can also easily incorporate a third producible factor (ideas, general knowledge).

exhibits constant returns to scale). Raw labor, l_{1it}, is inelastically supplied, and we normalize the amount supplied in each period to one: $l_{1it} = 1$. The stochastic productivity process $\{A_{1it}\}_{t=0}^{\infty}$ satisfies $A_{1it} = A_{1it}(s_{it}, S^t)$, that is, the current productivity realization depends on the current idiosyncratic shock and the (partial) history of aggregate shocks. Physical capital employed in production site i depreciates in period t at the stochastic rate $\delta_{k_1 it}$ and human capital at the rate $\delta_{h_1 it}$. The stochastic depreciation processes $\{\delta_{k_1 it}\}_{t=0}^{\infty}$ and $\{\delta_{h_1 it}\}_{t=0}^{\infty}$ satisfy $\delta_{k_1 it} = \delta_{k_1 it}(s_{it}, S^t)$ and $\delta_{h_1 it} = \delta_{h_1 it}(s_{it}, S^t)$. We can interpret a negative depreciation shock as bankruptcy (business closure), an event that is likely to lead to a substantial loss of (the value of) installed physical capital. To the extent that the entrepreneur has acquired business- or sector-specific skills, the event of bankruptcy may also result in a significant reduction in human capital.

The second production sector of the economy consists of one big stock company which also combines physical and human capital to produce the one good. The aggregate production function of this sector is $Y_{2t} = A_{2t} F(K_{2t} E_{2t})$, where E_{2t} is the level of human-capital-weighted labor employed in the second sector, A_{2t} is a parameter measuring total factor productivity, and F is a standard neoclassical production function. We assume that the stochastic process $\{A_{2t}\}_{t=0}^{\infty}$ is defined by functions of the type $A_{2t} = A_{2t}(S^t)$ and that physical capital employed in the second sector depreciates according to $\{\delta_{k_2 t}\}_{t=0}^{\infty}$, $\delta_{k_2 t} = \delta_{k_2 t}(S^t)$.

There is one long-lived asset in positive net-supply (equity) and one short-lived security in zero net-supply (bond). The payoff process of equity (dividends) is defined by a sequence of functions (random variables) $\{D_t\}_{t=0}^{\infty}$, $D_t = D(S^t)$. The payoff process of the bond is constant and normalized to one (risk-free asset).[14] We assume that dividends are non-negative and have bounded growth rates, but allow for the possibility that for some histories (sample paths of the state process) the corresponding dividend sequences are unbounded.

Households can participate in the production process of the second sector by purchasing equity contracts and supplying effective labor in competitive markets. Hence, households are not

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[14] The extension to the case of bonds with different maturities or risky payoffs is straightforward.

only entrepreneurs, but also shareholders and workers. More precisely, for the type of equilibria considered here, in each period each household has physical and human capital employed in both sectors. One interpretation of this feature of the model is that one member of the household operates a "family business" (proprietary business) and a second member works for a large corporation (stock company).[15] In addition to the assumptions already made, we assume that human capital employed in the second sector by household *i* depreciates according to $\{\delta_{h_2 it}\}_{t=0}^{\infty}$, $\delta_{h_2 it} = \delta_{h_2 t}(s_{it}, S^t)$. A negative individual-specific depreciation shock could be due to the event of a job loss (closing of an establishment/ plant) in period *t* with subsequent new employment in period *t+1* assuming that the household had acquired job-specific or sector-specific skills before the job loss.

The assumptions made so far ensure that the future productivity shocks, $A_{1i,t+n}$, and the future depreciation rates, $\delta_{h_1 i,t+n}$, $\delta_{k_1 i,t+n}$, and $\delta_{h_2 i,t+n}$, are identically distributed across households conditional on (s^t, S^{t+n}). A particular example of an economy for which these assumptions are satisfied is one in which the productivity and depreciation variables are the sum of an idiosyncratic and an aggregate component and the idiosyncratic component is unpredictable. As will be shown below (proposition), the assumptions made so far in conjunction with the assumption of log-utility preferences imply that in equilibrium households will not use borrowing and lending (bond-trading) to smooth out idiosyncratic shocks. As in Constantinides and Duffie (1996), in equilibrium the future consumption and income growth rates will be identically distributed across households conditional on (s^t, S^{t+n}).

Households have identical preferences over stochastic consumption sequences, $\{c_{it}\}_{t=0}^{\infty}$, which allow for a time-additive expected utility representation with logarithmic one-period utility function

$$U(\{c_{it}\}_{t=0}^{\infty}) = \lim_{T\to\infty} E\left[\sum_{t=0}^{T} \beta^t \log c_{it}\right], \tag{1}$$

[15]An alternative interpretation is that the first sector corresponds to home production (Benhabib et al. 1991 and Greenwood and Hercowitz 1991).

where $\beta \in (0,1)$ denotes the common pure discount factor.

b) Equilibrium

We denote the equity price by P_t, the bond price by Q_t, and the wage per unit of effective labor (human capital) employed in the second sector by W_t. In our definition of equilibrium, we confine attention to asset prices and wages of the form $Q_t = Q_t(S^t)$, $P_t = P(S^t)$, and $W_t = W_t(S^t)$. We denote household $i's$ beginning-of-period per capita holdings of equity by θ_{it} and the corresponding beginning-of-period per capita holdings of bonds by b_{it}. Households' common information set in period t is (s^t,S^t).[16] Thus, household i chooses $\{c_{it},b_{it},\theta_{it}\}_{t=0}^{\infty}$, with $c_{it} = c_{it}(s^t,S^t)$, $b_{i,t+1} = b_{i,t+1}(s^t,S^t)$, and $\theta_{i,t+1} = \theta_{i,t+1}(s^t,S^t)$.

The feasible choices of $\{c_{it},b_{it},\theta_{it}\}_{t=0}^{\infty}$ are defined by the sequential budget constraint:[17]

$$\forall\ t:$$

$$\begin{aligned}
&\text{i)}\quad c_{it} + x_{k_1 it} + x_{h_1 it} + x_{k_2 it} + x_{h_2 it} + x_{bit} = A_{1it} f(k_{1it},h_{1it}) + W_t h_{2it} + D_t\theta_{it} + b_{it}\\
&\text{ii)}\quad k_{1i,t+1} = (1-\delta_{k_1 it})k_{1it} + x_{k_1 it} \;,\; k_{1it} \geq 0 \;,\; k_{1i0} = k_{10} \text{ given}\\
&\text{iii)}\quad h_{1i,t+1} = (1-\delta_{h_1 it})h_{1it} + x_{h_1 it} \;,\; h_{1it} \geq 0 \;,\; h_{1i0} = h_{10} \text{ given}\\
&\text{iv)}\quad P_t\theta_{i,t+1} = P_t\theta_{it} + x_{k_2 it} \;,\; \theta_{it} \geq -\theta_l \;,\; \theta_{i0} = \tfrac{1}{I} \text{ given}\\
&\text{v)}\quad h_{2i,t+1} = (1-\delta_{h_2 it})h_{2it} + x_{h_2 it} \;,\; h_{2it} \geq 0 \;,\; h_{2it} = h_{20} \text{ given}\\
&\text{vi)}\quad Q_t b_{i,t+1} = x_{bit} \;,\; b_{it} \geq -b_l \;,\; b_{i0} = 0 \text{ given}\,.
\end{aligned} \tag{2}$$

In equation (2), x-variables denote investment variables and we imposed arbitrary lower bounds, $b_l > 0$ and $\theta_l > 0$, on bond and stock holdings to render the maximization problem well-defined. Our

[16] Our results remain unchanged if household i only observes (s_i^t,S^t).

[17] All statements involving random variables are supposed to hold almost surely. The notation $\forall\ t$ is therefore simply shorthand for the following statement: for any period t, $t = 1,2,...,$ and any $(s^t,S^t) \in \mathbf{s}^t \times \mathbf{S}^t$ with $\pi_t(s^t,S^t) > 0$. Notice that in equilibrium all variables in period t can be expressed as functions of (s^t,S^t).

formulation of the budget constraint assumes that in each period households can freely move the one good across different uses (consumption, physical capital, human capital) and sectors, but that in each period *t* households have to make the capital allocation decision relevant for production in period *t+1*.

The budget constraint (2) does not impose non-negativity constraints on investment. In particular for human capital investment, this assumption could be problematic since it is in general impossible to sell human capital. In many applications, however, the non-negativity constraint on human capital investment will not bind in equilibrium and introducing the constraints would therefore not change the analysis. Moreover, it follows from the formula for equilibrium investment in human capital (see proposition) that for the case in which aggregate depreciation rates of physical and human capital are positive and equal, this constraint only binds when a household receives a large positive idiosyncratic shock to human capital.

The stock company is operated by a manager who chooses an investment policy, $\{X_{k_2 t}\}_{t=0}^{\infty}$, $X_{k_2 t} = X_{k_2 t}(S^t)$, an employment policy, $\{E_{2t}\}_{t=0}^{\infty}$, $E_{2t} = E_{2t}(S^t)$, and a dividend policy, $\{D_t\}_{t=0}^{\infty}$, $D_t = D(S^t)$, subject to the feasibility constraints

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\begin{aligned}
K_{2,t+1} &= (1 - \delta_{k_2 t}) K_{2t} + X_{k_2 t} \\
A_{2t} F(K_{2t}, E_{2t}) &= W_t E_{2t} + X_{k_2 t} + D_t \ ,
\end{aligned}
\tag{3}
$$

where D_t denotes dividend payment per outstanding share. In (3) we have normalized the number of outstanding shares to one and have ruled out debt-financing of investment. In other words, we have picked an arbitrary financial policy, namely the policy which finances investment through retained earnings. This is justified since for the equilibrium constructed below, a Modigliani-Miller theorem holds, that is, changes in the financial policy do not affect the equilibrium allocation.

Labor markets are competitive and the stock company hires labor on a period by period basis. Hence, the optimal choice of labor is a static maximization problem and will result in the equalization of the marginal product of labor and the wage rate (see proposition). The firm's overall objective is to maximize the present value of dividends (including current dividend payments)

$$\lim_{T\to\infty} E\left[\sum_{t=0}^{T} \beta^t \left(\sum_{i=1}^{I} \mu_i (c_{it}/c_{i0})^{-1}\right) D_t\right], \tag{4}$$

where the term $\beta^t (c_{it}/c_{i0})^{-1}$ is the intertemporal marginal rate of substitution (IMRS) between periods 0 and t of household i in equilibrium, and $\mu_i \geq 0$ stands for an arbitrary weighting function with $\sum_i \mu_i = 1$.[18] Below we will show that there is an equilibrium, and that this equilibrium is the same regardless of the choice of the weights μ_i. Thus, there is no disagreement among households regarding the investment policy of the firm, at least for the equilibria considered here. The dividend process $\{D_t\}_{t=0}^{\infty}$ entering into (4) is indirectly determined through the choice of $\{X_{2t}\}_{t=0}^{\infty}$, $\{E_{2t}\}_{t=0}^{\infty}$ using (3). The intertemporal marginal rates of substitution used in (4) are taken as given by the manager when evaluating the performance of different investment (dividend) policies. In the absence of bubbles, the current stock price is equal to the expected present discounted value of future dividends, and maximizing (4) therefore amounts to maximizing the value of outstanding shares, $P_0 + D_0$ (or, for that matter, $P_t + D_t$).[19]

We use the standard definition of equilibrium in a sequential economy with competitive markets and fulfilled (rational) expectations (Radner 1972, Lucas 1978).

Definition.

A sequential market equilibrium (SME) is list of sequences of functions (stochastic processes) $\{a_{it}\}_{t=0}^{\infty} \doteq \{c_{it}, k_{1it}, h_{1it}, \theta_{it}, h_{2it}, b_{it}\}_{t=0}^{\infty}$, $\{K_{2t}, X_{2t}, E_{2t}\}_{t=0}^{\infty}$, and $\{Q_t, P_t, D_t, W_t\}_{t=0}^{\infty}$ such that

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[18] The analysis remains unchanged if we use $\beta(c_{it}/c_{i,t-1})^{-1}$ to discount dividend payment D_t. Similarly, instead of assuming constant weights μ_i, we could have assumed weights that vary over time according to the relative ownership shares of individual households in the stock company. Our choice of the objective function (4) corresponds to the assumption that initial shareholders determine the firm's investment policy.

[19] We rule out bubbles by assumption. Santos and Woodford (1997) show that in economies with bounded aggregate output there can be no bubbles for assets in zero net supply. In our economy, however, aggregate output is unbounded and equity is, of course, in positive net supply.

i) Households: for given $\{Q_t, P_t, D_t, W_t\}_{t=0}^{\infty}$ and any i, the plan $\{a_{it}\}_{t=0}^{\infty}$ maximizes (1) subject to (2).

ii) Stock company: for given $\{W_t\}_{t=0}^{\infty}$ and $\{\beta^t \sum_i \mu_i (c_{it}/c_{i0})^{-1}\}_{t=0}^{\infty}$, the plan $\{K_{2t}, X_{2t}, E_{2t}, D_t\}_{t=0}^{\infty}$ maximizes (4) subject to (3).

iii) Market clearing:

$$\sum_{i=1}^{I} \theta_{it} = 1 \ ; \ \sum_{i=1}^{I} b_{it} = 0 \ ; \ \sum_{i=1}^{I} h_{2it} = E_{2t} \ .$$

Notice that the above market clearing conditions in conjunction with the budget constraints (2) and (3) imply goods market clearing (Walras' law)

$$C_t + X_{k_1 t} + X_{h_1 t} + X_{k_2 t} + X_{h_2 t} = Y_{1t} + Y_{2t} \ , \tag{5}$$

where $C_t \doteq \sum_i c_{it}$ and corresponding definitions for the other aggregate variables. Moreover, the aggregate capital accumulation equations are automatically satisfied.

III. Existence and Characterization of Equilibrium

We focus attention on equilibria for which

$$\frac{D_t + P_t}{P_{t-1}} = 1 + A_{2t}(S^t) F_k(\tilde{K}_{2t}) - \delta_{k_2 t}(S^t)$$
$$W_t = A_{2t}(S^t) F_h(\tilde{K}_{2t}) \ , \tag{6}$$

where $\tilde{K}_{2t} = K_{2t}/E_{2t}$ is the capital-to-labor ratio in the corporate sector and F_k and F_h are the marginal products of physical and human capital in the corporate sector. Clearly, the second equation in (6) is the necessary condition for profit maximization. The first condition in (6) means that we only consider stock market equilibria that can also be interpreted as equilibria in which households own all capital and the corporate-sector firm rents physical (and human) capital in competitive markets.

The budget constraint (2) can be rewritten in a way that shows that each household's utility maximization problem is basically a standard intertemporal portfolio choice problem. To see this, let us introduce the following variables

$$w_{it} \doteq k_{1it} + h_{1it} + \theta_{it} P_{t-1} + h_{2it} + b_{it} Q_{t-1}$$

$$z_{it} \doteq \frac{k_{1it} + h_{1it}}{w_{it}} \quad , \quad \tilde{k}_{1it} \doteq \frac{k_{1it}}{h_{1it}} \quad , \quad \tilde{k}_{2it} \doteq \frac{\theta_{it} P_{t-1}}{h_{2it}} \; .$$

The variable w_{it} is the wealth level of household i at the beginning of period t ,z_{it} is the share of capital (physical and human) invested in the first sector, $\tilde{k}_{1it}$ the ratio of physical to human capital in sector 1 (the capital-to-labor ratio in sector 1), and $\tilde{k}_{2it}$ the ratio of physical to human capital in sector 2 (the capital-to-labor ratio in sector 2). The values of the variables $w_{it}, z_{it}, \tilde{k}_{1it}, \tilde{k}_{2it}$ are known at the end of period *t-1*. Let us further denote the marginal product of physical capital by $A_{1it} f_k(\tilde{k}_{1it})$ and the marginal product of human capital (labor) by $A_{1it} f_h(\tilde{k}_{1it})$ and define the following returns to investment:

$$r_{k_1 it} \doteq A_{1t}(s_{it}, S^t) f_k(\tilde{k}_{1it}) - \delta_{k_1 t}(s_{it}, S^t)$$

$$r_{h_1 it} \doteq A_{1t}(s_{it}, S^t) f_h(\tilde{k}_{1it}) - \delta_{h_1 t}(s_{it}, S^t)$$

$$r_{k_2 t} \doteq A_{2t}(S^t) F_k(\tilde{K}_{2t}) - \delta_{k_2 t}(S^t)$$

$$r_{h_2 t} \doteq A_{2t}(S^t) F_h(\tilde{K}_{2t}) - \delta_{h_2 t}(s_{it}, S^t) \; .$$

Using the new notation, equation (6), and the constant-returns-to-scale assumption, the budget constraint (2) reads

$$w_{i,t+1} = \left(1 + r_{it}(z_{it}, \tilde{k}_{1it}, \tilde{k}_{2it}, \tilde{K}_{2t}, s_{it}, S^t)\right) w_{it} - c_{it}$$

$$w_{it} \geq 0 \; , \; \tilde{k}_{1it} \geq 0 \; , \; \tilde{k}_{2it} \geq 0 \; , \; 0 \leq z_{it} \leq 1 \tag{7}$$

$$\left(w_{i0}, z_{i0}, \tilde{k}_{1i0}, \tilde{k}_{2i0}\right) \text{ given } ,$$

where $r_{it} = r_t(z_{it}, \tilde{k}_{1it}, \tilde{k}_{2it}, \tilde{K}_{2t}, s_{it}, S^t)$ is the total investment return given by

$$r_{it} \doteq z_{it}\left[\frac{\tilde{k}_{1it}}{1+\tilde{k}_{1it}} r_{k_1 t}(\tilde{k}_{1it}, s_{it}, S^t) + \frac{1}{1+\tilde{k}_{1it}} r_{h_1 t}(\tilde{k}_{1it}, s_{it}, S^t)\right]$$

$$+ (1 - z_{it})\left[\frac{\tilde{k}_{2it}}{1+\tilde{k}_{2it}} r_{k_2 t}(\tilde{K}_{2t}, S^t) + \frac{1}{1+\tilde{k}_{2it}} r_{h_2 t}(\tilde{K}_{2t}, s_{it}, S^t)\right].$$

Below we show that, under certain conditions, there exists a SME for which $z_{it} = z_t(S^{t-1})$, $\tilde{k}_{1it} = \tilde{k}_{1t}(S^{t-1})$, and $\tilde{k}_{2it} = \tilde{k}_{2t}(S^{t-1}) = \tilde{K}_{2t}(S^{t-1})$, that is, (relative) portfolio choices are independent of individual wealth levels and individual shock realizations. For any S^t, the values of these functions are defined implicitly by the Euler equations associated with the households's utility maximization problem (see the Appendix for details):

$$\text{i)}\quad E\left[\frac{r_{h_1,t+1}(\tilde{k}_1, s_{i,t+1}, S^{t+1}) - r_{k_1,t+1}(\tilde{k}_1, s_{i,t+1}, S^{t+1})}{1 + r_{t+1}(z, \tilde{k}_1, \tilde{k}_2, \tilde{k}_2, s_{i,t+1}, S^{t+1})} \mid S^t\right] = 0 \tag{8}$$

$$\text{ii)}\quad E\left[\frac{r_{h_2,t+1}(\tilde{k}_2, s_{i,t+1}, S^{t+1}) - r_{k_2,t+1}(\tilde{k}_2, S^{t+1})}{1 + r_{t+1}(z, \tilde{k}_1, \tilde{k}_2, \tilde{k}_2, s_{i,t+1}, S^{t+1})} \mid S^t\right] = 0$$

$$\text{iii)}\quad E\left[\frac{r_{k_1,t+1}(\tilde{k}_1, s_{i,t+1}, S^{t+1}) - r_{k_2,t+1}(\tilde{k}_2, S^{t+1})}{1 + r_{t+1}(z, \tilde{k}_1, \tilde{k}_2, \tilde{k}_2, s_{i,t+1}, S^{t+1})} \mid S^t\right] = 0 .$$

Notice that (8) is an equation system that is defined in terms of exogenous variables only. From a computational point of view, (8) provides a simple way of calculating the portfolio shares (and therefore the equilibrium) if the underlying state process has a Markovian structure. We will present an example of such a calculation in the next section. Clearly, it is possible that (8) has only a solution $z \neq [0,1]$ for some S^t. In this case, one can still use our method of proof to show that a no-trade equilibrium exists, but in this equilibrium agents are at a corner solution for some S^t, implying that for those S^t the three equalities (8) are replaced by two equalities and one inequality.[20]

Proposition.

Suppose that there exist portfolio choices $\{z_t\}_{t=0}^{\infty}$, $\{\tilde{k}_{1t}\}_{t=0}^{\infty}$, $\{\tilde{k}_{2t}\}_{t=0}^{\infty}$ solving (8) and satisfying $0 \leq (z_t, \tilde{k}_{1t}, \tilde{k}_{2t}) \leq 1$. Then there exists a sequential market equilibrium (SME) which is characterized as follows. The allocation is given by

$$w_{i,t+1} = \beta\left(1 + r_t(z_t, \tilde{k}_{1t}, \tilde{k}_{2t}, \tilde{k}_{2t}, s_{it}, S^t)\right) w_{it}$$

$$c_{it} = (1 - \beta)\left(1 + r_t(z_t, \tilde{k}_{1t}, \tilde{k}_{2t}, \tilde{k}_{2t}, s_{it}, S^t)\right) w_{it}$$

and

$$k_{1it} = z_t \frac{\tilde{k}_{1t}}{1 + \tilde{k}_{1t}} w_{it} \quad , \quad h_{1it} = z_t \frac{1}{1 + \tilde{k}_{1t}} w_{it}$$

$$k_{2it} = (1 - z_t) \frac{\tilde{k}_{2t}}{1 + \tilde{k}_{2t}} w_{it} \quad , \quad h_{2it} = (1 - z_t) \frac{1}{1 + \tilde{k}_{2t}} w_{it}$$

$$b_{it} = 0 \quad , \quad \theta_{it} = \frac{k_{2it}}{K_{2t}} \quad \text{with} \quad K_{2t} = \sum_i k_{2it} \; .$$

The dividend and wage policies are given by

$$D_t = \left(1 + A_{2t} F_k(\tilde{k}_{2t}) - \delta_{k_2 t}\right) K_{2t} - K_{2,t+1} \quad ; \quad W_t = A_{2t} F_h(\tilde{k}_{2t}) \; .$$

and bond and stock prices by[21]

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[20] With the standard Inada condition for production functions, the choices $\tilde{k}_1$ and $\tilde{k}_2$ are always interior solutions.

[21] Of course, stock returns (prices and dividends) satisfy the corresponding Euler equation (see appendix A1).

$$P_t = K_{2,t+1}$$

$$\begin{aligned} Q_t &= E[\,\beta\,(c_{i,t+1}/c_{it})^{-1} \mid S_t\,] \\ &= E\left[\left(1 + r_{t+1}(z_{t+1}, \tilde{k}_{1,t+1}, \tilde{k}_{2,t+1}, \tilde{k}_{2,t+1}, s_{i,t+1}, S^{t+1})\right)^{-1} \mid S^t\right]. \end{aligned}$$

Proof. See appendix.

In order to compare our framework with the model of an exchange economy discussed by Constantinides and Duffie (1996), let us calculate the growth rate of individual consumption in equilibrium using the proposition. Taking logs we find

$$\begin{aligned} \log c_{i,t+1} &= \log\beta + \log c_{it} + \log(1 + r_{i,t+1}) \\ &\approx \log\beta + \log c_{it} + r_{i,t+1} \end{aligned} \tag{9}$$

Notice that $s_{i,t+1}$ is the only idiosyncratic variable that affects the total return $r_{i,t+1} = r_{t+1}(z_{t+1}, \tilde{k}_{1,t+1}, \tilde{k}_{2,t+1}, \tilde{k}_{2,t+1}, s_{i,t+1}, S^{t+1})$. Since the idiosyncratic shock, $s_{i,t+1}$, is unpredictable, this means that consumption follows a logarithmic random walk. A similar argument shows that income follows (approximately) a logarithmic random walk (see section IV). Thus, as in Duffie and Constantinides (1996), income shocks are permanent, which is the reason why borrowing and lending is not the optimal response to income shocks (no-trade equilibrium). However, in contrast to Duffie and Constantinides (1996) the current model allows for any type of distributional assumption and also has positive aggregate saving.

IV. Quantitative Results

In this section, we compute the equilibrium of a calibrated model economy and study the asset return and business cycle implications.

a) Model Specification

We assume that $\{S_t\}_{t=0}^{\infty}$ is a two-state i.i.d. process. Thus, the state space is $\boldsymbol{S} \equiv \{L,H\}$, where L stands for the event of low economic activity (in both sectors) and H stands for the event of high economic activity (in both sectors). In addition, we assume that both aggregate states are equally likely: $\pi(L) = \pi(H) = 1/2$. The idiosyncratic state has two components, $s_{it} = (s_{1it}, s_{2it})$, where s_{1it} is the idiosyncratic shock to investment in the first sector and s_{2it} is the idiosyncratic shock to investment in the second sector. There are two possible idiosyncratic shock realizations: $s_{1it} \in \{l,h\}$ and $s_{2it} \in \{l,h\}$, and we assume that idiosyncratic shocks are uncorrelated across sectors and equally likely: $\pi(s_{1it}, s_{2it}) = \pi(s_{1it})\,\pi(s_{2it})$ and $\pi(l) = \pi(h) = 1/2$. Notice that even though the probability of a particular idiosyncratic shock does not depend on the aggregate state, there is still a correlation between idiosyncratic risk and the aggregate state because the magnitude of idiosyncratic depreciation shocks will depend on the aggregate state (see below).

We assume that the production functions in both sectors are the same and of the Cobb-Douglas type

$$\begin{aligned} y_{1it} &= A_t k_{1it}^{\alpha} h_{1it}^{(1-\alpha)} \\ Y_{2t} &= A_t K_{2t}^{\alpha} H_{2t}^{(1-\alpha)} \ . \end{aligned} \tag{10}$$

For simplicity, we have omitted idiosyncratic productivity shocks since we will introduce idiosyncratic depreciation shocks below, and these depreciation shocks suffice to permit arbitrary variations in idiosyncratic risk. The depreciation rates of physical and human capital in the first sector are equal and defined by $\delta_{1it} = \delta_1(s_{1it}, S_t)$. Similarly, depreciation rates of physical and human capital in the second sector are equal and defined by $\delta_{2it} = \delta_2(s_{2it}, S_t)$.

The Cobb-Douglas specification in conjunction with the assumption of equal depreciation rates for physical and human capital implies that $\tilde{k}_{1t} = \frac{\alpha}{1-\alpha}$. This can be seen by observing that this choice satisfies the first Euler equation in (8). Moreover, we have $z_t = z$ and $\tilde{k}_{2t} = \tilde{k}_2$ because the aggregate state is unpredictable. The two values z and $\tilde{k}_2$ are determined by the remaining two Euler equations.

The formula for equilibrium consumption (proposition) implies that per capita (aggregate) consumption growth is

$$\begin{aligned} \frac{C_{t+1}}{C_t} &= E\left[\frac{c_{i,t+1}}{c_{it}} \mid S^{t+1}\right] \\ &= \beta\left(1 + E[r_{i,t+1} \mid S^{t+1}]\right) \\ &= \beta\left(1 + r(z,\tilde{k}_1,\tilde{k}_2,,\tilde{k}_2,S_{t+1})\right) , \end{aligned} \tag{11}$$

where $r(z,\tilde{k}_1,\tilde{k}_2,\tilde{k}_2,S_{t+1}) = E[r(z,\tilde{k}_1,\tilde{k}_2,\tilde{k}_2,s_{i,t+1},S_{t+1}) \mid S_{t+1}]$ with $r_{i,t+1} = r(z,\tilde{k}_1,\tilde{k}_2,\tilde{k}_2,s_{i,t+1},S_{t+1})$ as defined in (7). Thus, aggregate consumption growth in our incomplete-markets economy is equal to aggregate consumption growth in the corresponding representative-agent economy with total investment return $r_{t+1} = r(z,\tilde{k}_1,\tilde{k}_2,S_{t+1})$. Similar formulas hold for aggregate variables like output and investment. Hence, the model's business-cycle implications are similar to the business cycle implications of its representative-agent counterpart. Since these business-cycle implications have been extensively studied by Jones et al. (2000), in this paper we focus on asset return predictions, which differ quite substantially from the predictions of the corresponding representative-agent model (see below).

A second implication of (11) is that aggregate consumption growth is i.i.d., that is, log-consumption growth follows (approximately) a random walk. Hence, the risk-free rate is constant. Annual data on consumption and real short-term interest rates show only small deviations from these two characteristics (Campbell and Cochrane, 1999). Since in this paper we use annual data to compare the model's prediction with the empirical facts, our assumption that the aggregate state process is i.i.d. seems therefore a reasonable first approximation.[22]

b) Calibration

We assume that the period length is one year (annual data). We use $\alpha = .36$ to match capital's share

[22] The real business cycle literature uses quarterly data, whereas most of the asset pricing literature uses annual data. In this paper we use annual data because our calibration relies on estimates of idiosyncratic income risk that are obtained from annual PSID data.

in income. This yields $\tilde{k}_1 = \alpha/(1+\alpha) = .5625$. The values of the preference parameter β, the two productivity parameters $A(L)$ and $A(H)$, and the eight depreciation parameters $\delta_j(s_j, S)$, $j = 1,2$, $s_j = l,h$, $S = L,H$, are determined in conjunction with the values for z and $\tilde{k}_2$ by the following restrictions:

- The two remaining Euler equations hold
- $E[\delta_{1t}] = E[\delta_{2t}] = .06$
- $E[Y_{t+1}/Y_t - 1] = .02$
- $E[X_{kt}/Y_t] = .25$
- $z = .50$
- $\sigma[C_{t+1}/C_t] = .03314$
- $\sigma[Y_{t+1}/Y_t] = .05584$
- $\sigma[y_{i,t+1}/y_{it} \mid S_{t+1} = L] = .24 \quad , \quad \sigma[y_{i,t+1}/y_{it} \mid S_{t+1} = H] = .12$
- $$\frac{\sigma[y_{1i,t+1}/y_{1it} \mid S_{t+1} = L]}{\sigma[y_{1i,t+1}/y_{1it} \mid S_{t+1} = H]} = \frac{\sigma[y_{2i,t+1}/y_{2it} \mid S_{t+1} = L]}{\sigma[y_{2i,t+1}/y_{2it} \mid S_{t+1} = H]}$$

Notice that we do not require the aggregate depreciation rates, $E[\delta_{1it} \mid S_t]$ and $E[\delta_{2it} \mid S_t]$, to be constant, that is, we allow for aggregate depreciation shocks. In our heterogeneous-agent economy, these aggregate fluctuations in depreciation rates correspond to fluctuations in the aggregate rate of business failure and job displacement.

The first of the above restrictions ensures that the portfolio choices z and $\tilde{k}_2$ are equilibrium choices. The next restriction pins down the average aggregate depreciation rate. The value of .06 is a compromise between the probably higher depreciation rate of physical capital[23] and the probably lower depreciation rate of human capital. This value is also assumed by Jones et al. (2000). The next two restrictions ensure that the implied average values for per capita output growth and the saving rate match their empirical counterpart for the U.S. economy. The next restriction says that the corporate sector is half of the entire economy. The following two restrictions require the implied

[23] A common choice is $\delta_k = .10$. However, Cooley and Prescott (1995) argue that $\delta_k = .05$ is a more reasonable value.

volatility of aggregate consumption and output growth to match the US experience for the period 1890-2000.[24] The last two equations restrict idiosyncratic labor income risk. The second of these two restrictions is imposed to reduce the number of free parameters. The first one ensures that the implied process of income risk is consistent with evidence from microeconomic data, to which we now turn.

According to the model, total income of household i is

$$\begin{aligned} y_{it} &= y_{1it} + y_{2it} \\ &= A(S_t)\,\alpha\,\tilde{k}_1^{\alpha}\,h_{1it} + \left[(r_{k_2t} + \delta_{2t})k_{2it} + (r_{h_2t} + \delta_{2t})\,h_{2it}\right] \\ &= A(S_t)\left[\alpha\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} z + \alpha\,\tilde{k}_2^{\alpha-1}(1-z)\frac{\tilde{k}_2}{1+\tilde{k}_2} + (1-\alpha)\tilde{k}_2^{\alpha}(1-z)\frac{1}{1+\tilde{k}_2}\right] w_{it}\,. \end{aligned} \tag{12}$$

This implies

$$\frac{y_{i,t+1}}{y_{it}} = \frac{A(S_{t+1})}{A(S_t)}\,\beta\left(1 + r(z,\tilde{k}_1,\tilde{k}_2,\tilde{k}_2,s_{it},S_t)\right)\,. \tag{13}$$

Using the approximation $\log(1+r) \approx r$, we find

$$\log y_{i,t+1} = \varphi(z,\tilde{k}_1,\tilde{k}_2,\tilde{k}_2,S_t,S_{t+1}) + \log y_{it} + \eta_{it}\,, \tag{14}$$

where $\eta_{it} = z\left(\delta_{1it} - E[\delta_{1it}|S_t]\right) + \frac{1-z}{1+\tilde{k}_2}\left(\delta_{2it} - E[\delta_{2it}|S_t]\right)$. Thus, conditional on the history of aggregate states, individual log-income follows (approximately) a random walk.[25] The random walk

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[24] Real per capita GNP figures are taken from *Historical Statistics of the United States, Colonial Times to 1970* for the period 1890-1970 and from The St. Louis Fed's *FRED* database for the period 1971-2000. Real per capita consumption figures are taken from the online data section of Robert Shiller's home page for the period 1890-1970 and from *FRED* for the period 1971-2000.

[25] We have η_{it} instead of $\eta_{i,t+1}$ in equation (15), where the latter is the common specification for a random walk. However, if the econometrician observes the idiosyncratic depreciation shocks with a one-period lag, then (15) is the correct equation form the household's point of view, but a modified version of (15) with $\eta_{i,t+1}$ instead of η_{it} is the specification estimated by the econometrician.

specification is often used by the empirical literature to model the permanent component of the income process (Carroll and Samwick, 1997, Hubbard, Skinner, and Zeldes, 1995, Meghir and Pistaferri, 2001, Storesletten et. al., 2001),[26] and this literature therefore provides us with an estimate of $\sigma[\eta_{it} | S_t] = \sigma[y_{i,t+1}/y_{it} | S_t]$. Carroll and Samwick (1997) and Hubbard et al. (1995) estimate that the mean of this standard deviation, $E[\sigma[\eta_{it} | S_t]]$, has a value of .15, Meghir and Pistaferri (2001) find .19, and Storesletten et al. (2001) estimate .25. We choose $E[\sigma[\eta_{it} | S_t]] = .18$. Meghir and Pistaferri (2001) and Storesletten et al. (2001) are the only studies so far that allow the standard deviation of η_{it} to vary with the aggregate state, S_t. Meghir and Pistaferri (2001) find that the variation of this standard deviation, measured by $\sigma[\sigma[\eta_{it} | S_t]]$, is equal to $.05$ for all education groups and $.06$ for college-educated individuals. The college-educated group of households is likely to be the more relevant group since it includes most of the stock holders. Storesletten et al. (2001) find $\sigma[\sigma[\eta_{it} | S_t]] = .10$. In the baseline economy, we assume $\sigma[\sigma[\eta_{it} | S_t]] = .06$. Given that there are two equally probable aggregate states, the two restrictions $E[\sigma[\eta_{it} | S_t]] = .18$ and $\sigma[\sigma[\eta_{it} | S_t]] = .06$ translate into the two equations assumed in the calibration exercise, $\sigma[\eta_{it} | S_t = L] = .24$ and $\sigma[\eta_{it} | S_t = H] = .12$.

There is at least one reason why the above procedure might underestimate idiosyncratic income risk: the actual distribution of income growth might have a fatter lower tail than suggested by the normal distribution (Brav et al., 2002, and Geweke and Keane, 2000). In this paper, we do not consider the implications of such possible deviations from the normal-distribution framework, but simply note that in principle the incomplete-markets model can generate any equity premium even if the standard deviation of income shocks is constant and/or close to zero (Krebs, 2001).

In addition to the implications for labor income risk, the model also has implications for individual consumption volatility. More precisely, in the baseline economy we have $E[\sigma[c_{i,t+1}/c_{it} | S_t]] = .18$. In comparison, CEX data on consumption of non-durables and services Brav et al. (2002) find a standard deviation of quarterly consumption growth ranges from of .06 to

[26] Notice that Hubbard et al. (1995) and Storesletten et al. (2001) do not impose the random walk restriction, but estimate an autocorrelation coefficient close to one for the permanent income shocks. Heaton and Lucas (1996) do not allow for shocks of different persistence.

.12 for different household groups with an average of .09 . If consumption growth is i.i.d., then this corresponds to an (average) annual standard deviation of .18. Thus, consumption volatility is roughly in line with the data.

c) Results

The model is calibrated so that it matches certain features of the real sector of the U.S. economy. More specifically, it matches the observed average per capita output growth and saving rates as well as the volatility of output and consumption. In contrast, the calibration method does not constrain the asset return implications of the model. In this section, we compare the model's asset return predictions with U.S. financial data.

In table 1, we report the mean and standard deviation of the equity premium, the Sharpe ratio, and the standard deviations of aggregate per capita consumption and output growth produced by several versions of the model economy. The baseline economy (M1) features both aggregate productivity and depreciation shocks. We also consider a calibration with only aggregate productivity shocks (M2), and a calibration with aggregate productivity and depreciation shocks that matches observed stock return volatility by construction (M3). Results for several different magnitudes of variation in idiosyncratic risk are reported (a, b, and c). For purposes of comparison, we have also added results for the complete-markets versions of our production model and for the Mehra-Prescott economy (complete-markets exchange economy).

Table 1 shows that for the baseline economy (M1a) the implied mean of the equity premium, $E[r_{k_2 t} - r_b]$, is still far below the mean of the observed equity premium: $.18\%$ vs. 6.54%. This value increases to $.23\%$ for an economy (M1c) with variations in idiosyncratic risk comparable to the ones found by Storesletten et al. (2001). In comparison, the value for the corresponding complete-markets economy is $.11\%$ (CM1). Notice that a mean equity premium of $.11\%$ is much higher than the mean equity premium of the corresponding exchange economy (Mehra-Prescott economy, CM4), which is a meager $.006\%$. The main reason for this large discrepancy is the fact that stock return volatility in the exchange economy is much lower than stock return volatility in the production economy (table 1).[27] In summary, our results show that market-incompleteness roughly doubles the mean equity premium, but moving from an exchange economy to a production economy increases the mean equity premium by a factor of 50.

Although the introduction of production improves the model's ability to match the first and second moment of aggregate stock returns, both theoretical moments still fall short of what is observed in the data. The model's low volatility of stock returns, which in a sense is responsible for the low mean of the equity premium, is mainly a consequence of the following two model features. First, equity returns are equal to the marginal product of physical capital net of depreciation. Second, the model is calibrated so as to match the observed volatility of output growth. Since in the data output growth is far less volatile than equity returns, the calibrated model implies a stock return volatility that falls far below the observed volatility. Given this "volatility problem", a comparison of Sharpe ratios is perhaps a more informative way of assessing the model's ability to generate a realistic price of aggregate risk.[28] A glance at table 1 shows that with respect to the Sharpe ratio, the model's performance is quite good: model M1a implies a Sharpe ratio of .0555 and model M1c implies a Sharpe ratio of .0701, whereas in the data the Sharpe ratio is .3522.

Table 1 also reports the implications of a model without aggregate depreciation shocks (M2a-M2c). In this case, productivity shocks are the only source of aggregate uncertainty. Table 1 reveals that for this model aggregate consumption growth is far too smooth compared to the data, and that the already low stock return volatility drops even further.[29] Thus, we conclude that the introduction of aggregate depreciation shocks improves the model's ability to match the consumption and stock return data significantly.

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[27] Of course, in both economies the consumption Euler equation holds. However, following Mehra and Prescott (1985) we set dividends equal to output in the exchange economy, which implies that the dividend series in the exchange economy is much smoother than the dividend series in our production economy.

[28] This is the strategy pursued in Tallarini (2000).

[29] The counter-factually low volatility of aggregate consumption in the model with only productivity shocks has also been noticed by Jones et al. (2000).

Finally, we report the mean equity premium for a model economy in which equity returns are as volatile as observed in the data. As mentioned in the Introduction, in this case we consider an economy without a stock market sector ($z = 1$) so that we are still able to match the volatility of aggregate consumption and output growth. Even though this model has no stock market sector, we can still discuss its equity premium implication, that is, we can ask what average equity premium (stock return) makes $z = 1$ an equilibrium outcome for a given level of stock return volatility. Table 1 reveals that with realistic variations in stock returns, the equity premium is substantial: 1.01% for moderate variations in idiosyncratic risk (M3a), and 1.27% if idiosyncratic risk is highly variable (M3c).

V. Conclusion

In this paper, we developed a stochastic endogenous growth model with incomplete markets and proved the existence of a highly tractable equilibrium. We also provided a first assessment of the empirical performance of a calibrated version of the model. Given its simplicity, the calibrated model economy is relatively successful in reproducing some important features of observed asset returns and business cycle fluctuations.

There are several extensions of the present model which could provide promising avenues for future research. First, the preference specification should be generalized, at a minimum to the general case of time- and state-additive preferences with CRRA one-period utility function. Such an extension allows one to study whether the model is capable of matching the risk premium exactly for moderate degrees of relative risk aversion (more than one, the case considered here, but less than four). Second, some additional "market friction" can be introduced to break the tight (and counter-factual) link between the marginal product of capital in the corporate sector and stock returns. Finally, a more general specification of the productivity process will in general lead to equilibria with bond trading. This type of extension reintroduces analytical complexity, but is indispensable for a study of trading volume in asset markets.

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Appendix A1.

Clearly, the allocation satisfies the market clearing condition. Hence, it suffices to show that households and firms are choosing optimal policies. Consider first households. Each household maximizes (1) subject to (2). The Euler equations for the optimal choice of $\{k_{1it}\}_{t=0}^{\infty}$, $\{h_{1it}\}_{t=0}^{\infty}$, $\{\theta_{it}\}_{t=0}^{\infty}$, $\{h_{2it}\}_{t=0}^{\infty}$, $\{b_{it}\}_{t=0}^{\infty}$ are

$$
\begin{aligned}
&\forall\, t:\\
c_{it}^{-1} &= \beta\, E\left[(1+r_{k_1 i,t+1})\, c_{i,t+1}^{-1} \mid s^t, S^t\right]\\
c_{it}^{-1} &= \beta\, E\left[(1+r_{h_1 i,t+1}) c_{i,t+1}^{-1} \mid s^t, S^t\right]\\
c_{it}^{-1} &= \beta\, E\left[(1+r_{k_2,t+1}) c_{i,t+1}^{-1} \mid s^t, S^t\right] \qquad \text{(A1)}\\
c_{it}^{-1} &= \beta\, E\left[(1+r_{h_2 i,t+1}) c_{i,t+1}^{-1} \mid s^t, S^t\right]\\
c_{it}^{-1} Q_t &= \beta\, E\left[c_{i,t+1}^{-1} \mid s^t, S^t\right],
\end{aligned}
$$

where the returns are defined as in Section III. The corresponding transversality condition reads

$$
\forall\, t: \quad \lim_{T\to\infty} \beta^T \left\{ E\left[\left(r_{k_1 i,t+T} + r_{h_1 i,t+T} + r_{k_2,t+T} + r_{h_2,t+T} + 5\right) c_{i,t+T}^{-1} \mid s^t, S^t\right]\right\} = 0\,. \qquad \text{(A2)}
$$

Since the Euler equations in conjunction with the transversality condition are sufficient, the household's choice as specified in the proposition is optimal if it satisfies the budget constraint (2) as well as (A1) and (A2). Clearly, the consumption and investment plans specified in the proposition satisfy the budget constraint (7). Since (7) is equivalent to (2), the plan satisfies (2). Thus, it is left to show that (A1) and (A2) hold.

To see that the Euler equations (A1) are satisfied, notice first that for the proposed equilibrium consumption plan we have

$$\left(\frac{c_{i,t+1}}{c_{it}}\right)^{-1} = \left(\beta\left(1+r_{i,t+1}\right)\right)^{-1} , \tag{A3}$$

where $r_{i,t+1}$ is defined as in Section III. Using (A3) and the property that s^t is not useful in predicting $(s_{i,t+1}, S_{t+1})$ (which implies that s^t is not useful in predicting returns), we can rewrite the Euler equations (A1) as

$$\forall\, t :$$

$$1 = E\left[\frac{1+r_{k_1 i,t+1}}{1+r_{i,t+1}} \mid S^t\right] , \quad 1 = E\left[\frac{1+r_{h_1 i,t+1}}{1+r_{i,t+1}} \mid S^t\right]$$

$$1 = E\left[\frac{1+r_{k_2,t+1}}{1+r_{i,t+1}} \mid S^t\right] , \quad 1 = E\left[\frac{1+r_{h_2 i,t+1}}{1+r_{i,t+1}} \mid S^t\right] \tag{A4}$$

$$Q_t = E\left[(1+r_{i,t+1})^{-1} \mid S^t\right].$$

The last equation in (A4) is satisfied by construction (see the expression for equilibrium bond prices). The first four equations of (A4) are equivalent to condition (8). To see this, notice first that (A4) clearly implies (8). Conversely, (8) implies (A4) because by construction

$$r_{i,t+1} = x_{1t}\, r_{k_1 i,t+1} + x_{2t}\, r_{h_2 i,+1t} + x_{3t}\, r_{k2 i,t+1} + (1 - x_{1t} - x_{2t} - x_{3t})\, r_{h_2 i,t+1} , \tag{A5}$$

where x_{1t}, x_{2t} and x_{3t} are positive numbers known at time t. Since condition (8) holds by assumption, the Euler equations (A1) are satisfied.

In order to show that the transversality condition (A2) holds, we first note that the conditioning variable s^t can again be dropped. This follows immediately from an argument similar to the one made above. Straightforward calculation then shows that (A2) holds.

The two constraints (3) of the firm's maximization problem can be combined to

$$D_t = A_{2t} F(K_{2t}, E_{2t}) - W_t E_t - K_{2,t+1} + (1 - \delta_{k_2 t}) K_{2t} \tag{A6}$$

Substituting (A6) into the objective function (4) shows that the firm's maximization problem reduces to choosing sequences $\{K_{2t}\}_{t=0}^{\infty}$ and $\{E_{2t}\}_{t=0}^{\infty}$ maximizing

$$\lim_{T\to\infty} E\left[\sum_{t=0}^{T} \beta^t \left(\sum_{i=1}^{I} \mu_i (c_{it}/c_{i0})^{-1}\right) \left(A_{2t} F(K_{2t}, E_{2t}) - W_t E_{2t} - K_{2,t+1} + (1 - \delta_{k_2 t}) K_{2t}\right)\right]. \tag{A7}$$

The optimal choice of $\{E_{2t}\}_{t=0}^{\infty}$ yields the equalization of the marginal product of labor and the real wage. The optimal choice of $\{K_{2t}\}_{t=0}^{\infty}$ results in the Euler equation

$$1 = \beta\, E\left[\sum_{i=1}^{I} \mu_i \left(\frac{c_{i,t+1}}{c_{it}}\right)^{-1} (1 + r_{k_2,t+1}) \mid S^t\right], \tag{A8}$$

which is satisfied because of (A3) and (A5). Finally, straightforward calculation demonstrates that the transversality condition corresponding to the firm's maximization problem is also satisfied.

Table 1.

	μ_{EP}	σ_{EP}	S	σ_c	σ_y
US Economy	6.56%	18.41%	.357	3.31%	5.58%
M1a	.18%	3.30%	.056	3.31%	5.58%
M1b	.21%	3.29%	.063	3.31%	5.58%
M1c	.23%	3.27%	.070	3.31%	5.58%
CM1	.11%	3.25%	.032	3.31%	5.58%
M2a	.010%	.398%	.026	.52%	5.58%
M2b	.013%	.393%	.033	.53%	5.58%
M2c	.016%	.386%	.040	.54%	5.58%
CM2	.002%	.461%	.004	.45%	5.58%
M3a	1.01%	18.41%	.055	3.31%	5.58%
M3b	1.14%	18.41%	.062	3.31%	5.58%
M3c	1.27%	18.41%	.069	3.31%	5.58%
CM3	.59%	18.41%	.032	3.31%	5.58%
CM4	.006%	.19%	.033	3.31%	3.31%

M: incomplete-markets economy

CM: complete-markets economy

1: economy with aggregate productivity and depreciation shocks calibrated to match observed aggregate output and consumption volatility

2: economy with only aggregate productivity shocks calibrated to match observed output volatility

3: economy with aggregate productivity and depreciation shocks calibrated to match observed stock return volatility

4: exchange economy

a: idiosyncratic risk satisfying $\sigma[y_{i,t+1}/y_{it} \mid S_{t+1} = L] = .24$ and $\sigma[y_{i,t+1}/y_{it} \mid S_{t+1} = H] = .12$

b: idiosyncratic risk satisfying $\sigma[y_{i,t+1}/y_{it} \mid S_{t+1} = L] = .26$ and $\sigma[y_{i,t+1}/y_{it} \mid S_{t+1} = H] = .10$

c: idiosyncratic risk satisfying $\sigma[y_{i,t+1}/y_{it} \mid S_{t+1} = L] = .28$ and $\sigma[y_{i,t+1}/y_{it} \mid S_{t+1} = H] = .08$

μ_{EP}: mean of equity premium

σ_{EP}: standard deviation of equity premium

S: Sharpe ratio

σ_c: standard deviation of aggregate consumption growth

σ_y: standard deviation of aggregate output growth